—NOTES—

A SOLUTION OF VAN DER POL'S DIFFERENTIAL EQUATION*

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The object of the present note is to form an elliptic function of the second order:

$$x = \wp(t \mid 2\omega_1 , 2\omega_3),$$

ω_1 being real and ω_3 purely imaginary, which satisfies the well-known differential equation of Van der Pol:

$$x'' + \mu(x^2 - 1)x' + \kappa^2 x = 0, \qquad (' = d/dt) \tag{1}$$

under the initial condition that $x = x_0$ when $t = t_0$.

Substituting in (1) the following relations with the usual notations [1] in the theory of elliptic functions:

$$x = \wp(t \mid 2\omega_1 , 2\omega_3), \qquad (\wp')^2 = 4\wp^3 - g_2\wp - g_3 , \qquad \wp'' = 6\wp^2 - \tfrac{1}{2}g_2 , \qquad (' = d/dt)$$

we get

$$6\wp^2 - \tfrac{1}{2}g_2 + \mu(\wp^2 - 1)(4\wp^3 - g_2\wp - g_3)^{1/2} + \kappa^2\wp = 0,$$

or, by clearing off the radical,

$$4\mu^2\wp^7 - (8 + g_2)\mu^2\wp^5 - (36 + \mu^2 g_3)\wp^4 + \{(4 + 2g_2)\mu^2 - 12\kappa^2\}\wp^3 + (6g_2 + 2\mu^2 g_3 - \kappa^4)\wp^2 - (\mu^2 - \kappa^2)g_2\wp - \mu^2 g_3 - \tfrac{1}{4}g_2^2 = 0. \tag{2}$$

Next, for the purpose of settling the fundamental periods $2\omega_1$, $2\omega_3$, let us take

$$-\mu^2 g_3 = \tfrac{1}{4}g_2^2 , \tag{3}$$

then from (2) and (3), we obtain

$$4\mu^2\wp^6 - (8 + g_2)\mu^2\wp^4 - (36 - \tfrac{1}{4}g_2^2)\wp^3 + \{(4 + 2g_2)\mu^2 - 12\kappa^2\}\wp^2 - (\kappa^4 - 6g_2 + \tfrac{1}{2}g_2^2)\wp - (\mu^2 - \kappa^2)g_2 = 0, \tag{4}$$

or, by the substitution of the initial condition,

$$x_0(x_0^2 - 2)g_2^2 + 24\{x_0 + \tfrac{1}{6}\kappa^2 - \tfrac{1}{6}\mu^2(x_0^2 - 1)^2\}g_2 - 144\{x_0^3 + \tfrac{1}{3}\kappa^2 x_0^2 + \tfrac{1}{36}\kappa^4 x_0 - \tfrac{1}{9}\mu^2 x_0^2(x_0^2 - 1)^2\} = 0. \tag{5}$$

In the present case, both g_2 and g_3 must be real; besides, g_3 must also be negative. Now, we have the expressions:

$$g_2 = \left(\frac{\pi}{\omega_1}\right)^4\left\{\frac{1}{12} + 20\sum_{n=1}^{\infty}\frac{n^3 q^{2n}}{1 - q^{2n}}\right\},$$

$$-g_3 = \left(\frac{\pi}{\omega_1}\right)^6\left\{\frac{7}{3}\sum_{n=1}^{\infty}\frac{n^5 q^{2n}}{1 - q^{2n}} - \frac{1}{216}\right\}, \quad q = \exp(\omega_3\pi i/\omega_1), \quad i = (-1)^{1/2}$$

where q will be later seen a positive quantity less than 1.

*Received June 5, 1966; revised manuscript received August 12, 1966.

If q is approximately greater than $(505)^{-1/2}$, then $-g_3 > 0$ and (3) holds. If

$$\{x_0 + \tfrac{1}{6}\kappa^2 - \tfrac{1}{6}\mu^2(x_0^2 - 1)^2\}^2 + x_0(x_0^2 - 2)\{x_0^3 + \tfrac{1}{3}\kappa^2 x_0^2 + \tfrac{1}{36}\kappa^4 x_0 - \tfrac{1}{9}\mu^2 x_0^2(x_0^2 - 1)^2\} \geqq 0,$$

we see by (5) that g_2 is real. In the case that both μ and κ are comparatively smaller than x_0, we see that the condition is satisfied, for

$$x_0^2 + x_0^4(x_0^2 - 2) = x_0^2(x_0^2 - 1)^2 \geqq 0.$$

Under the above restrictions, we can thus calculate ω_1 and ω_3 with the values of g_2 and g_3.

By solving the three equations:

$$g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1),$$
$$g_3 = 4e_1e_2e_3,$$
$$0 = e_1 + e_2 + e_3$$

we obtain a set of real quantities e_1, e_2, e_3 such that $e_1 > e_2 > e_3$, if $0 < g_2 < (16/27)\mu^4$; consequently

$$k^2 = (e_2 - e_3)/(e_1 - e_3), \qquad k'^2 = (e_1 - e_2)/(e_1 - e_3).$$

Finally we can compute K, K′, ω_1, ω_3 by the formulae:

$$\mathrm{K} = (\pi/2)F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2), \qquad \mathrm{K}' = (\pi/2)F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k'^2),$$
$$\omega_1 = \mathrm{K}(e_1 - e_3)^{-1/2}, \qquad \omega_3 = i\mathrm{K}'(e_1 - e_3)^{-1/2}.$$

$q = \exp(\omega_3\pi i/\omega_1) = \exp(-\mathrm{K}'\pi/\mathrm{K})$, which is real and less than 1.

Various properties of the solutions of (1) usually discussed can be derived directly from the explicit expression of the solution we have obtained, namely

$$x = \wp(t \mid 2\omega_1, 2\omega_3)$$
$$= -\frac{\eta_1}{\omega_1} + \left(\frac{\pi}{2\omega_1}\right)^2 \operatorname{cosec}^2 \pi v - 2\left(\frac{\pi}{\omega_1}\right)^2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \cos 2n\pi v$$

where

$$v = \frac{t}{2\omega_1}, \qquad \eta_1 = \zeta(\omega_1) = \frac{\pi^2}{\omega_1}\left\{\frac{1}{12} - 2\sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}}\right\}.$$

Consider, for example, the path on the phase plane. Now

$$y \equiv x' = \wp'(t \mid 2\omega_1, 2\omega_3), \qquad (' = d/dt)$$

and the equation of the path runs as follows,

$$y^2 = 4x^3 - g_2x + (1/4\mu^2)g_2^2 = 4(x - e_1)(x - e_2)(x - e_3).$$

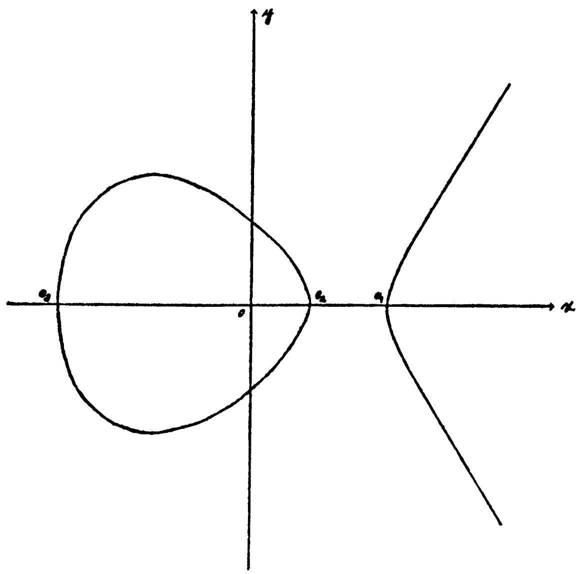
Since

$$e_1 + e_2 + e_3 = 0,$$
$$e_1 > e_2 > e_3,$$
$$e_1e_2e_3 = -(1/4\mu^2)g_2^2 < 0,$$

we find easily

$$e_3 < 0, \qquad e_1 > e_2 > 0;$$

the ordinates are then imaginary for $x < e_3$, $e_2 < x < e_1$. Hence we see that there exists always, and evidently only, one cycle on the phase plane for each solution under consideration as shown in the figure.



Reference

[1] F. Oberhettinger und W. Magnus, *Anwendung der Elliptischen Funktionen in Physik und Technik,* Springer-Verlag, Berlin, 1949; first chapter (where $\omega = \omega_1$, $\omega' = \omega_3$).