FINITE AMPLITUDE OSCILLATIONS IN CURVILINEARLY AEOLOTROPIC ELASTIC CYLINDERS*

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1. Introduction. In 1954, Ericksen and Rivlin [1] formulated the theory of finite deformations of transversely isotropic materials. Certain problems were solved by them assuming the materials to possess rectilinear aeolotropy. Following this Green and Adkins [2] added a few more results to the above set of solutions and recently Huilgol [3] added a further one when he showed that the deformation of Singh and Pipkin [4] was possible in transversely isotropic materials as well. This seems to cover the total list of static deformations.

The present paper deals with the dynamic problem of axially symmetric oscillations of an infinitely long cylindrical circular tube of incompressible material which is curvilinearly transverse-isotropic. The anisotropy, when defined along the axis, has been treated by the author [5]. Here, the anisotropy is assumed to exist in the radial direction. In Sec. 2, the equations governing the motion are described and Sec. 3 treats the static inflation of a cylindrical tube. Though this problem has been considered by Green and Adkins [2], the results are derived afresh, since the present treatment differs from that of [2] and also because use is made of Truesdell's theorem on quasi-equilibrated motions [11], [12] in Sec. 4. The above theorem requires that the static deformation be considered in material and spatial co-ordinates and not convected co-ordinates.

In Sec. 4, the differential equation governing the motion of the cylindrical tube is derived and is reduced to a form identical to that of [6], [7] for isotropic materials. Certain restrictions, suggested by the treatment in Sec. 4, on the strain energy functions are also noted.

2. Governing equations. The constitutive equation for a transversely isotropic elastic material is [1]:

$$t^i{}_j = -p\delta^i{}_j + 2\left\{\frac{\partial\Sigma}{\partial\mathrm{I}}(c^{-1})^i{}_j - \frac{\partial\Sigma}{\partial\mathrm{II}}c^i{}_j + \frac{\partial\Sigma}{\partial\mathrm{I}'}h^ih_j + \frac{\partial\Sigma}{\partial\mathrm{II}'}[(c^{-1})^i{}_kh_j + (c^{-1})_{jk}h^i]h^k\right\}. \quad (2.1)$$

In (1.1), $\mathbf{t}$ is the symmetrical stress tensor, p the hydrostatic pressure, $\Sigma = \Sigma(\mathrm{I}, \mathrm{II}, \mathrm{I}', \mathrm{II}')$ is the strain energy functions, and[1]

$$(c^{-1})^{ij} = G^{\alpha\beta}x^i{}_{,\alpha}x^j{}_{,\beta}\,. \quad (2.2)$$

Also, $\mathbf{c}$ is defined as the inverse of $\mathbf{c}^{-1}$ and

$$h^i = H^\alpha x^i{}_{,\alpha}\,. \quad (2.3)$$

X^α is the material co-ordinate system with the metric tensor $G_{\alpha\beta}$ and x^i is the spatial co-ordinate system with the metric tensor g_{ij}. Now

$$\mathrm{I} = (c^{-1})^i{}_i\,, \quad (2.4)$$

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[1] $x^i{}_{,\alpha}$ defines the partial derivative $\partial x^i/\partial X^\alpha$ in (2.2) and (2.3).

$$\mathrm{II} = \tfrac{1}{2}\{[(c^{-1})^i{}_i]^2 - (c^{-1})^i{}_j(c^{-1})^j{}_i\}, \tag{2.5}$$

$$\mathrm{I}' = g_{ij}h^ih^j, \tag{2.6}$$

$$\text{and} \quad \mathrm{II}' = h_ih_j(c^{-1})^{ij}. \tag{2.7}$$

Further H^α defines the direction of the anisotropic director. In [1], this director was considered to lie along the z-axis of the co-ordinate system for some solutions and other directions for a few more. In this note, the director H^α is chosen so that, in the material co-ordinate system (R, Θ, Z),

$$H^R = 1, \qquad H^\Theta = H^Z = 0. \tag{2.8}$$

The equations of motion in the absence of body forces are [1]

$$t^{ij}{}_{;j} = \rho\, dv^i/dt. \tag{2.9}$$

Now Truesdell [11], [12] has shown that if $\mathbf{T}_0$ is the equilibrium stress corresponding to a static deformation, then the stress due to a quasi-equilibrated motion causing the same deformation is given by

$$\mathbf{T} = \mathbf{T}_0 - \rho\zeta\mathbf{I}, \tag{2.10}$$

where ζ is a single valued acceleration potential. He has determined the acceleration potential for a cylindrical tube and the deformation field specified in Eq. (14) of [11] is exactly that assumed in the present problem as well. Thus, Sec. 3 is directed towards determining $\mathbf{T}_0$.

3. Inflation of a hollow cylindrical tube. Consider an infinitely long cylindrical tube of radii R_1 and R_2 $(R_2 > R_1)$, made of incompressible homogeneous material with a radial transverse-isotropy. Denote the displacement field by

$$r = r(R), \qquad \theta = \Theta, \qquad z = Z, \tag{3.1}$$

where (R, Θ, Z) and (r, θ, z) are respectively the material and spatial co-ordinates. The strain measure is:

$$||(c^{-1})^{ij}|| = \begin{bmatrix} (dr/dR)^2 & 0 & 0 \\ 0 & 1/R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.2}$$

The incompressibility condition reads

$$r^2 - r_1^2 = R^2 - R_1^2, \tag{3.3}$$

where r_1 is the inner radius after deformation.

Denoting $H^R = 1$, $H^\Theta = H^Z = 0$, (2.3) yields:

$$h^r = R/r, \qquad h^\theta = h^z = 0. \tag{3.4}$$

The constitutive equation (2.1) yields the stresses:

$$t^r{}_r = -p + 2\left(\frac{\partial\Sigma}{\partial \mathrm{I}} + \frac{\partial\Sigma}{\partial \mathrm{I}'}\right)\frac{R^2}{r^2} - 2\,\frac{\partial\Sigma}{\partial \mathrm{II}}\frac{r^2}{R^2} + 4\,\frac{\partial\Sigma}{\partial \mathrm{II}'}\frac{R^4}{r^4}, \tag{3.5}$$

$$t^\theta{}_\theta = -p + 2\,\frac{\partial\Sigma}{\partial \mathrm{I}}\frac{r^2}{R^2} - 2\,\frac{\partial\Sigma}{\partial \mathrm{II}}\frac{R^2}{r^2}, \tag{3.6}$$

$$t^z{}_z = -p + 2\frac{\partial\Sigma}{\partial \mathrm{I}} - 2\frac{\partial\Sigma}{\partial \mathrm{II}}. \tag{3.7}$$

From the equilibrium equations, in the absence of body forces,

$$t^r{}_r = -\int \frac{1}{r}(t^r{}_r - t^\theta{}_\theta)\, dr \tag{3.8}$$

$$= -\int \frac{2}{r}\left[\frac{R^2}{r^2}\left(\frac{\partial\Sigma}{\partial \mathrm{I}} + \frac{\partial\Sigma}{\partial \mathrm{I}'} + \frac{\partial\Sigma}{\partial \mathrm{II}}\right) - \frac{r^2}{R^2}\left(\frac{\partial\Sigma}{\partial \mathrm{I}} + \frac{\partial\Sigma}{\partial \mathrm{II}}\right) + 2\frac{\partial\Sigma}{\partial \mathrm{II}'}\frac{R^4}{r^4}\right] dr. \tag{3.9}$$

This determines the stress field required to solve the problem in Sec. 4 and the other two follow from substitution into (3.6) and (3.7).

As Knowles [6] remarked, (3.7) implies that at infinity a normal force is required over the ends of the cylinder. Finally, the invariants are:

$$\mathrm{I} = \mathrm{II} = 1 + R^2/r^2 + r^2/R^2, \tag{3.10}$$

$$\mathrm{I}' = R^2/r^2, \tag{3.11}$$

$$\mathrm{II}' = R^4/r^4. \tag{3.12}$$

4. Oscillatory motion. If it is now assumed that, under pressures of $P_1(t)$ and $P_2(t)$ in the inner and outer radii respectively and certain initial conditions, the tube oscillates radially, then

$$r = r(R, t), \qquad \theta = \Theta, \qquad z = Z. \tag{4.1}$$

Thus, using Truesdell's theorem [11], [12], the acceleration potential is:

$$-\zeta = (r_1\dot{r}_1)^{\cdot} \log r + r_1{}^2\dot{r}_1{}^2/2r^2. \tag{4.2}$$

Hence, through (2.10) and (4.2), the radial stress is

$$t^r{}_r = \rho[(r_1\ddot{r}_1 + \dot{r}_1{}^2)\log r + r_1{}^2\dot{r}_1{}^2/2r^2] + \psi(t)$$
$$- \int \frac{2}{r}\left[\left(\frac{\partial\Sigma}{\partial \mathrm{I}} + \frac{\partial\Sigma}{\partial \mathrm{II}}\right)\left(\frac{R^2}{r^2} - \frac{r^2}{R^2}\right) + \frac{R^2}{r^2}\left(\frac{\partial\Sigma}{\partial \mathrm{I}'} + 2\frac{\partial\Sigma}{\partial \mathrm{II}'}\frac{R^2}{r^2}\right)\right] dr. \tag{4.3}$$

Following the procedure in [12] and evaluating the stresses at $r = r_1$ and $r = r_2$ in (4.3) and subtracting the second from the first, and using

$$x = r_1(t)/R_1, \qquad u = r^2/R^2, \tag{4.4}$$

$$\gamma = R_2{}^2/R_1{}^2 - 1 > 0, \tag{4.5}$$

$$f(x, \gamma) = \frac{2}{\rho R_1{}^2}\int_{(\gamma+x^2)/(\gamma+1)}^{x^2} \left\{\frac{1+u}{u^2}\left(\frac{\partial\Sigma}{\partial \mathrm{I}} + \frac{\partial\Sigma}{\partial \mathrm{II}}\right) + \frac{1}{u^2(1-u)}\left(\frac{\partial\Sigma}{\partial \mathrm{I}'} + \frac{2}{u}\frac{\partial\Sigma}{\partial \mathrm{II}'}\right)\right\} du \tag{4.6}$$

the differential equation governing the motion is:

$$x \log\left(1 + \frac{\gamma}{x^2}\right)\ddot{x} + \left[\log\left(1 + \frac{\gamma}{x^2}\right) - \frac{\gamma}{\gamma + x^2}\right]\dot{x}^2 + f(x, \gamma) = \frac{P_1(t) - P_2(t)}{\frac{1}{2}\rho R_1{}^2}. \tag{4.7}$$

Now define

$$\Sigma_0(u) = \frac{2}{\rho R_1{}^2}\cdot\Sigma(\mathrm{I}, \mathrm{II}, \mathrm{I}', \mathrm{II}')_{\mathrm{I}=\mathrm{II}=1+u+1/u,\, \mathrm{I}'=1/u,\, \mathrm{II}'=1/u^2}\,. \tag{4.8}$$

Then

$$\frac{d\Sigma_0}{du} = \frac{2}{\rho R_1^{\,2}}\cdot\frac{1}{u^2}\left[(u^2 - 1)\left(\frac{\partial\Sigma}{\partial \mathrm{I}} + \frac{\partial\Sigma}{\partial \mathrm{II}}\right) - \left(\frac{\partial\Sigma}{\partial \mathrm{I}'} + \frac{2}{u}\frac{\partial\Sigma}{\partial \mathrm{II}'}\right)\right]. \tag{4.9}$$

Thus

$$f(x, \gamma) = \int_{(\gamma + x^2)/(\gamma+1)}^{x^2} \frac{1}{(u-1)}\frac{d\Sigma_0(u)}{du}\cdot du, \tag{4.10}$$

which is the form derived by Knowles [6], [7].

Next, define

$$F(x, \gamma) = \int_1^x \xi f(\xi, \gamma)\, d\xi = \tfrac{1}{2}(x^2 - 1)\int_{(\gamma+x^2)/(\gamma+1)}^{x^2} (u-1)^{-2}\Sigma_0(u)\, du, \tag{4.11}$$

and thus (4.7) will assume forms identical to Eq. (4.2) of [6] and Eq. (12) of [7] respectively under the respective conditions therein. Knowles [6], [7] has solved certain problems under the assumption, which was proposed in [13], [14] and verified experimentally by Rivlin and Saunders [15], that $d\Sigma_0/du$ has the same sign as $(u - 1)$ for $u > 0$. This hypothesis is adopted here. An examination of (4.9) shows that $d\Sigma_0/du$ will have the same sign as $(u - 1)$ provided

$$\text{(i)} \qquad \left(\frac{\partial\Sigma}{\partial \mathrm{I}} + \frac{\partial\Sigma}{\partial \mathrm{II}}\right)_{\mathrm{I}=\mathrm{II},\mathrm{II}'=(\mathrm{I}')^2} \geqq 0, \tag{4.12}$$

and

$$\text{(ii)} \qquad \frac{1}{1-u}\left(\frac{\partial\Sigma}{\partial \mathrm{I}'} + \frac{2}{u}\frac{\partial\Sigma}{\partial \mathrm{II}'}\right)_{\mathrm{I}=\mathrm{II},\mathrm{II}'=(\mathrm{I}')^2} \geqq 0. \tag{4.13}$$

The first condition arises when torsion of a radially transverse-isotropic cylinder is examined with $r = R$, $\theta = \Theta$, $z = Z$ and $H^R = 1$, $H^\Theta = H^Z = 0$. (4.12) implies that the shear modulus is positive, while (4.12) is a new restriction. The latter holds whenever,

$$\text{(a)} \qquad u > 1 \quad \text{and} \quad \frac{\partial\Sigma}{\partial \mathrm{I}'} + \frac{2}{u}\frac{\partial\Sigma}{\partial \mathrm{II}'} \leqq 0, \tag{4.14}$$

$$\text{(b)} \qquad 0 < u < 1 \quad \text{and} \quad \frac{\partial\Sigma}{\partial \mathrm{I}'} + \frac{2}{u}\frac{\partial\Sigma}{\partial \mathrm{II}'} \geqq 0, \tag{4.15}$$

it being always assumed that $\Sigma(1) = 0$.

Below, we discuss briefly the form $\Sigma_0(u)$ has to assume so that periodic solutions may exist. Following Guo Zhong-Heng and Solecki [8], periodic solutions to free oscillations exist [6] if:

(a) As $u \to \infty$, $\Sigma_0(u) \to \infty$ arbitrarily, and (4.16)
(b) As $u \to 0$, $\Sigma_0(u) \sim Ku^{-k}$, $K > 0$, $k \geqq 1$. (4.17)

For the case of forced oscillations due to a pressure impulse [7], the restrictions are [8]:

(a) As $u \to \infty$, $\Sigma_0(u) \sim Mu^m$, $M > 0$, $m > 1$, and (4.18)
(b) As $u \to 0$, $\Sigma_0(u) \approx Nu^{-n}$, $N > 0$, $n \geqq 1$. (4.19)

Since $\Sigma_0(u) \neq \Sigma_0(1/u)$, (4.17) does not reduce to the conditions of Sec. 4 of [6]. This is an interesting deviation from the isotropic case.

Finally, for the calculation of the period of oscillation, the reader is referred to [6], [7] for the general formulae corresponding to the case of free oscillations and pressure impulse.

5. Concluding remarks. This paper has shown that oscillations in curvilinearly aeolotropic materials can be determined in a manner analogous to that of the isotropic theory. Extensions to the problems connected with the sphere [8], [9], [10] are under investigation and will be reported elsewhere.

Unlike the isotropic theory, the transversely isotropic theory does not possess an approximation analogous to that of the Mooney material. Despite Blackburn's work [16], which is in a spirit totally separate from that of [1], the author believes that an approximate theory backed by experiment is essential before problems of the type considered in [6], [7], [8] can be solved.

In passing it may be seen that if the tube was initially everted, then (3.3) reads

$$r^2 - r_1^2 = R_1^2 - R^2 \tag{5.1}$$

and r_1 denotes the *external radius* of the everted tube. In (3.4) we have

$$h^r = -R/r, \qquad h^\theta = h^z = 0. \tag{5.2}$$

but the stresses (3.5)–(3.9) are unchanged. In Sec. 4, ζ is unaltered, but under the substitution (4.4)–(4.5), the lower limit in the integral $f(x, \gamma)$ changes to $(x^2 - \gamma)/(\gamma + 1)$. The differential equation (4.7) for the external radius r_1 becomes

$$x \log\left(1 - \frac{\gamma}{x^2}\right)\ddot{x} + \left[\log\left(1 - \frac{\gamma}{x^2}\right) + \frac{\gamma}{x^2 - \gamma}\right]\dot{x}^2 + f(x, \gamma) = \frac{P_1(t) - P_2(t)}{\frac{1}{2}\rho R_1^2} \tag{5.3}$$

and with $F(x, \gamma)$ modified accordingly, an analysis similar to the rest of Sec. 4 follows.

Further it may be noted that results can also be obtained if the initial anisotropy is in the tangential direction. Thus, choosing $H^R = H^Z = 0$, $H^\Theta = 1/R$, and using (4.1) $\mathbf{c}^{-1}$ is unchanged, but

$$\mathrm{I}' = r^2/R^2, \tag{5.4}$$

$$\text{and} \quad \mathrm{II}' = r^4/R^4. \tag{5.5}$$

Also, $\partial\Sigma/\partial\mathrm{I}'$ and $\partial\Sigma/\partial\mathrm{II}'$ drop out of the stress $t^r{}_r$ and, while the equation (4.7) remains the same, $f(x, \gamma)$ involves $\partial\Sigma/\partial\mathrm{I}$ and $\partial\Sigma/\partial\mathrm{II}$ only. But, $F(x, \gamma)$ cannot be expressed in terms of $\Sigma_0(u)$.

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