ELASTOSTATIC BOUNDARY REGION PROBLEM IN SOLID CYLINDERS*

BY

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1. Introduction. We will consider the problem of a semi-infinite solid elastic cylinder occupying the region $r \leq 1, 0 \leq z < +\infty$, where r and z are the cylindrical coordinates. The problem will be restricted to the axisymmetric case and σ_r, σ_θ, σ_z will be the normal stresses and $\tau = \tau_{rz}$ the nonzero shear stress. Let u and w be the displacements in the radial and z directions respectively. We will consider the curved surface $r = 1$ to be free of stress

$$r = 1, \qquad 0 < z < \infty, \qquad \sigma_r = \tau = 0. \tag{1.1}$$

At the edge $z = 0$, we will specify one of the following conditions:

$$\sigma_z = \sigma_{zb}, \qquad u = ((1 + \nu)/E)u_b, \tag{1.2a}$$

$$\tau = \tau_b, \qquad w = ((1 + \nu)/E)w_b, \tag{1.2b}$$

$$\sigma_z = \sigma_{zb}, \qquad \tau = \tau_b, \tag{1.2c}$$

$$u = ((1 + \nu)/E)u_b, \qquad w = ((1 + \nu)/E)w_b, \tag{1.2d}$$

where the subscript b indicates a given function. It will be assumed that the stresses are self-equilibrating and the displacements satisfy sufficient conditions to ensure decaying solutions.

$$\text{solution} \to 0 \quad \text{as} \quad z \to \infty.$$

The problems involving specification of (1.2a) or (1.2b) will be referred to as mixed problems, while (1.2c) represents the stress problem and (1.2d) represents the displacement problem. The stress problem is the most important involving the complete investigation of the Saint Venant boundary layer for this geometry in the case that τ_b and σ_{zb} are self-equilibrating axisymmetric tractions.

The stress problem assumes the same fundamental importance as does the strip problem of plane elasticity [1] and serves as the predecessor to the investigation of boundary layers in cylindrical shells. The stress problem in solid cylinders has previously been investigated by Horvay and Mirabal [2], Hodgkins [3] and Mendelson and Roberts [4]. Although mention is made in these articles of an eigenfunction expansion appropriate for solution of this problem, this method was not used due to the difficulty of obtaining the eigenvalues and the fact that the eigenfunctions are not orthogonal.

Murray [5] in the mathematically similar thermal stress problem formulated his solution in terms of the eigenfunctions but retained only the first two terms of the series and used approximate methods to choose the coefficients. Horvay, Giaver and Mirabal [6] and Youngdahl and Sternberg [7] also gave consideration to this thermal problem.

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2. Formulation of problem. The problem will be solved by use of the Love strain function [8]. The governing differential equation for this function is

$$\nabla^2\nabla^2\psi = 0 \tag{2.1}$$

where ∇^2 is the Laplacian operator, in cylindrical coordinates:

$$\nabla^2(\,) = (\,)_{,rr} + \frac{1}{r}(\,)_{,r} + (\,)_{,zz} . \tag{2.2}$$

The nonzero stresses and deformations are expressed in terms of ψ as follows;

$$\sigma_r = [\nu\nabla^2\psi - \psi,_{rr}],z, \tag{2.3a}$$

$$\sigma_\theta = \left[\nu\nabla^2\psi - \frac{1}{r}\psi,_r\right],z, \tag{2.3b}$$

$$\sigma_z = [(2-\nu)\nabla^2\psi - \psi,_{zz}],z, \tag{2.3c}$$

$$\tau = [(1-\nu)\nabla^2\psi - \psi,_{zz}],r, \tag{2.3d}$$

$$u = -((1+\nu)/E)\psi,_{rz}, \tag{2.3e}$$

$$w = ((1+\nu)/E)[2(1-\nu\nabla)^2_{}\psi - \psi,_{zz}]. \tag{2.3f}$$

We will assume a solution of the form

$$\psi = \sum_i^\infty a_i M_i(r) \exp(-\gamma_i z). \tag{2.4}$$

Substitution into the biharmonic equation yields

$$\left(d^2/dr^2 + \frac{1}{r}\frac{d}{dr} + \gamma_i^2\right)^2 M(r) = 0. \tag{2.5}$$

The solution bounded at $r = 0$ is:

$$M(r) = AJ_0(\gamma_i r) + B\gamma_i r J_1(\gamma_i r). \tag{2.6}$$

Satisfying the boundary conditions, (1.1) yields

$$M(r) = -[2(1-\nu)(J_1(\gamma_i)/\gamma_i) + J_0(\gamma_i)]J_0(\gamma_i r) - J_1(\gamma_i)rJ_1(\gamma_i r) \tag{2.7}$$

where γ_i are the roots of the transcendental equation (See Table I)

$$J_0^2(\gamma) + [1 - (2(1-\nu)/\gamma^2)]J_1^2(\gamma) = 0. \tag{2.8}$$

The summation indicated in (2.4) is done over the eigenvalues in the right half complex plane (γ_i, γ_i^*), where * denotes the complex conjugate.

We may now identify a vector **f** related to the proper derivatives of M_i as follows;

$$\mathbf{f} = \sum_i a_i \phi_i(r) \exp(-\gamma_i z) \tag{2.9}$$

where

$$\mathbf{f} = \begin{bmatrix} \sigma_z \\ \tau \\ Eu/(1+\nu) \\ Ew/(1+\nu) \end{bmatrix}. \tag{2.10}$$

The components of $\boldsymbol{\phi}_j(r)$ are:

$$\phi_j^{(1)} = [2\gamma_j^2 J_1(\gamma_j) - \gamma_j^3 J_0(\gamma_j)]J_0(\gamma_j r) - \gamma_j^3 J_1(\gamma_j) r J_1(\gamma_j r), \tag{2.11a}$$

$$\phi_j^{(2)} = \gamma_j^3 J_1(\gamma_j) r J_0(\gamma_j r) - \gamma_j^3 J_0(\gamma_j) J_1(\gamma_j r), \tag{2.11b}$$

$$\phi_j^{(3)} = -\gamma_j^2 J_1(\gamma_j) r J_0(\gamma_j r) + [2(1-\nu)\gamma_j J_1(\gamma_j) + \gamma_j^2 J_0(\gamma_j)]J_1(\gamma_j r), \tag{2.11c}$$

$$\phi_j^{(4)} = [\gamma_j^2 J_0(\gamma_j) - 2(1-\nu)\gamma_j J_1(\gamma_j)]J_0(\gamma_j r) + \gamma_j^2 r J_1(\gamma_j) J_1(\gamma_j r). \tag{2.11d}$$

In this form the boundary values of **f** may be represented in an eigenfunction expansion,

$$\mathbf{f}_b = \sum_j a_j \boldsymbol{\phi}_j(r). \tag{2.12}$$

The functions $\boldsymbol{\phi}_j(r)$ are not orthogonal but a set of biorthogonal functions $\mathbf{W}_j(r)$ will be developed in the next section. The biorthogonality relation is:

$$\int_0^1 \mathbf{W}_k(r)\cdot\boldsymbol{\phi}_j(r) r\, dr = 0 \quad \text{if} \quad j \neq k, \qquad = N_j \quad \text{if} \quad j = k, \tag{2.13a}$$

where

$$N_j = (1-\nu)[-4\gamma_j J_0^2(\gamma_j) - 2\gamma_j J_1^2(\gamma_j) + 4(1-\nu)J_0(\gamma_j)J_1(\gamma_j)]. \tag{2.13b}$$

The biorthogonal vector has the following components:

$$W_j^{(1)} = \frac{1}{2\gamma_j J_1^2(\gamma_j)}\{[\gamma_j J_0(\gamma_j) - 2(1-\nu)J_1(\gamma_j)]J_0(\gamma_j r) + \gamma_j J_1(\gamma_j) r J_1(\gamma_j r)\}, \tag{2.14a}$$

$$W_j^{(2)} = \frac{1}{2\gamma_j J_1^2(\gamma_j)}\{\gamma_j r J_1(\gamma_j) J_0(\gamma_j r) - [2(1-\nu)J_1(\gamma_j) + \gamma_j J_0(\gamma_j)]J_1(\gamma_j r)\}, \tag{2.14b}$$

$$W_j^{(3)} = \frac{1}{2\gamma_j J_1^2(\gamma_j)}\{-\gamma_j^2 r J_1(\gamma_j) J_0(\gamma_j r) + \gamma_j^2 J_0(\gamma_j) J_1(\gamma_j r)\}, \tag{2.14c}$$

$$W_j^{(4)} = \frac{1}{2\gamma_j J_1^2(\gamma_j)}\{[2\gamma_j J_1(\gamma_j) - \gamma_j^2 J_0(\gamma_j)]J_0(\gamma_j r) - \gamma_j J_1(\gamma_j)\gamma_j r J_1(\gamma_j r)\}. \tag{2.14d}$$

The coefficients a_j may now be obtained by use of the biorthogonality condition;

$$a_j = \frac{1}{N_j}\int_0^1 \mathbf{W}_j \cdot \mathbf{f}_b r\, dr, \tag{2.15}$$

$$a_j = \frac{1}{N_j}\int_0^1 \{W_j^{(1)} f_b^{(1)} + W_j^{(2)} f_b^{(2)} + W_j^{(3)} f_b^{(3)} + W_j^{(4)} f_b^{(4)}\} r\, dr. \tag{2.16}$$

3. Biorthogonal vector. The biorthogonal vectors will be obtained in an indirect manner through solution of the two mixed boundary problems. The solution of the biharmonic equation will be taken in the following forms [9]:

$$\psi = \sum_m (A_m + B_m \beta_m z) J_0(\beta_m r) \exp(-\beta_m z)$$
$$+ \int_0^\infty \{C(\alpha) I_0(\alpha r) + D(\alpha) r I_1(\alpha r)\} \cos \alpha z \, d\alpha \tag{3.1a}$$

or

$$\psi = \sum_m (A_m + B_m\beta_m z)J_0(\beta_m r)\exp(-\beta_m z) + \int_0^\infty \{C(\alpha)I_0(\alpha r) + D(\alpha)\alpha r I_1(\alpha r)\}\sin\alpha z\,d\alpha \tag{3.1b}$$

depending upon whether the sine or cosine Fourier integral is desired. To specify the conditions (1.2a), we will assume the solution in the form of (3.1a). Differentiation and substitution of (3.1a) into (2.3d) yields

$$\tau = \sum_m \beta_m^3[A_m + B_m(\beta_m z - 2\nu)]J_1(\beta_m r)\exp(-\beta_m z) + \int_0^\infty \{C(\alpha)I_1(\alpha r) + D(\alpha)[\alpha r I_0(\alpha r) + 2(1-\nu)I_1(\alpha r)]\}\alpha^3\cos\alpha z\,d\alpha. \tag{3.2}$$

Specifying $\tau|_{r=1} = 0$ yields the following conditions;

$$J_1(\beta_m) = 0, \tag{3.3}$$

$$C = -D[2(1-\nu) + \alpha I_0(\alpha)/I_1(\alpha)]. \tag{3.4}$$

Substitution of (3.1a) into (2.3a) yields the following;

$$\sigma_r = \sum_m \beta_m^3\left[(-A_m - B_m(\beta_m z - 1 - 2\nu))J_0(\beta_m r) + (A_m + B_m(\beta_m z - 1)\frac{J_1(\beta_m r)}{\beta_m r}\right]\exp(-\beta_m z) + \int_0^\infty\left\{C(\alpha)\alpha^3\left[I_0(\alpha r) - \frac{I_1(\alpha r)}{\alpha r}\right] + D(\alpha)\alpha^3[(1-2\nu)I_0(\alpha r) + \alpha r I_1(\alpha r)]\right\}\sin\alpha z\,d\alpha. \tag{3.5}$$

The boundary condition $\sigma_r|_{r=1} = 0$ leads to the equation

$$\int_0^\infty\left\{C(\alpha)\left[I_0(\alpha) - \frac{I_1(\alpha)}{\alpha}\right] + D(\alpha)[(1-2\nu)I_0(\alpha) + \alpha I_1(\alpha)]\right\}\alpha^3\sin\alpha z\,d\alpha = \sum_m[-A_m + B_m(1+2\nu) - B_m\beta_m z]B_m^3 J_0(\beta_m)\exp(-\beta_m z). \tag{3.6}$$

Taking the inverse Fourier transform yields:

$$D = \frac{2}{\pi}\sum_m \frac{\beta_m^3 J_0(\beta_m)I_1(\alpha)[A_m(\alpha^2+\beta_m^2) + B_m((1-2\nu)(\alpha^2+\beta_m^2) - 2\alpha^2)]}{\alpha^3(\alpha^2+\beta_m^2)^2 T(\alpha)} \tag{3.7}$$

where

$$T(\alpha) = -I_0^2(\alpha) + I_1^2(\alpha)[1 + 2(1-\nu)/\alpha^2]. \tag{3.8}$$

One may now note that the α's are related to the γ's by $\alpha_n = i\gamma_n$. The Love function becomes

$$\psi = \sum_m (A_m + B_m\beta_m z)J_0(\beta_m r)\exp(-\beta_m z) + \frac{2}{\pi}\sum_m\int_0^\infty \frac{\beta_m^3 J_0(\beta_m)L(\alpha,\beta_m)M(\alpha,r)\cos(\alpha z)\,d\alpha}{\alpha^2(\alpha^2+\beta_m^2)^2 T(\alpha)} \tag{3.9a}$$

or, in terms of an infinite integral

$$\psi = \sum_m (A_m + B_m\beta_m z)J_0(\beta_m r) \exp(-\beta_m z)$$

$$+ \frac{1}{\pi} \sum_m \int_{-\infty}^{\infty} \frac{\beta_m^3 J_0(\beta_m)L(\alpha, \beta_m)M(\alpha, r)e^{i\alpha z}\, d\alpha}{\alpha^2(\alpha^2 + \beta_m^2)^2 T(\alpha)} \tag{3.9b}$$

where

$$L(\alpha, \beta_m) = A_m(\alpha^2 + \beta_m^2) + B_m((1 - 2\nu)(\alpha^2 + \beta_m^2) - 2\alpha^2), \tag{3.10}$$

$$M(\alpha, r) = I_1(\alpha)rI_1(\alpha r) - [2(1 - \nu)I_1(\alpha)/\alpha + I_0(\alpha)]I_0(\alpha r). \tag{3.11}$$

The infinite integral may be evaluated by calculus of residues taking the contour integration around the poles in the upper half plane. The pole of order two at the origin does not contribute to the solution. There are poles of order two at $\alpha = \pm i\beta_m$ and simple poles at the eigenvalues of the transcendental $T(\alpha)$.

If the first quadrant eigenvalues of $T(\alpha)$ are denoted by α_j , there will also be eigenvalues at α_j^* , $-\alpha_j^*$ and $-\alpha_j$. Only the poles α_j and $-\alpha_j^*$ are in the upper half plane.

The residue due to the poles at $\pm i\beta_m$ cancel the Fourier–Bessel series term by term, and the final value of ψ is obtained due to the residues at the eigenvalues of $T(\alpha)$. The form of ψ becomes

$$\psi = \sum_j \sum_m \frac{i\beta_m^3 J_0(\beta_m)L_{jm}M_j \exp(i\alpha_j z)}{(\alpha_j^2 + \beta_m^2)^2 F_j} \tag{3.12}$$

where

$$F_j = -2\alpha_j I_0^2(\alpha_j) + \alpha_j I_1^2(\alpha_j) + 2(1 - \nu)I_0(\alpha_j)I_1(\alpha_j). \tag{3.13}$$

It should be noted again that the summation of j is done over the eigenvalues of $T(\alpha_j)$ in the first and second quadrants.

The boundary conditions on the finite end will be satisfied by use of (3.1a). Substitution of this equation into the equations for σ_z and u and evaluating at $z = 0$ yields:

$$\sigma_z|_{z=0} = \sum_m \{A_m + B_m(1 - 2\nu)\}\beta_m^3 J_0(\beta_m r), \tag{3.14}$$

$$u|_{z=0} = \sum_m \{A_m - B_m\}\beta_m^2 J_1(\beta_m r)((1 + \nu)/E). \tag{3.15}$$

Using the boundary condition (1.2a), the coefficients A_m and B_m become;

$$A_m = \int_0^1 [\sigma_{zb}J_0(\beta_m r) - (1 - 2\nu)u_b J_1(\beta_m r)] \frac{r\, dr}{(1 - \nu)\beta_m^3 J_0^2(\beta_m)}, \tag{3.16}$$

$$B_m = \int_0^1 [\sigma_{zb}J_0(\beta_m r) + \beta_m u_b J_1(\beta_m r)] \frac{r\, dr}{(1 - \nu)\beta_m^3 J_0(\beta_m)}. \tag{3.17}$$

It is convenient to rewrite (3.12) in the following form

$$\psi = \sum_j \left[i \sum_m \frac{L_{jm}\beta_m^3 J_0(\beta_m)}{(\alpha_j^2 + \beta_m^2)^2} \right] \frac{M_j \exp(i\alpha_j z)}{F_j}. \tag{3.18}$$

Denoting c_i as follows,

$$c_i = i \sum_m \frac{L_{im}\beta_m^3 J_0(\beta_m)}{(\alpha_i^2 + \beta_m^2)^2}. \tag{3.19}$$

Substituting (3.16) and (3.17) into (3.19) and interchanging order of integration and summation yields:

$$c_i = i \int_0^1 \left\{ \frac{\sigma_{zb}}{(1-\nu)} \left[2(1-\nu) \sum_m \frac{J_0(\beta_m r)}{(\alpha_i^2 + \beta_m^2) J_0(\beta_m)} - 2\alpha_i^2 \sum_m \frac{J_0(\beta_m r)}{(\alpha_i^2 + \beta_m^2)^2 J_0(\beta_m)} \right] + \frac{u_b}{(1-\nu)} \left[-2\alpha_i^2 \sum_m \frac{\beta_m J_1(\beta_m r)}{(\alpha_i^2 + \beta_m^2)^2 J_0(\beta_m)} \right] \right\} r\, dr. \tag{3.20}$$

Noting the following transforms,

$$\sum_m \frac{\beta_m J_1(\beta_m r)}{(\alpha_i^2 + \beta_m^2)^2 J_0(\beta_m)} = \frac{r I_0(\alpha_i r)}{4\alpha_i I_1(\alpha_i)} - \frac{I_0(\alpha_i) I_1(\alpha_i r)}{4\alpha_i I_1^2(\alpha_i)}, \tag{3.21}$$

$$\sum_m \frac{J_0(\beta_m r)}{(\alpha_i^2 + \beta_m^2)^2 J_0(\beta_m)} = \frac{I_0(\alpha_i) I_0(\alpha_i r)}{4\alpha_i^2 I_1(\alpha_i)} - \frac{r I_1(\alpha_i r)}{4\alpha_i^2 I_1(\alpha_i)}, \tag{3.22}$$

$$\sum_m \frac{J_0(\beta_m r)}{(\alpha_i^2 + \beta_m^2) J_0(\beta_m)} = \frac{I_0(\alpha_i r)}{2\alpha_i I_1(\alpha_i)}, \tag{3.23}$$

the expression for c_i may be written as follows:

$$c_i = i \int_0^1 \frac{\sigma_{zb}}{(1-\nu)} \left[\left\{ \frac{(1-\nu)}{\alpha_i I_1(\alpha_i)} - \frac{I_0(\alpha_i)}{2I_1^2(\alpha_i)} \right\} I_0(\alpha_i r) + \frac{r I_1(\alpha_i r)}{2I_1(\alpha_i)} \right] r\, dr + i \int_0^1 \frac{u_b}{(1-\nu)} \left[\frac{\alpha_i r I_0(\alpha_i r)}{2I_1(\alpha_i)} + \frac{\alpha_i I_0(\alpha_i) I_1(\alpha_i r)}{2I_1^2(\alpha_i)} \right] r\, dr. \tag{3.24}$$

The expression for ψ becomes

$$\psi = \sum_i \frac{c_i M_i \exp(i\alpha_i z)}{F_i}. \tag{3.25}$$

The second mixed boundary condition given by (1.2b) may be handled in a similar manner. Beginning with (3.1b) the following expression for ψ is obtained:

$$\psi = \sum_i \frac{d_i M_i \exp(i\alpha_i z)}{F_i} \tag{3.26}$$

where

$$d_i = \int_0^1 \frac{\tau_b}{(1-\nu)} \left\{ \frac{-r I_0(\alpha_i r)}{2I_1(\alpha_i)} + \left[\frac{I_0(\alpha_i)}{2I_1^2(\alpha_i)} + \frac{(1-\nu)}{\alpha_i I_1(\alpha_i)} \right] I_1(\alpha_i r) \right\} r\, dr + \int_0^1 \frac{w_b}{(1-\nu)} \left\{ \left[\frac{\alpha_i I_0(\alpha_i)}{2I_1^2(\alpha_i)} - \frac{1}{I_1(\alpha_i)} \right] I_0(\alpha_i r) - \frac{\alpha_i}{2I_1(\alpha_i)} r I_1(\alpha_i r) \right\} r\, dr. \tag{3.27}$$

Substitution of $\alpha_i = i\gamma_i$ into Eqs. (3.24) and (3.27) yield the desired form from which the biorthogonal eigenvectors $\mathbf{W}_i(r)$ may be deduced. The constants F_i are related to the normalization constants N_i by the following equation:

$$N_i = -2i(1-\nu)F_i\,. \tag{3.28}$$

4. Solution of specific boundary conditions. To reduce the general solution (2.12) to the mixed problem (1.2a), we will specify the following for $\mathbf{f}_b$,

$$f_b^{(1)} = \sigma_b , \tag{4.1a}$$

$$f_b^{(2)} = \sum_i a_i \phi_i^{(2)}(r), \tag{4.1b}$$

$$f_b^{(3)} = u_b , \tag{4.1c}$$

$$f_b^{(4)} = \sum_i a_i \phi_i^{(4)}(r). \tag{4.1d}$$

Using Eq. (2.16), the constants a_i become

$$a_i = \frac{1}{N_i} \int_0^1 (W_i^{(1)} \sigma_b + W_i^{(3)} u_b) r \, dr + \frac{1}{N_i} \int_0^1 \sum_k a_k [W_i^{(2)} \phi_k^{(2)} + W_i^{(4)} \phi_k^{(4)}] r \, dr. \tag{4.2}$$

Substitution from (2.11) and (2.14) shows that

$$\begin{aligned} \int_0^1 [W_j^{(2)} \phi_k^{(2)} + W_j^{(4)} \phi_k^{(4)}] r \, dr &= 0 \quad \text{if} \quad j \neq k, \\ &= \frac{N_i}{2} \quad \text{if} \quad j = k. \end{aligned} \tag{4.3}$$

Using this inner biorthogonality condition (4.2) becomes;

$$a_i = \frac{2}{N_i} \int_0^1 [W_i^{(1)} \sigma_b + W_i^{(3)} u_b] r \, dr. \tag{4.4}$$

This equation is the same as (3.24) except for notational changes.

The other mixed problem (1.2b) also exhibits an inner biorthogonality condition and is obtained from the general solution in a similar manner. The result for the coefficients a_i is;

$$a_i = \frac{2}{N_i} \int_0^1 [W_i^{(2)} \tau_b + W_i^{(4)} w_b] r \, dr. \tag{4.5}$$

The stress and displacement problems do not exhibit an inner biorthogonality property and, therefore, do not reduce to a form where the coefficients a_i may be obtained directly. For the stress boundary problem, (1.2c) we specify:

$$\begin{aligned} f_b^{(1)} &= \sigma_b , & f_b^{(3)} &= \sum_i a_i \phi_i^{(3)}(r), \\ f_b^{(2)} &= \tau_b , & f_b^{(4)} &= \sum_i a_i \phi_i^{(4)}(r). \end{aligned} \tag{4.6}$$

Substitution into (2.15) yields the following expression for the constants a_i ;

$$a_i = \frac{1}{N_i} \int_0^1 [W_i^{(1)} \sigma_b + W_i^{(2)} \tau_b] r \, dr + \frac{1}{N_i} \sum_k a_k \int_0^1 [W_i^{(3)} \phi_k^{(3)} + W_i^{(4)} \phi_k^{(4)}] r \, dr, \tag{4.7}$$

TABLE I
Roots of the Transcendental Equation (2.8)

n	$\nu = 0.$		n	$\nu = 0.25$	
1	2.5567699	$+i$1.3889670	1	2.6976518	$+i$1.3673570
2	6.0058627	1.6387025	2	6.0512222	1.6381471
3	9.2331665	1.8290585	3	9.2612734	1.8285342
4	12.417892	1.9678845	4	12.438444	1.9674283
5	15.585955	2.0768032	5	15.602204	2.0764211
6	18.745600	2.1663589	6	18.759055	2.1660392
7	21.900357	2.2423784	7	21.911845	2.2421081
8	25.052005	2.3084045	8	25.062031	2.3081733
9	28.201546	2.3667575	9	28.210443	2.3665585
10	31.349590	2.4190317	10	31.357587	2.4188579
11	34.496531	2.4663634	11	34.503796	2.4662104
12	37.642634	2.5096184	12	37.649288	2.5094822
13	40.788084	2.5494371	13	40.794222	2.5493159
14	43.933016	2.5863247	14	43.938715	2.5862151
15	47.077530	2.6206835	15	47.082846	2.6205834
16	50.221701	2.6528348	16	50.226683	2.6527455
17	53.365584	2.6830475	17	53.370274	2.6829654
18	56.509231	2.7115412	18	56.513658	2.7114654
19	59.652673	2.7384999	19	59.656867	2.7384311
20	62.795941	2.7640825	20	62.799924	2.7640179

n	$\nu = 0.30$		n	$\nu = 0.50$	
1	2.7221755	$+i$1.3621971	1	2.8105617	$+i$1.3399331
2	6.0600832	1.6376243	2	6.0947291	1.6342958
3	9.2668352	1.8282558	3	9.2888501	1.8265905
4	12.442529	1.9672411	4	12.458767	1.9661816
5	15.605440	2.0762837	5	15.618332	2.0755346
6	18.761738	2.1659330	6	18.772437	2.1653696
7	21.914138	2.2420234	7	21.923286	2.2415813
8	25.064033	2.3081035	8	25.072026	2.3077463
9	28.212220	2.3664995	9	28.219317	2.3662036
10	31.359185	2.4188080	10	31.365568	2.4185578
11	34.505246	2.4661675	11	34.511046	2.4659532
12	37.650618	2.5094447	12	37.655931	2.5092599
13	40.795451	2.5492821	13	40.800353	2.5491195
14	43.939854	2.5861858	14	43.944407	2.5860418
15	47.083910	2.6205568	15	47.088157	2.6204285
16	50.227679	2.6527203	16	50.231661	2.6526054
17	53.371211	2.6829441	17	53.374957	2.6828392
18	56.514543	2.7114455	18	56.518082	2.7113512
19	59.657704	2.7384134	19	59.661057	2.7383264
20	62.800720	2.7640011	20	62.803905	2.7639218

Asymptotic formula $\gamma_n \sim \left\{ n\pi - \frac{\ln(4n\pi)}{4n\pi} \right\} + \frac{i}{2} \ln(4n\pi)$

Convergence of Eigenfunction Expansions

TABLE II

r	Specified Function $\sigma_s = 1 - 2r^2$	Number of paired eigenvalues		
		5	10	20
0.	1.0	0.975	1.008	0.993
0.1	0.98	0.985	0.982	0.982
0.2	0.92	0.937	0.914	0.920
0.3	0.82	0.794	0.829	0.819
0.4	0.68	0.667	0.668	0.681
0.5	0.50	0.547	0.513	0.500
0.6	0.28	0.283	0.265	0.279
0.7	0.02	−0.048	0.035	+0.021
0.8	−0.28	−0.261	−0.294	−0.278
0.9	−0.62	−0.561	−0.607	−0.626
1.0	−1.00	−1.087	−1.047	−1.022

This may be rewritten as follows:

$$a_i = F_i + \sum_k S_{ik} a_k \tag{4.8}$$

where

$$F_i = \frac{1}{N_i} \int_0^1 [W_i^{(1)} \sigma_b + W_i^{(2)} \tau_b] r \, dr, \tag{4.9}$$

$$S_{ik} = \frac{1}{N_i} \int_0^1 [W_i^{(3)} \phi_k^{(3)} + W_i^{(4)} \phi_k^{(4)}] r \, dr. \tag{4.10}$$

Equation (4.8) is a system of infinite equations in infinitely many unknowns which can be solved by truncation to obtain values of a_i to any desired degree of accuracy.

TABLE III

r	Specified Function $\tau = 2.4r - 2.6r^3 + 0.2r^5$	Number of paired eigenvalues		
		5	10	20
0.	.0	0.	0.	0.
0.1	.237	0.222	0.243	0.239
0.2	.459	0.479	0.454	0.457
0.3	.650	0.665	0.654	0.651
0.4	.796	0.755	0.795	0.796
0.5	.881	0.870	0.877	0.880
0.6	.894	0.957	0.903	0.894
0.7	.822	0.828	0.805	0.823
0.8	.654	0.577	0.677	0.651
0.9	.383	0.400	0.357	0.382
1.0	0.	0.	0.	0.

The less important displacement problem leads to a system of equations of the same form as (4.8).

5. Example problem. To examine numerically the convergence of the series to the given boundary values, we will consider the stress problem:

$$\sigma_{zb} = 1 - 2r^2, \tag{5.1}$$

$$\tau_b = 2.4r - 2.6r^3 + 0.2r^5. \tag{5.2}$$

The tractions, (5.1) and (5.2), were picked because they are fairly general in form, the shearing stresses are zero at the center and outside of the bar, and both have maximum orders of magnitude of unity.

All numerical results are for Poisson's ratio equal to 0.3, except for Table I. We have truncated the system of equations (4.8) to the first five, ten, and twenty pairs of eigenvalues. Tables II and III show how well (4.6) satisfies the specified tractions via each truncated system (4.8).

The decay properties of the stresses for this case are shown in Figs. 1 and 2.

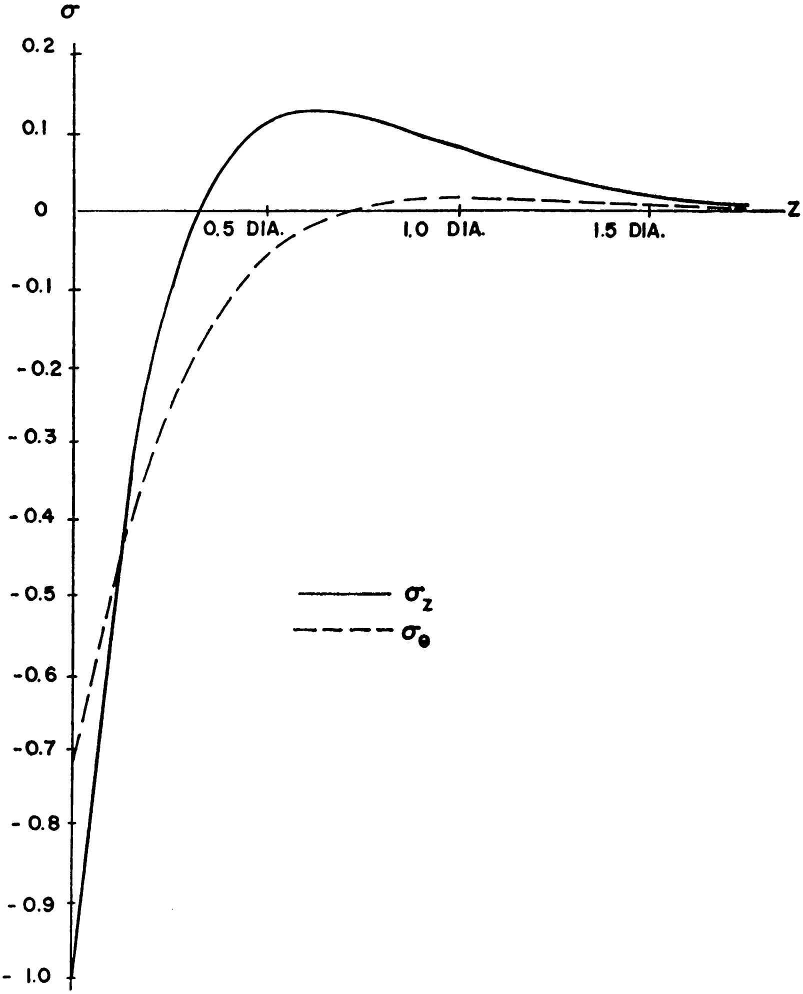

FIG. 1. Decay properties of σ_z and σ_θ at $r = 1$.

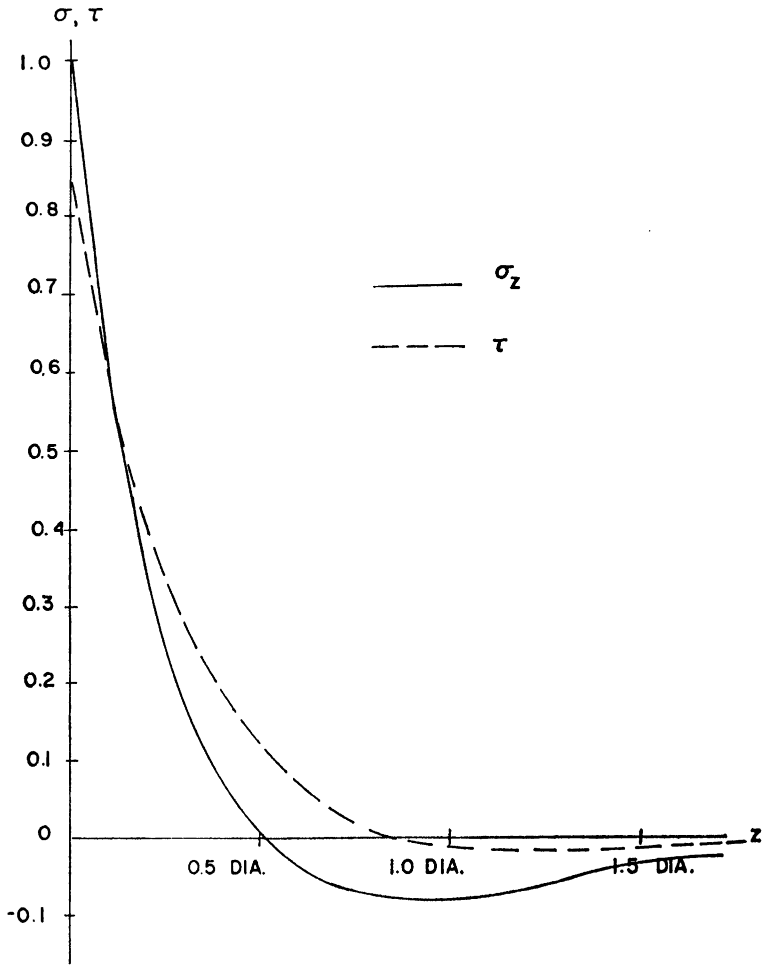

FIG. 2. Decay properties of σ_z at $r = 0$ and τ at $r = 0.5$.

6. Conclusions. The solution of the cylinder problem has been reduced to an eigenfunction expansion in terms of the end conditions. This form of solution has been known for some time but not utilized due to reasons mentioned previously and discussed by Horvay [2]. By use of the biorthogonal vectors deduced in an indirect manner, the problem may be solved with the aid of a digital computer without great difficulty.

The necessary conditions on the stresses to ensure a decaying solution are known but those on the displacements are yet to be determined. M. I. Gusein–Zade has recently obtained these conditions for the related two-dimensional semi-infinite strip problem [10] and similar development is required in this case.

The biorthogonality vectors were obtained for the desired end stress and displacement functions and not for the Love function directly. A "generalized biorthogonality" condition for the functions $M_i(r)$ (Eq. 2.7) could have been obtained as was done by Papkovich in the strip problem [11], [12] for cases when σ_r is not specified on the curved surfaces. (See Appendix) This approach, although more direct, and therefore more desirable, did not prove fruitful in the case of stress free surfaces.

Appendix I. The solution for the Love strain function for the semi-infinite cylinder may be assumed in the form

$$\psi = \sum A_i M_i(r) \exp(-\gamma_i z). \tag{A.1}$$

The equation for $M_i(r)$ involves the non-self-adjoint interated Bessel operator, as shown:

$$(L + \gamma_i^2)^2 M_i(r) = 0 \tag{A.2}$$

where

$$L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right). \tag{A.3}$$

Defining the adjoint operator as L^*, where

$$L^* = \frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \frac{1}{r^2} = \frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{1}{r}\right)\right] \tag{A.4}$$

we may write the adjoint equation as follows:

$$(L^* + \gamma_i^2)^2 \phi_i(r) = 0. \tag{A.5}$$

The boundary conditions on $\phi_i(r)$ will be chosen such that the $\phi_i(r)$ will satisfy a generalized biorthogonality condition with $M_i(r)$. The biorthogonality condition may be written as follows:

$$\int_0^1 \{\gamma_k^2 \phi_k (L + \gamma_i^2)^2 M_i - \gamma_i^2 M_i (L^* + \gamma_k^2)^2 \phi_k\}\, dr = 0. \tag{A.6}$$

Expanding and integrating by parts yields

$$(\gamma_k^2 - \gamma_i^2) \int_0^1 [(L^*\phi_k)(LM_i) - \gamma_i^2\gamma_k^2 \phi_k M_i]\, dr = -\gamma_k^2\left\{\phi_k \frac{d}{dr}(LM_i) - \frac{D}{Dr}(\phi_k) LM_i\right\}$$
$$+ \gamma_i^2\left\{M_i \frac{D}{Dr}(L^*\phi_k) - \frac{d}{dr}(M_i) L^*\phi_k\right\} - 2\gamma_k^2\gamma_i^2\left\{\phi_k \frac{d}{dr}(M_i) - M_i \frac{D}{Dr}(\phi_k)\right\} \tag{A.7}$$

where D/Dr is the differential operator

$$\frac{D}{Dr} = r\frac{d}{dr}\left(\frac{1}{r}\right). \tag{A.8}$$

On the surface $r = 1$, the following boundary conditions may be specified

$$\sigma_r = 0, \qquad \tau = 0, \tag{A.9a}$$

$$\sigma_r = 0, \qquad w = 0, \tag{A.9b}$$

$$\tau = 0, \qquad u = 0, \tag{A.9c}$$

$$u = 0, \qquad w = 0. \tag{A.9d}$$

The conditions that the stresses or displacements are zero are equivalent to the following conditions upon M_i at $r = 1$.

$$\sigma_r = 0, \qquad LM_i = \frac{\nu}{(1-\nu)}\gamma_i^2 M_i + \frac{1}{(1-\nu)}\frac{dM_i}{dr}, \tag{A.10}$$

$$\tau = 0, \qquad \frac{d}{dr}(LM_i) = \frac{\nu}{(1-\nu)}\gamma_i^2 \frac{dM_i}{dr}, \tag{A.11}$$

$$u = 0, \qquad \frac{dM_i}{dr} = 0, \tag{A.12}$$

$$w = 0, \qquad LM_i = -\frac{(1-2\nu)}{2(1-\nu)}\gamma_i^2 M_i . \tag{A.13}$$

Examination of equations (A.10), (A.11), and (A.13) when substituted into (A.7) shows that the biorthogonality conditions may not be determined in this manner for the boundary condition (A.9a) and (A.9b). The indirect method previously discussed was therefore employed. The remaining boundary conditions (A.9c) and (A.9d) are such that biorthogonal functions ϕ_k may be determined due to the fact that proper conditions may be found such that the right hand side of (A.7) vanishes.
The boundary conditions corresponding to (A.9c) are as follows:

$$\frac{d}{dr}(LM_i) = \frac{\nu}{(1-\nu)}\gamma_i^2 \frac{dM_i}{dr}, \tag{A.11}$$

$$dM_i/dr = 0. \tag{A.12}$$

These may be written

$$\frac{d}{dr}(LM_i) = 0, \tag{A.14a}$$

$$\frac{d}{dr}(M_i) = 0. \tag{A.14b}$$

The boundary conditions of ϕ_k are therefore:

$$\frac{D}{Dr}(\phi_k) = 0, \tag{A.15a}$$

$$\frac{D}{Dr}(L^*\phi_k) + 2\gamma_k^2 \frac{D}{Dr}(\phi_k) = 0. \tag{A.15b}$$

These may be written

$$\frac{D}{Dr}(\phi_k) = 0, \tag{A.16}$$

$$\frac{D}{Dr}(L^*\phi_k) = 0. \tag{A.17}$$

The functions M_i and the eigenvalues are different in this case than those given by (2.7) and (2.8) and are given as follows:

$$M_i = \frac{\{\gamma_i J_0(\gamma_i)J_0(\gamma_i r) + \gamma_i r J_1(\gamma_i)J_1(\gamma_i r)\}}{J_1(\gamma_i)} \tag{A.18}$$

where γ_i are the zeros of $J_1(\gamma_i)$. The biorthogonal functions for this case become:

$$\phi_k = \gamma_k r M_k \tag{A.19}$$

and the functions are orthogonal subject to the weighting function r.

The boundary conditions corresponding to (A.9d) are as follows:

$$LM_i = -\frac{(1-2\nu)}{2(1-\nu)}\gamma_i^2 M_i\,, \tag{A.13}$$

$$dM_i/dr = 0. \tag{A.12}$$

The boundary conditions on ϕ_k become:

$$\phi_k = 0, \tag{A.20}$$

$$\frac{D}{Dr}\left[L^*\phi + \frac{3-2\nu}{2(1-\nu)}\gamma_k^2\phi\right] = 0. \tag{A.21}$$

The eigenfunction M_i and the eigenvalues for this case are as follows:

$$M_i = \frac{1}{J_1(\gamma_i)}\left[\gamma_i J_0(\gamma_i)J_0(\gamma_i r) + \gamma_i r J_1(\gamma_i)J_1(\gamma_i r)\right] \tag{A.22}$$

where γ_i is defined by the transcendental equation

$$\gamma_i[J_0^2(\gamma_i) + J_1^2(\gamma_i)] - 4(1-\nu)J_0(\gamma_i)J_1(\gamma_i) = 0. \tag{A.23}$$

The biorthogonal functions become:

$$\phi_k = -\frac{1}{J_0(\gamma_k)}\left[\gamma_k^2 r J_1(\gamma_k)J_0(\gamma_k r) - (\gamma_k r)^2 J_0(\gamma_k)J_1(\gamma_k r)\right]. \tag{A.24}$$

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