A PERTURBATION METHOD FOR BOUNDARY VALUE PROBLEMS IN DYNAMIC ELASTICITY*

BY

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1. Introduction. Dynamic disturbances propagate in isotropic elastic solids in the form of dilatational and distortional waves, each of which is characterized by its propagation velocity. In real materials these speeds are always distinct. Although each type of wave can propagate independently in an infinite medium, they both occur, in general, after either of them strikes the surface of a bounded medium or a surface of discontinuity. The coexistence of these two waves traveling with different velocities makes the analysis of waves in an elastic solid much more complicated than for other wave propagation phenomena such as electromagnetic waves in dielectrics or acoustical waves in air.

As an example of the difficulties one may encounter, consider the steady-state propagation of elastic waves in a medium which has an interior or exterior boundary that coincides with one of the familiar curvilinear coordinate surfaces. It is well known that in many cases the governing field equations, a scalar and a vector wave equation, can be solved by the separation of variables method. Solutions are then expressed in terms of a series of the special wave functions which arise for the particular coordinate system under consideration. Since two types of waves coexist, there are two series of wave functions, one for dilatational and the other for distortional waves and each will contain a different wave speed as a parameter. Combination of these wave functions and their derivatives gives rise to displacements and stresses which are the usual quantities specified at a boundary. Because wave functions with different wave speeds are not orthogonal to each other, the boundary conditions, in general, cannot be exactly satisfied. Only at a cylindrical or spherical boundary are the wave functions, $\exp i(n\theta + mz)$ and $P_m^n(\cos\theta)\cos m\varphi$, respectively, independent of the wave speeds. Thus, boundary value problems for cylindrical or spherical geometries are the only ones that can be solved exactly by the so-called "method of separation of variables."

It is to be noted that separation of variables is not the only method that is ineffective for solving elastodynamic boundary value problems. For example, in using the Wiener–Hopf method to treat the scattering of elastic waves by a half plane, great difficulty is encountered at the critical stage of factoring the kernel into two functions which are analytic in two different, but overlapping regions. This, again, stems from the occurrence of two different wave velocities. In general, it can be said that elastodynamic problems would be easier to analyze if the dilatational and distortional wave speeds were equal and this fact lead us to develop the perturbation method presented in this paper.

In essence, this method, as outlined in Sec. II, is to replace the two unequal wave numbers (frequency/wave velocity) which appear in the two steady state wave equations by their root-mean-square average value. The difference between the actual wave numbers and the root-mean-square average one happens to be a small number which can be used as a perturbation parameter. The wave equations and boundary conditions

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are then all perturbed with the result that, for each order of the perturbation expansion, only one wave number, the rms average, is involved.

In Secs. III and IV, this method is illustrated and tested for studying diffractions of elastic waves by a semi-infinite rigid-smooth plane and by a rigid circular cylinder, for both of which the exact solutions are available. Two term perturbation solutions are obtained and, when compared with the exact solutions, they are found to be valid at low frequencies and in the region near the scatterer.

Finally, this method is applied in Sec. V to the diffraction of harmonic compressional and shear waves by a semi-infinite rigid-clamped plane. Previous attempts to solve this problem by the separation of variables method [1] and the Wiener–Hopf method [2], were seriously hampered by the difficulties discussed above, and thus, very little information on the stresses and displacements has been obtained. However, by applying the method of perturbation and utilizing the well known Sommerfeld solutions for the diffraction of electromagnetic waves [3], we construct with ease a two term perturbation solution, from which the stresses near the tip of the clamped plane are explicitly determined.

II. Equations of elastodynamics and the perturbation method. Waves in elastic solids can be expressed by two wave potentials, $\varphi(\mathbf{r}, t)$ and $\mathbf{h}(\mathbf{r}, t)$, which are related to the displacement vector $\mathbf{u}(\mathbf{r}, t)$ by

$$\mathbf{u} = \operatorname{grad} \varphi + \operatorname{curl} \mathbf{h}. \tag{1}$$

For steady state motion, the time dependent part of $\mathbf{u}$, φ, and $\mathbf{h}$, a harmonic function $\exp(-i\omega t)$, is separated from the part which depends on the position vector $\mathbf{r}$, and then $\varphi(\mathbf{r})$ and $\mathbf{h}(\mathbf{r})$ satisfy Helmholtz equations.

$$(\nabla^2 + k_1^2)\varphi = 0, \qquad (\nabla^2 + k_2^2)\mathbf{h} = 0 \tag{2}$$

where ∇^2 is the Laplacian operator and

$$k_1 = \omega/c_1 , \qquad k_2 = \omega/c_2 .$$

The ratio of the dilatational wave speed c_1 (P wave) to the distortional wave speed c_2 (S wave) depends on the Poisson's ratio, ν, of the medium,

$$\kappa = \frac{c_1}{c_2} = \frac{k_2}{k_1} = \left(\frac{2 - 2\nu}{1 - 2\nu}\right)^{1/2}. \tag{3a}$$

Stresses in the medium are related to the displacements by

$$\boldsymbol{\tau} = \mu[(\kappa^2 - 2)(\operatorname{div} \mathbf{u})\mathbf{I} + \operatorname{grad} \mathbf{u} + \mathbf{u} \operatorname{grad}] \tag{4}$$

where $\boldsymbol{\tau}$ and $\mathbf{I}$ are the stress dyadic and idemfactor [4], respectively, and μ is the shear modulus of the medium.

In an infinite body, P and S waves which are represented by the wave potentials φ and $\mathbf{h}$, respectively, can propagate independently, but, in a medium with a boundary or surface of discontinuity, these two waves usually are coupled. This is because boundary conditions are most often prescribed in terms of displacements and stresses which are comprised of both waves as can be seen from Eqs. (1) and (4). The fact that each wave is characterized by distinct wave speeds, c_1 and c_2, or distinct wave numbers, k_1 and k_2, and that both waves concurrently exist complicates the analysis of waves in a bounded elastic medium.

To circumvent such difficulties, we replace both wave numbers by their root-mean-square average value, k,

$$k^2 = \tfrac{1}{2}(k_1^2 + k_2^2), \qquad k_1^2 = k^2(1 - 2\epsilon), \qquad k_2^2 = k^2(1 + 2\epsilon) \tag{5a}$$

where

$$\epsilon = \frac{k_2^2 - k_1^2}{2(k_2^2 + k_1^2)} = \frac{1}{2(3 - 4\nu)}. \tag{6a}$$

Since Poisson's ratio, ν, has a numerical value lying between zero and one-half, the parameter, ϵ, is limited between 1/6 and 1/2. It thus can be used as the parameter in regular perturbation expansions for φ, $\mathbf{h}$, $\mathbf{u}$, and $\boldsymbol{\tau}$.

$$\begin{aligned} \varphi &= \sum \epsilon^n \varphi^{(n)}, & \mathbf{h} &= \sum \epsilon^n \mathbf{h}^{(n)}, \\ \mathbf{u} &= \sum \epsilon^n \mathbf{u}^{(n)}, & \boldsymbol{\tau} &= \sum \epsilon^n \boldsymbol{\tau}^{(n)}. \end{aligned} \tag{7}$$

All summations are from $n = 0$ to ∞.

In terms of k and ϵ, Eqs. (2) are written as

$$[\nabla^2 + k^2(1 - 2\epsilon)]\varphi = 0, \qquad [\nabla^2 + k^2(1 + 2\epsilon)]\mathbf{h} = 0.$$

Substituting (7) into the above equations and equating coefficients of like powers of ϵ, we obtain the field equations for the nth order potentials.

$$(\nabla^2 + k^2)\varphi^{(n)} = 2k^2\varphi^{(n-1)}, \qquad (\nabla^2 + k^2)\mathbf{h}^{(n)} = -2k^2\mathbf{h}^{(n-1)}. \tag{8}$$

Similarly, the nth order displacements and stresses can be computed from $\varphi^{(n)}$ and $\mathbf{h}^{(n)}$ by

$$\mathbf{u}^{(n)} = \operatorname{grad} \varphi^{(n)} + \operatorname{curl} \mathbf{h}^{(n)}, \tag{9}$$

$$\boldsymbol{\tau}^{(n)} = \mu[(\kappa^2 - 2)(\operatorname{div} \mathbf{u}^{(n)})\mathbf{I} + \operatorname{grad} \mathbf{u}^{(n)} + \mathbf{u}^{(n)} \operatorname{grad}]. \tag{10}$$

The boundary conditions are treated in like manner. We note then that the field equations (8) and boundary conditions for each order of perturbation depend on the single wave number k.

The zeroth order field equations are

$$(\nabla^2 + k^2)\varphi^{(0)} = 0, \qquad (\nabla^2 + k^2)\mathbf{h}^{(0)} = 0. \tag{11}$$

These are the first approximations for (2) by assuming that both P and S waves travel with equal speeds. Solutions of (11) satisfying zeroth order boundary conditions can be found in a manner similar to solving problems of acoustic waves in fluids for which only one wave potential is required to describe the motion.

Proceeding successively to higher order field equations, we decompose the solutions of (8) into two parts: the complementary solutions, $\varphi_c^{(n)}$ and $\mathbf{h}_c^{(n)}$ and the particular solutions, $\varphi_p^{(n)}$ and $\mathbf{h}_p^{(n)}$. The former satisfy the homogeneous equations,

$$(\nabla^2 + k^2)\varphi_c^{(n)} = 0, \qquad (\nabla^2 + k^2)\mathbf{h}_c^{(n)} = 0 \tag{12}$$

while the latter can be constructed from the solutions of lower orders. General solutions are then given by

$$\varphi^{(n)} = \varphi_c^{(n)} + \varphi_p^{(n)}, \qquad \mathbf{h}^{(n)} = \mathbf{h}_c^{(n)} + \mathbf{h}_p^{(n)}. \tag{13}$$

Consider the first order perturbation. From (8) and (11) the field equations become

$$(\nabla^2 + k^2)\varphi^{(1)} = -2\nabla^2\varphi^{(0)}, \qquad (\nabla^2 + k^2)\mathbf{h}^{(1)} = 2\nabla^2\mathbf{h}^{(0)}. \tag{14}$$

A particular solution for each equation is

$$\varphi_p^{(1)} = -\mathbf{r}\cdot\text{grad}\,\varphi^{(0)}, \qquad \mathbf{h}_p^{(1)} = \mathbf{r}\cdot\text{grad}\,\mathbf{h}^{(0)} \tag{15}$$

which can be verified by direct substitution into (14).

The second order equations follow from (8), (13), and (15).

$$\begin{aligned}(\nabla^2 + k^2)\varphi^{(2)} &= -2k^2\mathbf{r}\cdot\text{grad}\,\varphi^{(0)} - 2\nabla^2\varphi_c^{(1)},\\ (\nabla^2 + k^2)\mathbf{h}^{(2)} &= -2k^2\mathbf{r}\cdot\text{grad}\,\mathbf{h}^{(0)} + 2\nabla^2\mathbf{h}_c^{(1)}.\end{aligned} \tag{16}$$

Particular solutions in m dimensions are

$$\begin{aligned}\varphi_p^{(2)} &= -\mathbf{r}\cdot\text{grad}\left(\frac{m}{2}\varphi^{(0)} + \varphi_c^{(1)}\right) - \tfrac{1}{2}k^2r^2\varphi^{(0)},\\ \mathbf{h}_p^{(2)} &= -\mathbf{r}\cdot\text{grad}\left(\frac{m}{2}\mathbf{h}^{(0)} - \mathbf{h}_c^{(1)}\right) - \tfrac{1}{2}k^2r^2\mathbf{h}^{(0)}.\end{aligned} \tag{17}$$

We shall not generate solutions for higher order field equations. Not only would the particular solutions become quite lengthy, but also it becomes increasingly difficult to determine the complementary solutions since they must satisfy the boundary conditions which, in turn, involve the corresponding order particular solutions. In practice, we shall be content with results of the first few orders.

For plane problems of elastodynamics, the above method can be applied to *plane strain* without modification. If the x_3-axis is perpendicular to the plane considered, plane strain means that $\mathbf{u} = (u_1, u_2, 0)$ and u_i is independent of the x_3-coordinate. For such a problem we may take $\mathbf{h} = (0, 0, h)$ and both wave potentials, φ and h, are functions of x_1, x_2, and t.

In *generalized plane stress* problems of thin plates, $\mathbf{u}(x_1, x_2, t)$ represents the average value of displacement across the x_3-coordinate and u_3 can be computed from u_1 and u_2. The shear wave velocity in such a medium is still c_2, but the corresponding dilatational wave, generally known as an extensional wave [5], has a velocity, $\bar{c}_1 = \bar{\kappa}c_2$, where

$$\bar{\kappa}^2 = 2/(1 - \nu). \tag{3b}$$

Thus, the k_1 in (2) should be replaced by $\bar{k}_1 = \omega/\bar{c}_1$ and κ in (4) and (10) should be replaced by $\bar{\kappa}$. The rms average wave number, k, should be changed to $\bar{k}$ with

$$\bar{k}^2 = \tfrac{1}{2}(\bar{k}_1^2 + k_2^2), \qquad \bar{k}_1 = k_2/\bar{\kappa} \tag{5b}$$

and ϵ to

$$\bar{\epsilon} = (1 + \nu)/2(3 - \nu). \tag{6b}$$

It is seen that $\bar{\epsilon}$, whose range is $1/6 < \bar{\epsilon} < 3/10$ for practical Poisson's ratios, still is suitable as a perturbation parameter.

In the next two sections, we shall illustrate the perturbation method outlined above for two problems of diffraction of elastic waves. Since the exact solutions for both cases are known, a comparison will show the accuracy and validity of this approximate method.

III. Diffraction of compressional waves by a semi-infinite rigid-smooth strip. We first apply the perturbation method to the analysis of the diffraction of plane harmonic waves in an infinite medium with a semi-infinite rigid-smooth plane barrier. A rigid-smooth plane is one which is free of shearing stresses and is immobile in its normal direction. The formulation of this problem and its exact solution which were presented in detail in a recent investigation [6] are summarized in the following subsection.

1. *Formulation of Problem and Exact Solution.* As shown in Fig. 1, a semi-infinite rigid-smooth surface which is taken as the positive xz plane of a Cartesian space forms a diffracting screen to an incident plane dilatational wave (P wave) propagating in the positive y-direction. The incident wave is represented by

$$\varphi^i = \varphi_0 \exp(ik_1 y), \qquad \mathbf{h}^i = 0 \tag{18}$$

with the time factor $\exp(-i\omega t)$ omitted throughout. Deformations will occur only in the x and y directions conforming to a state of plane strain with $\varphi = \varphi(x, y)$ and $\mathbf{h} = [0, 0, h(x, y)]$. In terms of the parabolic coordinates (ξ, η, z) where

$$x = \tfrac{1}{2}(\xi^2 - \eta^2), \qquad y = \xi\eta, \qquad z = z \tag{19}$$

and

$$r = (x^2 + y^2)^{1/2} = \tfrac{1}{2}(\xi^2 + \eta^2) = \tfrac{1}{2}g^2,$$

equations (1) and (2) take the following forms, respectively,

$$u_\xi = \frac{1}{g}\left(\frac{\partial\varphi}{\partial\xi} + \frac{\partial h}{\partial\eta}\right), \qquad u_\eta = \frac{1}{g}\left(\frac{\partial\varphi}{\partial\eta} - \frac{\partial h}{\partial\xi}\right), \qquad u_z = 0; \tag{20}$$

$$(\partial^2/\partial\xi^2 + \partial^2/\partial\eta^2 + g^2 k_{1,2}^2)(\varphi, h) = 0. \tag{21}$$

The boundary conditions on the rigid-smooth plane which is defined by $\eta = \eta_0 = 0$ in parabolic coordinates are

$$\tau_{\xi\eta}(\xi, 0) = u_\eta(\xi, 0) = 0 \tag{22}$$

which can be reduced to [6],

$$\partial\varphi(\xi, 0)/\partial\eta = h(\xi, 0) = 0, \qquad \xi \neq 0. \tag{23}$$

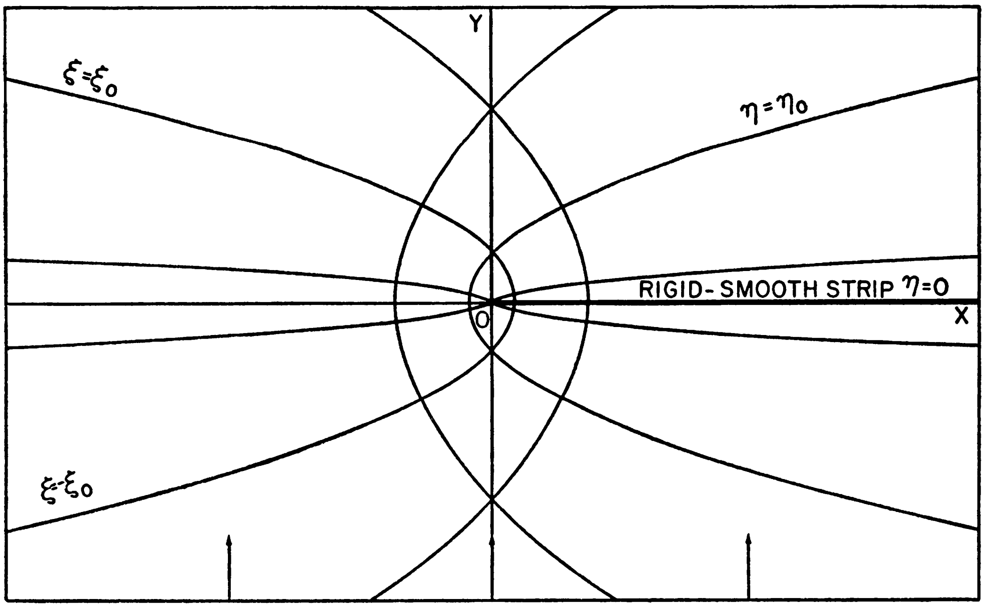

FIG. 1. Parabolic Coordinates and Plane Wave Incident Upon Rigid-Smooth Strip.

At the edge of the strip ($\eta = \xi = 0$), the displacement components must be finite.

Upon striking the strip, the incident wave is scattered into compressional and shear waves. The outgoing scattered waves are constructed from the solutions of Eqs. (21) in terms of Weber's functions $D_0(z)$ and $D_{-1}(z)$ where

$$D_0(z) = \exp(-z^2/4),$$
$$D_{-1}(z) = 2^{1/2} \exp(z^2/4) \int_\alpha^\infty \exp(-s^2)\, ds, \qquad \alpha = 2^{-1/2} z.$$

The combined incident and outgoing waves should satisfy the boundary conditions (23), the edge conditions, and the Sommerfeld radiation condition at infinity.

An exact solution was found in [6] as

$$\begin{aligned} \varphi &= [A\xi/\lambda_1(\xi^2 + \eta^2)]D_0(\lambda_1\eta)D_0(\lambda_1\xi) + W_1(\xi, \eta; \lambda_1), \\ h &= [A\eta/\lambda_1(\xi^2 + \eta^2)]D_0(\lambda_2\eta)D_0(\lambda_2\xi) \end{aligned} \tag{24}$$

where $A = -4\pi^{-1/2}\varphi_0/(1 + \kappa)$,

$$W_{1,2}(\xi, \eta; \lambda_j) = (2\pi)^{-1/2}\{D_0[\lambda_j'(\xi + \eta)]D_{-1}[\lambda_j'(\xi - \eta)] \pm D_0[\lambda_j'(\xi - \eta)]D_{-1}[\lambda_j'(\xi + \eta)]\} \tag{25}$$

and

$$\lambda_j' = 2^{-1/2}\lambda_j\,; \qquad \lambda_j = (-2ik_j)^{1/2}, \qquad j = 1, 2.$$

In the above solution, W_1 is the combination of both the incident wave (18) and the outgoing scattered wave, satisfying the boundary conditions (23). The outgoing wave terms with the factor A identically satisfy (23) and are added to meet the edge condition. Because of the latter condition, the diffracted wave is comprised of both dilatational and shear waves.

2. *Perturbation Solution.* Without knowing the exact solution, we can derive an approximate solution to the above problem by finding the first two terms of the perturbation expansion. Following the development in Sec. II with $\mathbf{h}^{(n)} = (0, 0, h^{(n)})$, we take

$$\varphi \cong \varphi^{(0)} + \epsilon\varphi^{(1)}, \qquad h \cong h^{(0)} + \epsilon h^{(1)}. \tag{26}$$

The perturbed boundary conditions, as derived from (23) are

$$\partial\varphi^{(n)}(\xi, 0)/\partial\eta = h^{(n)}(\xi, 0) = 0, \qquad \xi \neq 0 \tag{27}$$

and $u_\xi^{(n)}$ and $u_\eta^{(n)}$ must be finite at the edge of the strip.

When k_1 is replaced by $k(1 - 2\epsilon)^{1/2}$ and the latter is approximated by its binomial expansion, the incident P wave (18) takes the form

$$\varphi^i = \varphi_0 \exp(iky)[1 - \epsilon ky \exp(i\pi/2) + O(\epsilon^2)]. \tag{28}$$

Thus, the first two order perturbations of φ^i are

$$\varphi^{i(0)} = \varphi_0 \exp(iky), \qquad \varphi^{i(1)} = -ky\varphi_0 \exp(i(ky + \pi/2)). \tag{29}$$

(A) *Zeroth Order Solution.* With a common wave number, k, for both φ and h, the zeroth order solution can be deduced from Sommerfeld's well known solutions for the diffractions of vertically and horizontally plane polarized light waves by a semi-infinite opaque screen [3]. In fact, functions like $W_{1,2}$ defined in (25) were first used by Lamb for solving these problems in parabolic coordinates [7]. The zeroth order solution to our problem becomes,

$$\begin{aligned} \varphi^{(0)} &= [A_0\xi/\lambda(\xi^2 + \eta^2)]D_0(\lambda\eta)D_0(\lambda\xi) + W_1(\xi, \eta; \lambda), \\ h^{(0)} &= [A_0\eta/\lambda(\xi^2 + \eta^2)]D_0(\lambda\eta)D_0(\lambda\xi) \end{aligned} \tag{30}$$

where the first terms on the right hand sides of the above are added to Sommerfeld's solution W_1 in order to satisfy the regularity condition for displacements at the sharp edge. No such modification is needed in the case of optical waves because singular derivatives of φ at the edge of a screen are physically permissible. Equations (30) are identical to (24) except that

$$\lambda = (-2ik)^{1/2}, \qquad A_0 = -2\varphi_0/\pi^{1/2}.$$

(B) *First Order Solution.* General solutions for the first order wave potentials are given by (13) and (15). With $\mathbf{h}^{(n)} = (0, 0, h^{(n)})$, they take the form

$$\varphi^{(1)} = \varphi_c^{(1)} - \mathbf{r}\cdot\text{grad}\,\varphi^{(0)}, \qquad h^{(1)} = h_c^{(1)} + \mathbf{r}\cdot\text{grad}\,h^{(0)}. \tag{31}$$

Since the first order incident wave $\varphi^{i(1)}$ in (29) is included in the expression for $\varphi_p^{(1)}$, i.e., $-\mathbf{r}\cdot\text{grad}\,\varphi^{i(0)}$, the as yet undetermined complementary solution will be comprised solely of outgoing scattered waves. The absence of plane waves in complementary solutions beyond the zeroth order is because the initial incident wave is perturbed through the particular solutions. In fact, plane wave solutions for $\varphi_c^{(n)}$ or $h_c^{(n)}$ $(n > 0)$ mean that there are additional incident waves impinging on the strip with amplitudes proportional to ϵ^n and such waves do not exist in the posed problem.

Instead of directly substituting (30) into (31) we shall first modify the particular solution to shorten the calculations of constants in the complementary solutions. By noting that $-\mathbf{r}\cdot\text{curl}\,\mathbf{h}^{(0)}$ also satisfies a homogeneous Helmholtz equation, we may write the first of (31) as

$$\varphi^{(1)} = \varphi_c^{(1)} - \mathbf{r}\cdot(\text{grad}\,\varphi^{(0)} + \text{curl}\,\mathbf{h}^{(0)}) \tag{32}$$

where $\varphi_c^{(1)}$ is a new complementary solution, still not yet found. The quantity inside the parenthesis is now the zeroth order displacement vector. In terms of parabolic coordinates and using (20)

$$\varphi^{(1)} = \varphi_c^{(1)} - \tfrac{1}{2}g(\xi u_\xi^{(0)} + \eta u_\eta^{(0)}) = \varphi_c^{(1)} + \tfrac{1}{2}\lambda^2\xi\eta W_2(\xi, \eta; \lambda). \tag{33}$$

It is true that $\mathbf{r}\cdot\text{curl}\,(0, 0, \varphi^{(0)})$ also satisfies a homogeneous Helmholtz equation and apparently can be used to modify the particular solution of $h^{(1)}$ in (31). However, $\varphi^{(0)}$ contains the zeroth order incident wave and so the inclusion of $\mathbf{r}\cdot\text{curl}\,(0, 0, \varphi^{(0)})$ would introduce incident shear waves which do not exist in this problem. Hence, we maintain the original form and substitute (30) directly into the second of (31) to obtain

$$h^{(1)} = h_c^{(1)} - \tfrac{1}{2}h^{(0)} - \tfrac{1}{4}\lambda\eta A_0 D_0(\lambda\eta)D_0(\lambda\xi).$$

Since $h_c^{(1)}$ and $h^{(0)}$ both satisfy the same homogeneous Helmholtz equation and the former is not yet specified, we combine both to form a new complementary solution,

$$h^{(1)} = h_c^{(1)} - \tfrac{1}{4}\lambda\eta A_0 D_0(\lambda\eta)D_0(\lambda\xi). \tag{34}$$

From the boundary condition (27) with $n = 1$ and because $W_2(\xi, 0; \lambda)$ is identically zero, the boundary conditions for $\varphi_c^{(1)}$ and $h_c^{(1)}$, respectively, in (33) and (34) are simply

$$\partial\varphi_c^{(1)}(\xi, 0)/\partial\eta = h_c^{(1)}(\xi, 0) = 0, \qquad \xi \neq 0. \tag{35}$$

Solutions for $\varphi_c^{(1)}$ and $h_c^{(1)}$ which satisfy homogeneous Helmholtz equations, homogeneous boundary conditions (35) and must vanish at infinity would ordinarily be $\varphi_c^{(1)} = h_c^{(1)} = 0$, but that would result in singular displacements at $\xi = \eta = 0$. Thus, we take two singular outgoing wave functions as solutions which satisfy (35).

$$\begin{aligned} \varphi_c^{(1)} &= [A_1\xi/\lambda(\xi^2 + \eta^2)]D_0(\lambda\eta)D_0(\lambda\xi), \\ h_c^{(1)} &= [B_1\eta/\lambda(\xi^2 + \eta^2)]D_0(\lambda\eta)D_0(\lambda\xi) \end{aligned} \tag{36}$$

where the constants A_1 and B_1 are determined such that $u_\xi^{(1)}$, $u_\eta^{(1)} \to 0(r^{1/2})$ as $r \to 0$. Their values, found by substituting (36) into (33) and (34) and then calculating $u_i^{(1)}$ according to (20), are

$$A_1 = B_1 = -\tfrac{1}{2}A_0 . \tag{37}$$

The first order solutions are thus completely determined.

3. *Comparison of Approximate and Exact Solutions.* Since the exact results for this problem are available [6], the accuracy of the approximate solution can be examined. In fact, if k_1 and k_2 in the exact results are expressed in terms of k and ϵ, expansions of φ and h in (24) in a MacLaurin series in ϵ would yield the various order perturbation solutions. The errors caused by truncating such a MacLaurin's series can, in theory, be estimated. Furthermore, the expansion of the incident wave, φ^i, in (28) provides a rough guide to the behavior of the approximate solution. If the perturbation expansion ends at the first order, the solution is certainly not valid if ϵky is large, and so usually the best results will be obtained at low frequencies and in the region near the origin.

A detailed comparison is made by computing $\tau_{\eta\eta}(\pm\xi, 0)$ and $\tau_{\xi\eta}(0, \eta)$ from (4) and (10). The former are the normal stresses (τ_{yy} at $y = \pm 0$, $x > 0$) along both sides of the semi-infinite strip and the latter is the shear stress (τ_{xy} at $y = 0$, $x < 0$) along the line extension in front of the strip. Results are shown for $\nu = 0.2$ ($\epsilon = 0.227$) in Fig. 2, A and B, together with the exact values obtained from (24). All stresses are normalized by the factor, $\mu k_2^2\varphi_0$, which is the maximum principal stress on the incident P wave front and is assumed approaching a nonzero constant as $\omega(= k_2c_2) \to 0$. The abscissa scale in Fig. 2A is $(k_2/2)^{1/2}\xi = (k_2x)^{1/2}$ at $\eta = 0$ and in Fig. 2B it is $(k_2/2)^{1/2}\eta = (-k_2x)^{1/2}$ at $\xi = 0$. Each scale is thus proportional to the square root of the ratio of distance along the x-axis to the wave length of shear waves.

Near the tip of the strip ($x = 0$, $y = 0$), the stress values are dominated by the $x^{-1/2}$ terms which arise from the scattered waves. Hence, in this region the discrepancy is mainly between the coefficients of those singular terms which are known as stress-intensity factors. The largest error occurring is 11.5 per cent for τ_{xy}, while for the normal stresses, the errors are less than 5 per cent.

Further away from the tip the accuracy of the approximate results appears quite good over the range shown. However, this is admittedly a special situation as the normally incident wave, $\varphi_0 \exp(ik_1y)$, is constant at $y = 0$ and hence is not perturbed there. Only the scattered waves are actually approximated by perturbation and, interestingly, along either side of the strip the perturbed scattered wave effects decrease as k_2x increases. In general, a scattered wave perturbation expansion, just as a plane wave expansion,

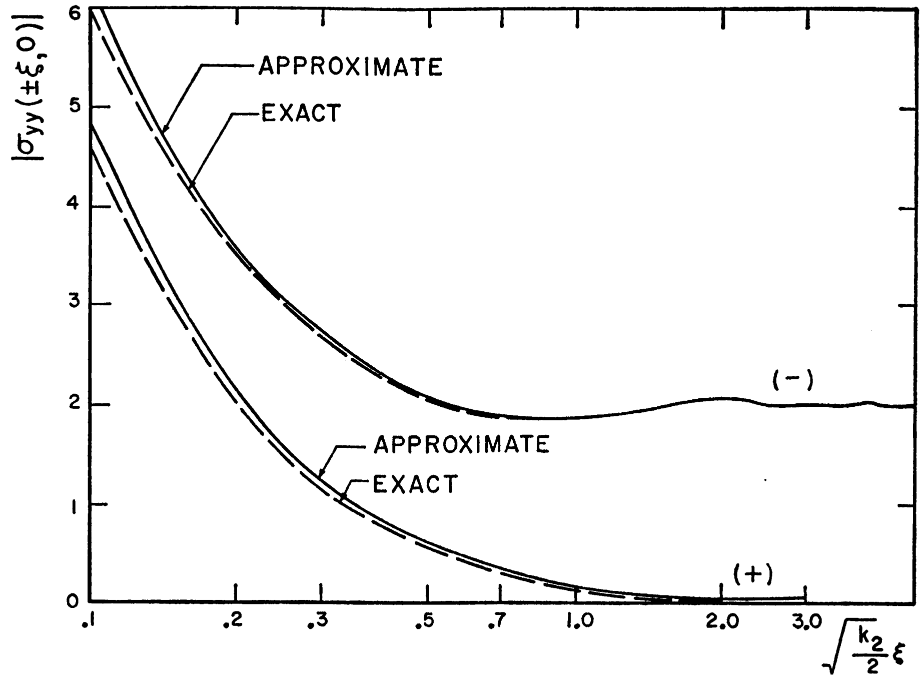

Fig. 2A. Normal stress, $|\sigma_{yy}(\pm\xi, 0)|$, along top and bottom surfaces of the strip.

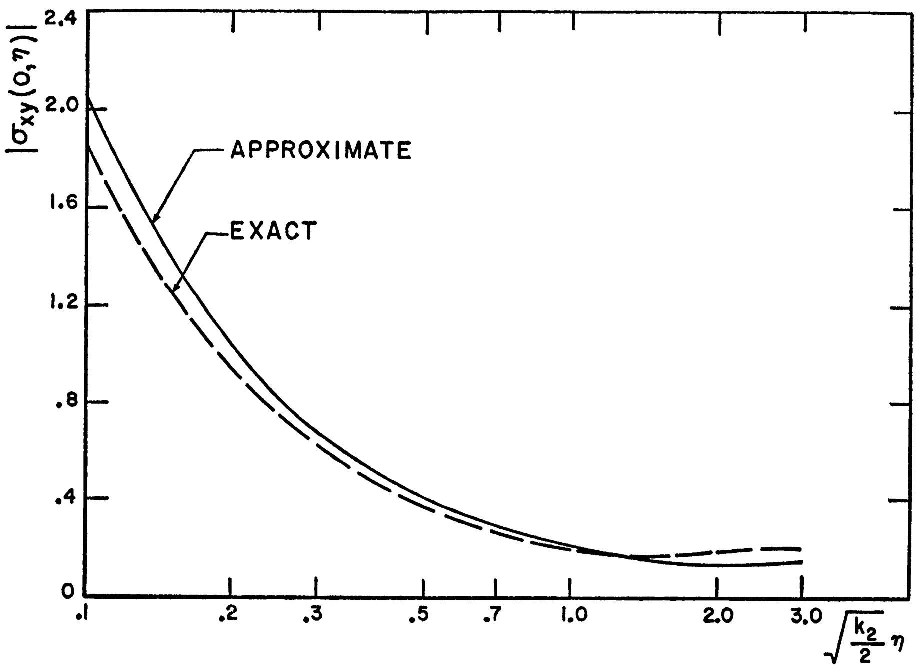

Fig. 2B. Shear stress, $|\sigma_{xy}(0, \eta)|$, in front of the strip.

Fig. 2. Normalized stresses $\sigma_{ij} = \tau_{ij}/\mu k_2^2\varphi_0$ for a semi-infinite rigid-smooth strip versus normalized shear wave number, $(k_2/2)^{1/2}\xi$ and $(k_2/2)^{1/2}\eta$, (or normalized distance) for a normally incident P wave and $\nu = 0.20$ (comparing exact and approximate solutions).

will be inaccurate for large values of wave number times distance. This indeed occurs for $\tau_{xy}(0, \eta)$ when $(k_2/2)^{1/2}\eta > 3$ in Fig. 2B and so these results are not compared further. However, the agreement of results in Fig. 2A will continue even as $(k_2/2)^{1/2}\xi \to \infty$.

The fact that the incident wave is constant everywhere along $y = 0$ also accounts for the somewhat larger error for the shear stress at the edge of the strip, since it arises solely from scattered waves. Normal stresses along the strip, on the other hand, are comprised of both incident and scattered parts and the former part is exact in this case.

IV. Diffraction of extensional waves by a circular rigid inclusion. In a thin plate, infinite in extent, the diffraction of plane extensional waves by a rigid, circular inclusion has been discussed in reference [8]. With the center of the rigid and fixed inclusion of radius, a, taken as the origin of cylindrical coordinates (r, θ, z) where z is along the thickness of the plate, the boundary conditions are

$$u_r(r, \theta) = u_\theta(r, \theta) = 0 \quad \text{at} \quad r = a. \tag{38}$$

The resulting radial stress of both the incident and scattered field on the inclusion boundary $(r = a)$ is [8]

$$\tau_{rr}(a, \theta) = (2\tau_0/i\pi) \sum \epsilon_n i^n k_2 a H_n'(k_2 a)\, \Delta_n^{-1} \cos n\theta \tag{39}$$

where τ_0 is the maximum normal stress on the incident wave front; $\epsilon_n = 1$ for $n = 0$ and $\epsilon_n = 2$ for $n > 0$; H_n is the Hankel function of the first kind with the prime indicating differentiation with respect to argument and

$$\Delta_n = \bar{k}_1 k_2 a^2 H_n'(\bar{k}_1 a) H_n'(k_2 a) - n^2 H_n(\bar{k}_1 a) H_n(k_2 a).$$

The perturbation solutions of the first two orders can be derived by following the same procedure outlined in Sec. III. The boundary conditions for the various order quantities are

$$u_r^{(n)} = u_\theta^{(n)} = 0 \quad \text{at} \quad r = a.$$

With $\bar{k}$ replacing $\bar{k}_1$ and k_2, the zeroth order solutions are derived as in the analogous diffraction problem for acoustical waves in air [9]. Particular solutions of the first order field equations are obtained by applying Eq. (15). We omit the details of the derivation and present final results for the radial stress at the inclusion:

$$\tau_{rr}(a, \theta) \approx \tau_{rr}^{(0)} + \epsilon \tau_{rr}^{(1)} \tag{40}$$

with

$$\begin{aligned} \tau_{rr}^{(0)} &= \frac{2i\tau_0}{\pi ka} \sum_{n=0}^{\infty} \frac{\epsilon_n i^n H_n'(ka)}{H_{n-1}(ka) H_{n+1}(ka)} \cos n\theta, \\ \tau_{rr}^{(1)} &= \frac{2i\tau_0}{\pi (ka)^2} \sum_{n=0}^{\infty} \frac{\epsilon_n i^n (n^2 - k^2 a^2) H_n(ka)}{H_{n-1}(ka) H_{n+1}(ka)} \cos n\theta. \end{aligned} \tag{41}$$

These can be identified as the coefficients of the first two terms of the expansion of $\tau_{rr}(a, \theta)$ into a MacLaurin series in ϵ.

Figs. 3A and 3B show, respectively, the variation of the normalized radial stress $|\tau_{rr}/\tau_0|$ $(= |\sigma_{rr}|)$ at the illuminous and shadow points on the inclusion ($\theta = \pi$ and $\theta = 0$, respectively) with the dimensionless wave number $\bar{k}_1 a$. The approximate stresses are computed from Eqs. (40) and (41) with Poisson's ratio equal to 0.15 and 0.35 and $\bar{\epsilon}$ accordingly equal to 0.202 and 0.255. The exact values, based on Eq. (39), are taken from reference [8].

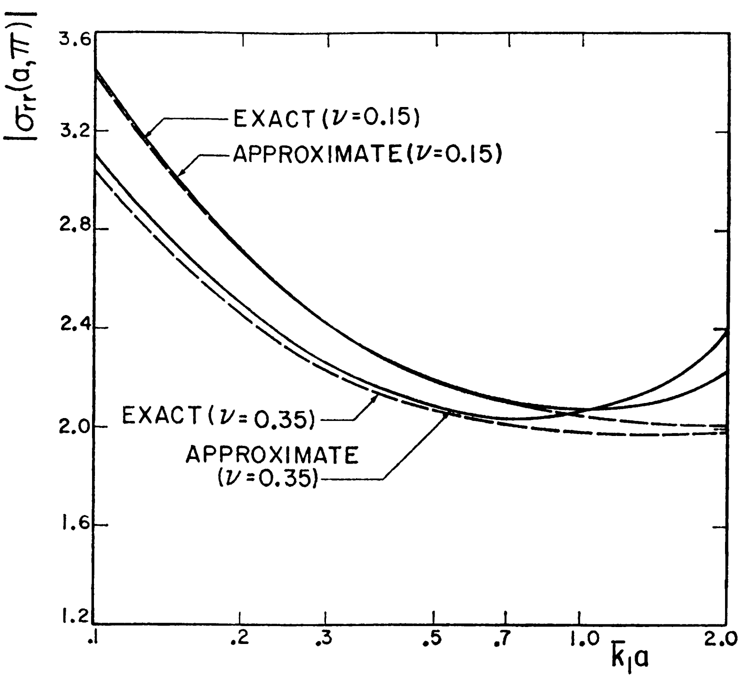

FIG. 3A. Radial stress at illuminous point, $|\sigma_{rr}(a, \pi)|$.

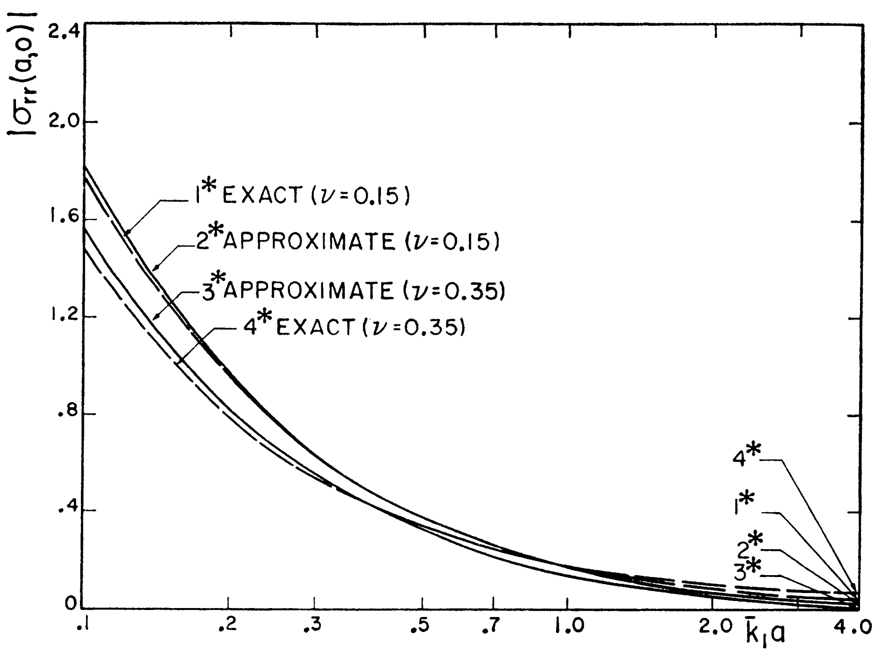

FIG. 3B. Radial stress at shadow point, $|\sigma_{rr}(a, 0)|$.

FIG. 3. Normalized radial stress at surface of a rigid-circular inclusion versus normalized extensional wave number, $\bar{k}_1 a$, (or normalized radius) for an incident P wave and various Poisson's ratios (comparing exact and approximate solutions).

For $0 < \bar{k}_1 a < 1$ in Fig. 3A, the two term perturbation solution agrees with the exact solution within 4 per cent. For $\bar{k}_1 a > 1$, the approximate results begin to diverge rapidly from the exact ones. Again, this confirms that the perturbation method yields good results in a region near the scatterer and at low frequencies. However, note that at $\bar{k}_1 a = 1$ the high frequency limit, $\tau_{rr}/\tau_0 = 2$, is nearly reached.

As compared to the rigid-smooth strip problem, the range of distance/wave length for which the approximate results are acceptable is, here, smaller. This is because there are contributions to the stress at the point $\theta = \pi$ from the incident plane wave, from a reflected plane wave, and from a cylindrical diffracted wave, each of which is perturbed. However, only contributions from diffracted waves had to be perturbed in the results of the previous problem.

At the shadow point ($\theta = 0$) the stress is caused only by a scattered wave whose amplitude decreases with increasing values of $\bar{k}_1 a$. The agreement between the exact and approximate results in Fig. 3B is therefore close over a greater range of frequencies and inclusion size.

V. Diffraction of elastic waves by a semi-infinite clamped strip. In the preceding sections, the perturbation method was applied to problems whose exact solutions are known, for the purpose of illustrating the method and checking its accuracy. Although the method indeed provides valid results in the low frequency range, the two term perturbation solutions, for the examples cited were no simpler to derive than the exact ones.

However, the perturbation method can be applied to problems not amenable to standard treatments. To illustrate this, we consider the diffraction of a plane harmonic elastic wave by a semi-infinite clamped strip. While the solution to this problem has been set up with the standard Wiener–Hopf technique [2], it is so complicated that no results for stresses and displacements have been obtained. Similarly, application of the method of separation of variables in parabolic coordinates provides a meager amount of information because the shear and compressional wave functions are not orthogonal to each other [1]. As shall be shown, a two-term perturbation solution, on the other hand, is easily constructed.

1. *Exact Formulation by the Wiener–Hopf Method.* As in Sec. III, we again consider a plane harmonic compressional wave in an infinite solid impinging upon a semi-infinite plane of discontinuity, taken as the xz-plane with $x > 0$. The plane is, however, fixed and rigid instead of rigid-smooth as previously considered. The propagation vector of the incident wave is perpendicular to the edge of the barrier so that the assumption of plane strain is applicable. Thus, the incident field becomes,

$$\varphi^{(i)} = \varphi_0 \exp(ik_1(x\cos\alpha + y\sin\alpha)), \qquad \mathbf{h}^{(i)} = (0, 0, h^{(i)}) = 0 \tag{42}$$

where α is the angle between the propagation vector and the x-axis.

The potentials for the waves scattered by the semi-infinite barrier, denoted by $\varphi^{(s)}$ and $h^{(s)}$, satisfy the field equations

$$(\nabla^2 + k_{1,2}^2)(\varphi^{(s)}, h^{(s)}) = 0 \tag{43}$$

and the boundary conditions

$$\begin{aligned} u_x^{(s)} &= \partial\varphi^{(s)}/\partial x + \partial h^{(s)}/\partial y = -ik_1\varphi_0 \cos\alpha \exp(ik_1 x\cos\alpha), \\ u_y^{(s)} &= \partial\varphi^{(s)}/\partial y - \partial h^{(s)}/\partial x = -ik_1\varphi_0 \sin\alpha \exp(ik_1 x\cos\alpha) \end{aligned} \tag{44}$$

along $y = 0^{\pm}$, $x > 0$. The above conditions imply from Eq. (1) that the total displacements along the clamped strip must vanish. The scattered wave functions, $\varphi^{(s)}$ and $h^{(s)}$, further should satisfy the Sommerfeld radiation condition and, at the sharp edge of the strip, should yield regular displacement functions, i.e., $u_x^{(s)}$ and $u_y^{(s)}$ are bounded as $r = (x^2 + y^2)^{1/2} \to 0$. This completes the mathematical statement of the problem.

A Wiener–Hopf type solution was formulated by Roseau [2] from the equivalent integral equations for the problem. Also, as outlined in reference [10], Jone's method can be adopted to derive the Wiener–Hopf type equations for this problem.

If $\bar{f}(\gamma)$ denotes the Fourier transform of a function $f(x)$, the transformed solutions of (43) may be written as,

$$\bar{\varphi}^{(s)}(\gamma, y) = \begin{cases} \tfrac{1}{2}(\bar{f}_- - i\gamma\bar{m}_-/\eta_1)\exp(-\eta_1 y), & y > 0, \\ -\tfrac{1}{2}(\bar{f}_- + i\gamma\bar{m}_-/\eta_1)\exp(\eta_1 y), & y < 0, \end{cases}$$

$$\bar{h}^{(s)}(\gamma, y) = \begin{cases} \tfrac{1}{2}(\bar{m}_- + i\gamma\bar{f}_-/\eta_2)\exp(-\eta_2 y), & y > 0, \\ -\tfrac{1}{2}(\bar{m}_- - i\gamma\bar{f}_-/\eta_2)\exp(\eta_2 y), & y < 0; \end{cases}$$

where

$$\eta_{1,2} = (\gamma^2 - k_{1,2})^{1/2},$$

and $\bar{f}_-$ and $\bar{m}_-$ satisfy the Wiener–Hopf type equations,

$$\begin{aligned} \left(\frac{\gamma^2}{\eta_2} - \eta_1\right)\bar{f}_- &= \frac{2k_1\varphi_0 \sin\alpha}{(\gamma - k_1\cos\alpha)_-} + 2\bar{q}_+ , \\ \left(\frac{\gamma^2}{\eta_1} - \eta_2\right)\bar{m}_- &= \frac{2k_1\varphi_0 \cos\alpha}{(\gamma - k_1\cos\alpha)_-} + 2\bar{p}_+ , \end{aligned} \tag{45}$$

in which $p(x)$ and $q(x)$ are zero for $x > 0$ and are equal to $u_x^{(s)}(x, 0)$ and $u_y^{(s)}(x, 0)$ respectively for $x < 0$. The $\pm$ subscripts indicate that a function is analytic in the upper or lower γ-planes, respectively.

The key step in the analysis at this point is to factor the expressions $(\gamma^2/\eta_2 - \eta_1)$ and $(\gamma^2/\eta_1 - \eta_2)$ in (45) into the product of functions which are analytic in upper and lower overlapping γ-planes. After this is done, the four unknown functions in (45) can be found with the function-theoretic methods associated with the Wiener–Hopf procedure [10], and then the transformed potentials are determined.

However, the plus and minus factors of $(\gamma^2/\eta_2 - \eta_1)$ cannot be found explicitly because $\eta_1 \neq \eta_2$, but instead are given in the form of complicated contour integrals [2]. Hence, the final solution in the γ-plane is expressed in such a cumbersome manner that apparently no results have been evaluated.

2. *Perturbation Solution.* From the above discussion, it is clear that an exact analysis for the diffraction of elastic waves by a rigid half-plane is made difficult because of the presence of two waves which are characterized by distinct wave numbers. However, if we take $k_1 = k_2 = k$, then $\gamma^2/\eta_2 - \eta_1 = \gamma^2/\eta_1 - \eta_2 = k^2(\gamma^2 - k^2)^{-1/2}$ which function is readily factored into plus and minus functions. In fact, as we shall show below, the zeroth order displacements so obtained are each identical to the solution for the diffraction of a plane polarized electromagnetic wave by a perfectly conducting semi-infinite screen, i.e., the Sommerfeld diffraction solution satisfying a Dirichlet boundary condition along the screen [3]. Furthermore, first order perturbation solutions can be generated from this same Sommerfeld solution.

(A) *Zeroth Order Solution.* We recall that the zeroth order solution of an elastodynamic problem is the same as the exact solution with k_1 and k_2 replaced by k. Therefore, all the steps leading to the derivation of (45) may be repeated with φ and h replaced by $\varphi^{(0)}$ and $h^{(0)}$, respectively, and $k_1 = k_2 = k$. The incident wave field is given by (42) while the diffracted field can be obtained after solving the Wiener–Hopf type equations (45) which now become

$$\begin{aligned} k^2(\gamma^2 - k^2)^{-1/2}\bar{f}_-^{(0)} &= 2k\varphi_0(\gamma - k\cos\alpha)_-^{-1}\sin\alpha + 2\bar{q}_+^{(0)}, \\ k^2(\gamma^2 - k^2)^{-1/2}\bar{m}_-^{(0)} &= 2k\varphi_0(\gamma - k\cos\alpha)_-^{-1}\cos\alpha + 2\bar{p}_+^{(0)}. \end{aligned} \tag{46}$$

Taking branch cuts for the functions $(\gamma - k)^{1/2}$ and $(\gamma + k)^{1/2}$ to extend from $\gamma = \pm k$ to $\gamma = \pm\infty$ in the upper and lower γ-planes, respectively, we can factor $(\gamma^2 - k^2)^{1/2}$ as $[(\gamma - k)^{1/2}]_-[(\gamma + k)^{1/2}]_+$. Then the first of Eqs. (46) may be changed into the form,

$$\begin{aligned} &k^2(\gamma - k)_-^{-1/2}\bar{f}_-^{(0)} - [2k^{3/2}\varphi_0(1 + \cos\alpha)^{1/2}(\gamma - k\cos\alpha)^{-1}\sin\alpha]_- \\ &\quad = \{2k\varphi_0[(\gamma + k)^{1/2} - (k + k\cos\alpha)^{1/2}](\gamma - k\cos\alpha)^{-1}\sin\alpha\}_+ + 2[(\gamma + k)^{1/2}]_+\bar{q}_+^{(0)} \end{aligned} \tag{47}$$

and a similar relationship follows from the second of Eqs. (46).

Now, we have obtained the desired result, an equation relating a minus function and a plus function. Using the usual analytic continuation step of equating each side of (47) to an entire function, in conjunction with the condition that the displacements must be finite at the edge of the strip, we can show that this entire function must be zero and therefore,

$$\begin{aligned} \bar{f}_-^{(0)} &= 2\varphi_0\sin\alpha\{[k^{-1}(1 + \cos\alpha)(\gamma - k)]^{1/2}(\gamma - k\cos\alpha)^{-1}\}_-, \\ \bar{m}_-^{(0)} &= 2\varphi_0\cos\alpha\{[k^{-1}(1 + \cos\alpha)(\gamma - k)]^{1/2}(\gamma - k\cos\alpha)^{-1}\}_-. \end{aligned} \tag{48}$$

The transformed potentials, $\bar{\varphi}^{(0)(s)}$ and $\bar{h}^{(0)(s)}$, can now be calculated and then the transforms of the zeroth order displacements are obtained from the Fourier transform of (9), i.e.,

$$\begin{aligned} \bar{u}_x^{(0)(s)} &= i\gamma\bar{\varphi}^{(0)(s)} + \partial\bar{h}^{(0)(s)}/\partial y, \\ \bar{u}_y^{(0)(s)} &= \partial\bar{\varphi}^{(0)(s)}/\partial y - i\gamma\bar{h}^{(0)(s)}. \end{aligned} \tag{49}$$

The results for the displacements are

$$\begin{aligned} \bar{u}_x^{(0)(s)}(\gamma, y) &= k\varphi_0\cos\alpha(k + k\cos\alpha)^{1/2}e^{-\eta|y|}/(\gamma + k)^{1/2}(\gamma - k\cos\alpha), \\ \bar{u}_y^{(0)(s)}(\gamma, y) &= k\varphi_0\sin\alpha(k + k\cos\alpha)^{1/2}e^{-\eta|y|}/(\gamma + k)^{1/2}(\gamma - k\cos\alpha), \end{aligned} \tag{50}$$

where $\eta = (\gamma^2 - k^2)^{1/2}$.

Equations (50) may be formally inverted to yield the actual displacements associated with the diffracted waves. However, by noting that these transformed expressions are identical with the corresponding ones in the Sommerfeld diffraction problem for an "absorbent" halfplane [10], we immediately can express the displacements in terms of Weber functions mentioned in Sec. III.

The Sommerfeld solution required here is W_2 (Eq. 25) which was defined for an incident angle $\alpha = \pi/2$. In terms of familiar polar coordinates (r, θ) and an incident wave with unit amplitude and arbitrary propagation direction it becomes,

$$W_2(r, \theta, \alpha) = \pi^{-1/2}\left[\exp(ikr\cos(\alpha - \theta)) \int_{a-}^{\infty} e^{-t^2}\, dt - \exp(ikr\cos(\alpha + \theta)) \int_{a+}^{\infty} e^{-t^2}\, dt\right] \tag{51}$$

where

$$a_\pm = (-2ikr)^{1/2} \sin \tfrac{1}{2}(\alpha \pm \theta).$$

Recall that W_2 contains the sum of the incident and diffracted waves.

Thus, the zeroth order displacements become,

$$u_x^{(0)} = ik\varphi_0 \cos \alpha W_2(r, \theta, \alpha), \qquad u_y^{(0)} = ik\varphi_0 \sin \alpha W_2(r, \theta, \alpha) \tag{52}$$

and the zeroth order stresses follow directly from (10).

For an incident shear wave at an angle β with the semi-infinite rigid strip,

$$\varphi^{(i)} = 0, \qquad h^{(i)} = h_0 \exp(ik_2(x\cos\beta + y\sin\beta)) \tag{53}$$

we can repeat the above analysis for the zeroth order field and can similarly derive the results,

$$u_x^{(0)} = ikh_0 \sin \beta W_2(r, \theta, \beta), \qquad u_y^{(0)} = -ikh_0 \cos \beta W_2(r, \theta, \beta). \tag{54}$$

Combination of (52) and (54) thus yields the zeroth order solution for the diffraction of arbitrary plane harmonic elastic waves by a clamped half-plane.

(B) *First Order Solution.* The first order field equations (14) may be written as

$$(\nabla^2 + k^2)\varphi^{(1)} = 2k^2\varphi^{(0)}, \qquad (\nabla^2 + k^2)h^{(1)} = -2k^2h^{(0)} \tag{55}$$

and particular solutions of them are given in (15). However, instead of using (14) and (15), we find it more convenient to derive and solve first order field equations for the displacements. The analysis, as will be seen below, is considerably shortened since the boundary conditions directly specify the displacements.

From (8) and (9) with $\mathbf{h} = (0, 0, h)$ for plane strain problems we obtain

$$\begin{aligned} k^2\varphi^{(n)} &= 2k^2\varphi^{(n-1)} - \partial u_x^{(n)}/\partial x - \partial u_y^{(n)}/\partial y, \\ k^2h^{(n)} &= -2k^2h^{(n-1)} + \partial u_y^{(n)}/\partial x - \partial u_x^{(n)}/\partial y \end{aligned} \tag{56}$$

and

$$\begin{aligned} (\nabla^2 + k^2)u_x^{(n)} &= 2k^2u_x^{(n-1)} - 4k^2\, \partial h^{(n-1)}/\partial y, \\ (\nabla^2 + k^2)u_y^{(n)} &= -2k^2u_y^{(n-1)} + 4k^2\, \partial \varphi^{(n-1)}/\partial y. \end{aligned} \tag{57}$$

It is seen that the zeroth order Cartesian displacements, $u_i^{(0)}$, satisfy homogeneous Helmholtz equations with wave number k, and so the first order field equations may be written as

$$\begin{aligned} (\nabla^2 + k^2)u_x^{(1)} &= -2\nabla^2u_x^{(0)} - 4k^2\, \partial h^{(0)}/\partial y, \\ (\nabla^2 + k^2)u_y^{(1)} &= 2\nabla^2u_y^{(0)} + 4k^2\, \partial \varphi^{(0)}/\partial y. \end{aligned} \tag{58}$$

It can be verified that particular solutions of these equations are

$$\begin{aligned} u_{xp}^{(1)} &= -\mathbf{r}\cdot\operatorname{grad} u_x^{(0)} - 2k^2yh^{(0)}, \\ u_{yp}^{(1)} &= \mathbf{r}\cdot\operatorname{grad} u_y^{(0)} + 2k^2y\varphi^{(0)} \end{aligned} \tag{59}$$

where the zeroth order potentials and displacements are related by (56) with $n = 0$.

In the problem of diffraction by a semi-infinite rigid strip, the zeroth order displacements, $u_x^{(0)}$ and $u_y^{(0)}$, vanish along $y = 0$, $x > 0$. Thus the first order displacements given by (59) also satisfy the boundary conditions along the rigid strip. Furthermore the right hand sides of (59) are bounded as $r = (x^2 + y^2)^{1/2} \to 0$. It then appears that the particular solutions (59) are the exact first order solutions to our problem. However, we must still calculate the first order potentials, $\varphi^{(1)}$ and $h^{(1)}$ to check whether or not the expressions in (59) introduce additional plane waves in the field. If so, they would have a magnitude of the order of ϵ. Such a plane wave corresponds to another incident wave which is not present in the posed problem and must be removed by complementary solutions of (58).

Letting $\mathbf{u}^{(1)} = \mathbf{u}_p^{(1)}$ as given by (59) and substituting it into (56) with $n = 1$, we find

$$k^2\varphi^{(1)} = \left(x\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial x\,\partial y}\right)u_x^{(0)} - \left(x\frac{\partial^2}{\partial x\,\partial y} + y\frac{\partial^2}{\partial y^2}\right)u_y^{(0)} + \frac{\partial u_x^{(0)}}{\partial x} - \frac{\partial u_y^{(0)}}{\partial y} - 2k^2yu_y^{(0)},$$

$$k^2h^{(1)} = \left(x\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial x\,\partial y}\right)u_y^{(0)} + \left(x\frac{\partial^2}{\partial x\,\partial y} + y\frac{\partial^2}{\partial y^2}\right)u_x^{(0)} + \frac{\partial u_x^{(0)}}{\partial y} + \frac{\partial u_y^{(0)}}{\partial x} + 2k^2yu_x^{(0)}.$$

Since $\mathbf{u}^{(0)}$ contains an incident plane wave

$$u_x^{(0)(i)} = ik(\varphi_0 \cos\alpha \exp(ikM) + h_0 \sin\beta \exp(ikN)),$$
$$u_y^{(0)(i)} = ik(\varphi_0 \sin\alpha \exp(ikM) - h_0 \cos\beta \exp(ikN)),$$
$$M = x\cos\alpha + y\sin\alpha, \qquad N = x\cos\beta + y\sin\beta,$$

which are derivable from (42) and (53) with $k_{1,2} = k$, the first order potentials do contain the undesired plane wave terms:

$$k^2\varphi^{(1)} = \frac{\partial u_x^{(0)(i)}}{\partial x} - \frac{\partial u_y^{(0)(i)}}{\partial y} = -k^2(\varphi_0 \cos 2\alpha \exp(ikM) + h_0 \sin 2\beta \exp(ikN)),$$

$$k^2h^{(1)} = \frac{\partial u_y^{(0)(i)}}{\partial x} + \frac{\partial u_x^{(0)(i)}}{\partial y} = -k^2(\varphi_0 \sin 2\alpha \exp(ikM) - h_0 \cos 2\beta \exp(ikN)).$$

These are eliminated by adding to $\mathbf{u}_p^{(1)}$ the complementary solutions $\mathbf{u}_c^{(1)}$ with

$$u_{xc}^{(1)} = u_x^{(0)}, \qquad u_{yc}^{(1)} = -u_y^{(0)} \tag{60}$$

which still satisfy the boundary and radiation conditions. The final first order solution is the sum of (59) and (60) with

$$\begin{aligned} u_x^{(1)} &= u_x^{(0)} - \mathbf{r}\cdot\operatorname{grad} u_x^{(0)} - 2k^2yh^{(0)}, \\ u_y^{(1)} &= -u_y^{(0)} + \mathbf{r}\cdot\operatorname{grad} u_y^{(0)} + 2k^2y\varphi^{(0)}. \end{aligned} \tag{61}$$

In summary, the displacement vector for the diffracted plane elastic waves by a semi-infinite rigid strip can be approximated by

$$\mathbf{u} = \mathbf{u}^{(0)}(x, y; k) + \epsilon\mathbf{u}^{(1)}(x, y; k) \tag{62}$$

where $\mathbf{u}^{(0)}$ is defined in (52) and (54) for incident compressional and shear waves, respectively, and $\mathbf{u}^{(1)}$ is given by (61).

Examination of (51), (52), (54), and (61) indicates that a fair amount of calculation is required to derive explicit expressions for the displacement functions and the stress field. Nevertheless, these calculations are simple when compared to the evaluation of the same results by the Wiener–Hopf or separation of variables procedure. Besides, an analytical result is derived here whereas recourse to numerical calculation must be made by the other methods.

The singular portion of the stress components along either side of the strip ($x > 0$) which are dominant close to the sharp edge are found to be

$$\begin{aligned}
\tau_{zz}(x, 0^{\pm}) &= \frac{\nu}{1+\nu}\,\tau_{yy}(x, 0^{\pm}),\\
\tau_{yy}(x, 0^{\pm}) &\approx \pm\left(1 - \frac{\epsilon}{2}\right)\left(\frac{2}{\pi k x}\right)^{1/2} \sin\alpha\,\cos\left(\tfrac{1}{2}\alpha\right)\exp\left(i\left(kx - \frac{3\pi}{4}\right)\right),\\
\tau_{xy}(x, 0^{\pm}) &\approx \pm\left(1 + \frac{\epsilon}{2}\right)\left(\frac{2}{\pi k x}\right)^{1/2} \cos\alpha\,\cos\left(\tfrac{1}{2}\alpha\right)\exp\left(i\left(kx - \frac{3\pi}{4}\right)\right)
\end{aligned}\tag{63}$$

for an incident compressional wave alone ($h^{(i)} = 0$), and

$$\begin{aligned}
\tau_{zz}(x, 0^{\pm}) &= \frac{\nu}{1+\nu}\,\tau_{yy}(x, 0^{\pm}),\\
\tau_{yy}(x, 0^{\pm}) &\approx \pm\left(1 - \frac{\epsilon}{2}\right)\left(\frac{2}{\pi k x}\right)^{1/2} \cos\beta\,\cos\left(\tfrac{1}{2}\beta\right)\exp\left(i\left(kx - \frac{3\pi}{4}\right)\right),\\
\tau_{xy}(x, 0^{\pm}) &\approx \mp\left(1 + \frac{\epsilon}{2}\right)\left(\frac{2}{\pi k x}\right)^{1/2} \sin\beta\,\cos\left(\tfrac{1}{2}\beta\right)\exp\left(i\left(kx - \frac{3\pi}{4}\right)\right)
\end{aligned}\tag{64}$$

for an incident distortional wave alone ($\varphi^{(i)} = 0$). In (63) the stresses are normalized by the value of the normal stress along the incident compressional wave front, while for (64) the normalization factor is the magnitude of the shear stress along the plane incident distortional wave.

VI. Conclusions. A perturbation method which reduces boundary value problems of elastodynamics to ones involving a single wave number has been presented. The perturbation solutions of the first few orders can then be generated nearly as easily as those for acoustic waves in air. So far, the method has been applied to investigate diffractions of elastic waves, and the resulting solutions, based on a two-term perturbation expansion, are in good agreement with the exact ones at low frequencies and in the near field. At high frequencies and/or in the far field, it is expected that good results may be obtained by different methods, e.g., geometric ray theory.

Perturbation solutions have also been obtained for the diffraction of elastic waves by a clamped half-plane, a problem which defies an exact analysis even when the powerful Wiener–Hopf method is applied. The results, again, are valid at low frequencies and near the edge of the plane.

For the examples cited, the perturbation results are valid up to the high frequency region, and so a complete frequency spectrum for the responses, either stresses or displacements, can be obtained approximately. This means that transient motions can be determined by using the Fourier integrals. Hence, the present perturbation method may serve as a useful step in the study of transient motion of an elastic solid.

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