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ON THE CONTACT PROBLEM OF LAYERED ELASTIC BODIES*

BY

TING-SHU WU (*The Ohio State University*)

AND

Y. P. CHIU (*S. K. F. Industries, King of Prussia, Pennsylvania*)

Abstract. The contact problem of elastic bodies, each consisting of a finite layer of uniform thickness rigidly adhering to a half-plane, is investigated on the basis of the two-dimensional theory of elasticity. The materials of the layer and the half-plane in the contact body are isotropic and homogeneous, yet each of them may have distinct elastic properties. The mixed boundary value problem is reduced to a single Fredholm integral equation of the second kind where the unknown variable is a fictitious surface deformation, through which the contact pressure can easily be obtained.

I. Introduction. One of the usual methods of solving mixed boundary value problems in the theory of elasticity is to use dual integral equations, which are then transformed into the form of Fredholm integral equations. Recently England [1] and Keer [2], using the method suggested by Collins [3, 4], have reduced their axially symmetric contact problems directly to Fredholm equations. In this paper, an approach is suggested for solving two-dimensional contact problems, particularly for the problems of two layered elastic bodies. The contact body is considered to be formed of an elastic layer of uniform thickness rigidly adhering to a semi-infinite elastic substrate, where the layer and the substrate generally have dissimilar elastic properties.

Using the complex variable solution for the mixed boundary value problem of a half-plane and the Fourier integral solutions for an elastic layer and for a half-plane, the present mixed boundary value problem of layered elastic substrates is reduced to a single Fredholm integral equation of the second kind. The unknown variable in this integral equation is a "fictitious" surface deformation through which the pressure at the contact region may be obtained without difficulty.

For the simplicity of presentation, mathematical derivations will be given in the second section of this paper for the contact of a layered elastic substrate and a rigid body. The result is generalized in part three for the case of two layered elastic bodies in contact.

II. The mixed boundary value problem of a layered half-plane. An elastic layer $-h \leq y \leq 0$ of infinite extent and an elastic substrate $-\infty < y \leq -h$, both isotropic and homogeneous but of different material properties, are rigidly adhered at $y = -h$. As a smooth, rigid, cylindrical punch of surface contour $f(x)$ is pressed against the layer on the surface $y = 0$ by a normal force P along the y-direction as shown in Fig. 1, a contact pressure $p(x)$ is developed over the region $-a \leq x \leq a$ on the boundary $y = 0$. Using u and v to denote the components of infinitesimal displacement in the x and y

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direction, τ_{yy} and τ_{yx} the normal and the shear stresses in the y direction, the boundary conditions can be expressed as;

(i) Along the boundary $y = 0$,

$$(\tau_{yy})_1 = 0, \qquad |x| \geq a, \tag{1a}$$

$$(\tau_{yx})_1 = 0, \tag{1b}$$

$$\left(\frac{\partial v}{\partial x}\right)_1 = f'(x), \qquad |x| \leq a. \tag{1c}$$

(ii) Along the interface $y = -h$ and for all values of x

$$(\tau_{yy})_1 = (\tau_{yy})_2 , \tag{2a}$$

$$(\tau_{yx})_1 = (\tau_{yx})_2 , \tag{2b}$$

$$\left(\frac{\partial u}{\partial x}\right)_1 = \left(\frac{\partial u}{\partial x}\right)_2 , \tag{2c}$$

$$\left(\frac{\partial v}{\partial x}\right)_1 = \left(\frac{\partial v}{\partial x}\right)_2 , \tag{2d}$$

where the subscript 1 has been used to denote the quantities of the layer and the subscript 2 those of the substrate.

For the sake of simplicity, we assume that $f(x)$ is symmetric and has no corners, such that the contact pressure $p(x)$ is zero at the ends of the contact length. The general case can be derived in a similar manner.

The general solutions for a half-plane $-\infty < y \leq -h$ and for a layer $-h \leq y \leq 0$ in the form of Fourier integrals are given by Sneddon [5].[1] Introducing the symbols $y_1 = y + h$, $\bar{y} = y/h$ and $\bar{y}_1 = y_1/h$, the general expressions for the stresses and displacements in plane strain for the symmetrically loaded layer and substrate can be written as

$$\begin{aligned}
\tau_{xx} &= \frac{2}{\pi h^3} \int_0^\infty \frac{\partial^2 G_i}{\partial \bar{y}^2} \cos\left(\xi \frac{x}{h}\right) d\xi, \\
\tau_{yy} &= -\frac{2}{\pi h^3} \int_0^\infty \xi^2 G_i \cos\left(\xi \frac{x}{h}\right) d\xi, \\
\tau_{yx} &= \frac{2}{\pi h^3} \int_0^\infty \xi \frac{\partial G_i}{\partial \bar{y}} \sin\left(\xi \frac{x}{h}\right) d\xi, \\
\frac{\partial u}{\partial x} &= \frac{2(1 + \nu_i)}{\pi E_i h^3} \int_0^\infty \left[(1 - \nu_i) \frac{\partial^2 G_i}{\partial \bar{y}^2} + \nu_i G_i \xi^2\right] \cos\left(\xi \frac{x}{h}\right) d\xi, \\
\frac{\partial v}{\partial x} &= -\frac{2(1 + \nu_i)}{\pi E_i h^3} \int_0^\infty \left[(1 - \nu_i) \frac{\partial^3 G_i}{\partial \bar{y}^3} + (\nu_i - 2) \frac{\partial G_i}{\partial \bar{y}}\right] \sin\left(\xi \frac{x}{h}\right) d\xi.
\end{aligned} \tag{3}$$

Where E and ν are the Young's modulus and Poisson's ratio respectively, $i = 1, 2$, and the function G is as follows:

(1) The layer

$$G_1 = (A_1 + B_1\bar{y}) \text{ ch } (\xi\bar{y}) + (c_1 + D_1\bar{y}) \text{ sh } (\xi\bar{y}). \tag{4a}$$

[1]The variable ξ and the functions B and D in [5] are replaced by the quantities ξ/h, B/h and D/h respectively to obtain Eqs. (3) and (4) in which ξ is a dimensionless dummy variable.

(2) The substrate

$$G_2 = (A_2 + B_2\bar{y}_1) \exp (\xi \bar{y}_1). \tag{4b}$$

One possible approach to obtain the solution for the layer and for the substrate is to substitute Eqs. (3) and (4) directly into the boundary conditions (1) and (2) for the boundary $y = 0$ and the interface, and then solve for the unknown functions of ξ and the contact pressure $p(x)$. This method finally requires a solution for dual integral equations. This paper presents an alternate method aimed at reducing the problem to a single Fredholm integral equation of the second kind which can then be solved by using a standard technique.

Consider that the solution for the layer in this problem consists of two parts: (a) the solution for a slab $-h \leq y \leq 0$ with zero surface traction on the surface $y = 0$ and (b) the solution for a half-plane $y < 0$ which is pressed upon by a smooth rigid punch having a contour $g(x)$ over the region $|x| \leq a$. (The function $g(x)$ is to be sought.) Here both the slab and the half plane have the same material properties as the original layer. The sum of the two solutions is then, according to the continuity condition given in Eq. (2), equated to the stresses and displacement derivatives of the substrate at $y = -h$ expressed in Eq. (3) and (4b).

The solution for the layer in part (a) is expressed in Fourier integral form as given in Eqs. (3) and (4a), and since there is no surface traction on $y = 0$, it follows immediately that

$$A_1(\xi) = 0, \tag{5}$$

$$B_1(\xi) + \xi C_1(\xi) = 0. \tag{6}$$

As for the half-plane $y < 0$ in part (b), its solution can be expressed in terms of a single complex function $\Phi(z)$ [6], $z = x + iy$. For the case that the contact pressure $p(x)$ vanishes at $z = \pm a$ and $g(x)$ is an even function,

$$\Phi(z) = \frac{E_1(a^2 - z^2)^{1/2}}{4\pi(1 - \nu_1^2)} \int_{-a}^{a} \frac{g'(t)\, dt}{(t - z)(a^2 - t^2)^{1/2}} = \frac{E_1'(a^2 - z^2)^{1/2}}{2\pi(1 - \nu_1^2)} \int_0^a \frac{tg'(t)dt}{(t^2 - z^2)(a^2 - t^2)^{1/2}} \tag{7}$$

where $g'(t) = dg/dx$ and $X(z) = (a^2 - z^2)^{1/2}$ is the branch, single-valued in the plane cut $x = -a$ to $x = +a$ along the x-axis, for which the argument is so selected that $X^+(t) = (a^2 - t^2)^{1/2}$ and $X^-(t) = -(a^2 - t^2)^{1/2}$, where the $(+)$ and $(-)$ denote the boundary values taken respectively on the upper and lower edges of the cut. For the points at a depth h below the surface, $z = x - ih \equiv z_0$, we obtain from Eq. (7)

$$(\tau_{yy})_{y=-h} = \frac{E_1}{\pi(1 - \nu_1^2)h} \int_0^a (T + S) \frac{tg'(t)\, dt}{(a^2 - t^2)^{1/2}}, \tag{8a}$$

$$\left(\frac{\partial u}{\partial x}\right)_{y=-h} = \frac{1}{2\pi(1 - \nu_1)h} \int_0^a [2(1 - \nu_1)T - S] \frac{tg'(t)\, dt}{(a^2 - t^2)^{1/2}}\,[2] \tag{8b}$$

where

$$T(x, t) = h \operatorname{Re} [F(z_0)],$$

$$S(x, t) = -h^2 \operatorname{Im} [F'(z_0)],$$

[2]Plane strain case.

$$F(z) = \frac{X(z)}{t^2 - z^2}, \qquad F'(z) = \frac{dF(z)}{dz}.$$

The integrals in Eq. (8) can also be expressed as Fourier integrals similar to those in Eq. (3), so that

$$\frac{E_1}{\pi(1-\nu_1^2)h}\int_0^a (T+S)\frac{tg'(t)\,dt}{(a^2-t^2)^{1/2}} = -\frac{2}{\pi h^3}\int_0^\infty \xi^2\alpha_1 \cos\left(\frac{\xi x}{h}\right) d\xi,$$
$$\frac{1}{2\pi(1-\nu_1)h}\int_0^a [2(1-\nu_1)T - S]\frac{tg'(t)\,dt}{(a^2-t^2)^{1/2}} = \frac{2(1+\nu_1)}{\pi E_1 h^3}\int_0^\infty [\xi^2\alpha_1 - 2(1-\nu_1)\xi\beta_1]\cos\left(\frac{\xi x}{h}\right) d\xi. \tag{9}$$

By taking the Fourier inverse transform on both sides of Eq. (8) and solving for α_1 and β_1, we have

$$\xi^2\alpha_1 = -\frac{E_1h^2}{\pi(1-\nu_1^2)}\int_0^\infty \cos\xi\eta\,d\eta \int_0^a (T+S)\frac{tg'(t)\,dt}{(a^2-t^2)^{1/2}},$$
$$\xi\beta_1 = \frac{E_1h^2}{4\pi(1-\nu_1)(1-\nu_1^2)}\int_0^\infty \cos\xi\eta\,d\eta\int_0^a [4(1-\nu_1)T+S]\frac{tg'(t)\,dt}{(a^2-t^2)^{1/2}}, \tag{10}$$

in which the variable x in T and S is replaced by ηh. After changing the order of integration, Eq. (10) can further be written as

$$\xi^2\alpha_1 = -\frac{E_1h^2}{\pi(1-\nu_1^2)}\int_0^a (I+H)\frac{tg'(t)\,dt}{(a^2-t^2)^{1/2}},$$
$$\xi\beta_1 = \frac{E_1h^2}{4\pi(1-\nu_1)(1-\nu_1^2)}\int_0^a [4(1-\nu_1)I + H]\frac{tg'(t)\,dt}{(a^2-t^2)^{1/2}}, \tag{11}$$

where I, H are, respectively, the Fourier cosine integrals of T and S, i.e.

$$I(\xi, t) = \int_0^\infty T(\eta h, t)\cos\xi\eta\,d\eta, \qquad H(\xi, t) = \int_0^\infty S(\eta h, t)\cos\xi\eta\,d\eta.$$

The integrals I and H are evaluated in the Appendix and the results are as follows:

$$H = \xi I = -\frac{\pi}{2}\xi e^{-\xi}\left[\frac{(a^2-t^2)^{1/2}}{t}\sin\left(\xi\frac{t}{h}\right) + \sum_{k=0,1}^{\infty}\frac{(-1)^k}{(2k)!}\left(\xi\frac{a}{h}\right)^{2k}P_k^*(t)\right], \tag{11a}$$

where

$$P_k^*(t) = \left(\frac{t}{a}\right)^{2k} - \frac{1}{2}\left(\frac{t}{a}\right)^{2k-2}\cdots - \frac{1\cdot1\cdot3\cdots(2k-3)}{2\cdot4\cdot6\cdots2k}, \qquad \xi \geq 0.$$

The substitution of the stresses and displacement derivatives given in Eqs. (3), (4) and (9) into Eq. (2) yield, with the aid of Eqs. (5) and (6), the following four simultaneous equations:

$$[B_1(\xi^2 \operatorname{ch}\xi - \xi\operatorname{sh}\xi) - D_1\xi^2\operatorname{sh}\xi - \xi^2\alpha_1] + \xi^2 A_2 = 0, \tag{12a}$$

$$[B_1\xi^2\operatorname{sh}\xi - D_1(\xi\operatorname{sh}\xi + \xi^2\operatorname{ch}\xi) + \xi\beta_1 + \xi^2\alpha_1] - \xi B_2 - \xi^2 A_2 = 0, \tag{12b}$$

$$\{B_1[(2\nu_1 - 1)\xi\operatorname{sh}\xi - \xi^2\operatorname{ch}\xi] + D_1[\xi^2\operatorname{sh}\xi + 2(1-\nu_1)\xi\operatorname{ch}\xi] + \xi^2\alpha_1 + 2(1-\nu_1)\xi\beta_1\} - \gamma\xi^2 A_2 - 2\gamma(1-\nu_2)\xi B_2 = 0, \tag{12c}$$

$$\{B_1[\xi^2 \operatorname{sh} \xi - 2(1 - \nu_1)\xi \operatorname{ch} \xi] + D_1[(1 - 2\nu_1)\xi \operatorname{sh} \xi - \xi^2 \operatorname{ch} \xi] + \xi^2\alpha_1 - (1 - 2\nu_1)\xi\beta_1\} - \gamma\xi^2 A_2 + (1 - 2\nu_2)\gamma\xi B_2 = 0, \qquad (12d)$$

where $\gamma = G_1/G_2$, the ratio of the shear moduli.

Eliminating A_2 and B_2 from Eq. (12), we obtain

$$\begin{aligned} eB_1 + bD_1 &= (\gamma - 1)\xi\alpha_1 + 2[(1 - \nu_2)\gamma - (1 - \nu_1)]\beta_1 , \\ cB_1 + dD_1 &= (\gamma - 1)\xi\alpha_1 - [(1 - 2\nu_2)\gamma - (1 - 2\nu_1)]\beta_1 , \end{aligned} \qquad (13)$$

where

$$\begin{aligned} e &= [(\gamma - 1)(1 - 2\nu_1) - 2\gamma(1 - \nu_2)\xi] \operatorname{sh} \xi - [1 + \gamma - 2\gamma\nu_2]\xi \operatorname{ch} \xi, \\ b &= [2\gamma(1 - \nu_2) + (1 + \gamma - 2\gamma\nu_2)\xi] \operatorname{sh} \xi + 2[(1 - \nu_1) + \gamma(1 - \nu_2)\xi] \operatorname{ch} \xi, \\ c &= [-2\gamma(1 - \nu_2) + (1 + \gamma - 2\gamma\nu_2)\xi] \operatorname{sh} \xi + 2[-(1 - \nu_1) + \gamma(1 - \nu_2)\xi] \operatorname{ch} \xi, \\ d &= [-(\gamma - 1)(1 - 2\nu_1) - 2\gamma(1 - \nu_2)\xi] \operatorname{sh} \xi - [1 + \gamma - 2\gamma\nu_2]\xi \operatorname{ch} \xi. \end{aligned} \qquad (14)$$

Using the result of Eq. (11), the function $B_1(\xi)$ can be obtained from Eq. (13),

$$B_1(\xi) = \frac{E_1 h^2}{\pi(1 - \nu_1^2)\xi} \frac{[dk_1 + bk_3 + \xi(dk_2 + bk_4)]}{de - bc} \int_0^a \frac{I(\xi, t)tg'(t)\, dt}{(a^2 - t^2)^{1/2}} , \qquad (15)$$

where

$$\begin{aligned} k_1 &= \gamma - 1 + 2(\nu_1 - \gamma\nu_2), \\ k_2 &= \tfrac{1}{2} - \gamma + \frac{(1 - \nu_2)\gamma}{2(1 - \nu_1)} , \\ k_3 &= 2(\gamma - 1 + \nu_1 - \nu_2\gamma), \\ k_4 &= \gamma - 1 + \frac{(1 - 2\nu_2)\gamma - (1 - 2\nu_1)}{4(1 - \nu_1)}. \end{aligned}$$

It still remains to satisfy the boundary condition (1c) by the superimposed solution for the layer. Since the prescribed displacement derivative of the auxiliary half-plane is $g'(x)$ while that from the general Fourier integral solution for a layer having zero surface traction on the surface $y = 0$ (i.e. part a) is taken from Eq. (3),

$$\left(\frac{\partial v}{\partial x}\right)_{y=0} = -\frac{4(1 - \nu_1^2)}{\pi h^3 E_1} \int_0^\infty \xi B_1(\xi) \sin\left(\frac{\xi x}{h}\right) d\xi, \qquad (16)$$

Eq. (1c) can be written as

$$g'(x) - \frac{4(1 - \nu_1^2)}{\pi h^3 E_1} \int_0^\infty \xi B_1(\xi) \sin\left(\frac{\xi x}{h}\right) d\xi = f'(x). \qquad (17)$$

After substituting Eq. (15) into Eq. (17) and interchanging orders of integration, Eq. (17) can be reduced to the following Fredholm integral equation of the second kind for $g'(x)$,

$$g'(x) - \frac{1}{\pi^2 h} \int_0^a K(x, t)g'(t)\, dt = f'(x) \qquad (18)$$

where the dimensionless kernel $K(x, t)$ is given by

$$K(x, t) = \frac{t}{(a^2 - t^2)^{1/2}} \int_0^\infty \frac{dk_1 + bk_3 + \xi(dk_2 + bk_4)}{de - bc} I(\xi, t) \sin\left(\xi \frac{x}{h}\right) d\xi.$$

Knowing the function of the kernel $K(x, t)$, the integral equation (18) may be solved by numerical or other means for the function $g'(x)$. The contact force P and the distribution of the contact pressure $p(x)$ can be determined from [6]

$$P = \frac{E_1}{2(1 - \nu_1^2)} \int_{-a}^{a} \frac{t g'(t)\, dt}{(a^2 - t^2)^{1/2}},$$
$$p(x) = \frac{E_1(a^2 - x^2)^{1/2}}{2\pi(1 - \nu_1^2)} \int_{-a}^{a} \frac{g'(t)\, dt}{(t - x)(a^2 - t^2)^{1/2}}. \tag{19}$$

When the layer and the substrate are of same material, that is, $E_1 = E_2$, $\nu_1 = \nu_2$, the problem is therefore reduced to a mixed boundary value problem of an elastic half-space. All the k's defined in Eq. (16) vanish, and the kernel K is zero for all values of x and t. Eq. (18) then gives $g'(x) = f'(x)$ as it should.

The kernel K for the case that the substrate is rigid, i.e. $\gamma = 0$, can be written in the following reduced form:

$$K(x, t) = \frac{t}{4(1 - \nu_1)(a^2 - t^2)^{1/2}} \int_0^\infty \Theta(\xi) I(\xi, t) \sin\left(\xi \frac{x}{h}\right) d\xi,$$

$$\Theta(\xi) = \frac{(1-\nu_1)][-4(2\nu_1-1)^2 \operatorname{sh} \xi - 16(1-\nu_1)^2 \operatorname{ch}\xi + (8\nu_1 - 6)\xi \operatorname{ch} \xi + (4\nu_1 - 6)\xi \operatorname{sh} \xi - 2\xi^2 \operatorname{ch} \xi] - (5 - 6\nu_1)\xi^2 \operatorname{sh} \xi}{\xi^2 + 4\nu_1^2 - 4\nu_1 + 1 - (4\nu_1 - 3) \operatorname{ch}^2 \xi}.$$

The foregoing analysis is based on the assumptions that the surface contour $f(x)$ of the punch is symmetric and that the contact pressure $p(x)$ is continuous and vanishes at the edges of the contact region. If $f(x)$ is not symmetric, the Fourier exponential transform should be used in the derivation. For the case that $p(x)$ does not vanish at the edges, the relationship between Φ and $g'(x)$ should assume the following general form [6]

$$\Phi(z) = \frac{E_1}{4\pi(1 - \nu_1^2)(a^2 - z^2)^{1/2}} \int_{-a}^{a} \frac{(a^2 - t^2)^{1/2} g'(t)\, dt}{t - z} + \frac{P}{2\pi(a^2 - z^2)^{1/2}}. \tag{20}$$

III. The contact problem of two layered elastic bodies. In addition to the assumptions made in the preceding section, we shall further assume here that the radii of curvature of both bodies are large in comparison with the dimensions of the area of contact and therefore each of the bodies can be substituted by a semi-infinite plane (Fig. 2).

Using I and II as the subscripts pertaining to the upper body and the lower body respectively, and 1 and 2 as used previously pertaining to the layer and the substrate, it is possible to denote, for instance, $E_{1\text{II}}$ as the Young's modulus of the layer of the lower body II, and $\nu_{2\text{I}}$ as the Poisson's ratio of the substrate of the upper body I. The relationship between the derivatives of displacement at the area of contact can be established as

$$\frac{\partial v_{1\text{I}}}{\partial x} + \frac{\partial v_{1\text{II}}}{\partial x} = \frac{\partial f_{\text{II}}(x)}{\partial x} - \frac{\partial f_{\text{I}}(x)}{\partial x} \equiv f'(x) \tag{21}$$

while the other boundary conditions have expressions similar to Eqs. (1) and (2).

Following the procedure used previously, we extend the layer of each contacting body to a half-plane, and denote the surface deformation of each fictitious half-plane under the yet to be determined contact pressure $p(x)$ by $g_{\mathrm{I}}(x)$ and $g_{\mathrm{II}}(x)$ respectively. Then, if we again assume that both $f_{\mathrm{I}}(x)$ and $f_{\mathrm{II}}(x)$ are symmetric functions and that the contact pressure $p(x)$ vanishes at the ends of the contact length, the actual surface deformations v_{II}, v_{III} of the layers can be expressed, respectively, in terms of the fictitious surface deformation functions $g_{\mathrm{I}}(x)$ and $g_{\mathrm{II}}(x)$,

$$\begin{aligned}\frac{\partial v_{\mathrm{II}}}{\partial x} &= g_{\mathrm{I}}'(x) - \frac{1}{\pi^2 h}\int_0^a K_{\mathrm{I}}(x, t) g_{\mathrm{I}}'(t)\, dt,\\ \frac{\partial v_{\mathrm{III}}}{\partial x} &= g_{\mathrm{II}}'(x) - \frac{1}{\pi^2 h}\int_0^a K_{\mathrm{II}}(x, t) g_{\mathrm{II}}'(t)\, dt,\end{aligned} \tag{22}$$

where the kernels K_{I} and K_{II} are similar to K defined in Eq. (19) except that the geometrical and material constants should be changed accordingly.

Substitution of Eq. (22) into Eq. (21) yields,

$$g_{\mathrm{I}}'(x) + g_{\mathrm{II}}'(x) - \frac{1}{\pi^2 h}\int_0^a [K_{\mathrm{I}}(x, t) g_{\mathrm{I}}'(t) + K_{\mathrm{II}}(x, t) g_{\mathrm{II}}'(t)]\, dt = f'(x). \tag{23}$$

Since the normal pressure acting on the upper body I along the area of contact coincides with the normal pressure acting on the lower body II, and $g_{\mathrm{I}}(x)$ and $g_{\mathrm{II}}(x)$ are the surface deformations in the contact region $|x| < a$ of the half-planes with elastic moduli E_{II}, ν_{II} and E_{III}, ν_{III}, therefore, from Eq. (20), $g_{\mathrm{I}}(x)$ and $g_{\mathrm{II}}(x)$ are related to each other through the equation

$$\frac{E_{\mathrm{II}}}{(1 - \nu_{\mathrm{II}}^2)} g_{\mathrm{I}}(x) = \frac{E_{\mathrm{III}}}{(1 - \nu_{\mathrm{III}}^2)} g_{\mathrm{II}}(x). \tag{24}$$

Introducing a new fictitious function $g_c(x)$ defined as

$$g_c(x) = g_{\mathrm{I}}(x) + g_{\mathrm{II}}(x), \tag{25}$$

then, the substitution of Eqs. (24) and (25) into Eq. (23) again yield a Fredholm integral equation of the second kind as follows

$$g_c'(x) - \frac{\int_0^a [(1 - \nu_{\mathrm{II}}^2)E_{\mathrm{III}}K_{\mathrm{I}} + (1 - \nu_{\mathrm{III}}^2)E_{\mathrm{II}}K_{\mathrm{II}}] g_c'(t)\, dt}{\pi^2[(1 - \nu_{\mathrm{II}}^2)E_{\mathrm{III}} + (1 - \nu_{\mathrm{III}}^2)E_{\mathrm{II}}]h} = f'(x) \tag{26}$$

while the contact pressure $p(x)$ and the contact force P can be computed from

$$p(x) = \frac{E_{\mathrm{II}}E_{\mathrm{III}}(a^2 - x^2)^{1/2}}{2[(1 - \nu_{\mathrm{II}}^2)E_{\mathrm{III}} + (1 - \nu_{\mathrm{III}}^2)E_{\mathrm{II}}]\pi}\int_{-a}^a \frac{g_c'(t)\, dt}{(t - x)(a^2 - t^2)^{1/2}}, \tag{27a}$$

$$P = \frac{E_{\mathrm{II}}E_{\mathrm{III}}}{2[(1 - \nu_{\mathrm{II}}^2)E_{\mathrm{III}} + (1 - \nu_{\mathrm{III}}^2)E_{\mathrm{II}}]}\int_{-a}^a \frac{t g_c'(t)\, dt}{(a^2 - t^2)^{1/2}}. \tag{27b}$$

Appendix

(A) *Evaluation of the Integral $I(\xi, t)$ given in Eq. (11).*

$$I(\xi, t) = \int_0^\infty T(z_0, t)\cos \xi\eta\, d\eta = \mathrm{Re}\left\{h\int_0^\infty \frac{(a^2 - z_0^2)^{1/2}}{t^2 - z_0^2}\cos\xi\eta\, d\eta\right\}. \tag{A1}$$

Using the identity

$$\cos \xi\eta = \tfrac{1}{2}(e^{-\xi}e^{i\xi(z_0/h)} + e^{\xi}e^{-i\xi z_0/h}), \qquad z_0/h = \eta - i, \tag{A2}$$

we obtain, from Eqs. (A1), (A2)

$$I = \operatorname{Re}\left\{\frac{e^{\xi}}{4} I_1 + \frac{\xi^{-\xi}}{4} I_2\right\}, \tag{A3}$$

where the integrals I_1 and I_2 are given by

$$I_1 = \int_S F(z)e^{-i\xi(z/h)}\, dz,$$

$$I_2 = \int_S F(z)e^{i\xi(z/h)}\, dz,$$

$$F(z) = \frac{(a^2 - z^2)^{1/2}}{t^2 - z^2},$$

and S, as shown in Fig. 3, is the path along $y = -h$ from $x = -\infty$ to ∞.

By applying the complex residue theorem and Jordan's lemma, the integration of I_1 and I_2 can proceed as follows: (see Fig. 3)

(1) Within the close path formed of S and $C_{R'}$, the complex function $F(z)$ is analytic and $F(z) \to 0$ as $|z| \to \infty$, therefore

$$\left(\int_{C_{R'}} + \int_S\right) F(z)e^{-i\xi(z/h)}\, dz = 0, \qquad \xi > 0.$$

Since $\int_{C_{R'}} = 0$ as $R \to \infty$, we have

$$I_1 = \int_S F(z)e^{-i\xi(z/h)}\, dz = 0, \qquad \xi > 0. \tag{A4}$$

(2) The function $F(z)e^{i\xi(z/h)}$ is analytic and single valued inside the closed path formed of S and $C_{R''}$ except at the branch cut, thus

$$\left(\int_S + \int_{C_{R''}} + \int_\Lambda\right) F(z)e^{i\xi(z/h)}\, dz = 0, \tag{A5}$$

where Λ is a closed contour surrounding the cut in the clockwise direction. Since

$$\int_{C_{R''}} F(z)e^{i\xi(z/h)}\, dz = 0 \quad \text{as} \quad R \to \infty,$$

we have

$$I_2 = \int_S F(z)e^{i\xi(z/h)}\, dz = -\int_\Lambda F(z)e^{i\xi(z/h)}\, dz.$$

By expanding $e^{i\xi(z/h)}$ into an infinite series and expressing $F(z)$ as the sum of two partial fractions, we obtain

$$I_2 = \frac{1}{2t}\left\{\sum_{k=0,1}^{\infty} \frac{(i\xi)^k}{k!}\,[L_k(t) - L_k(-t)]\right\}, \tag{A6}$$

where

$$L_k(t) = \int_\Lambda \frac{(a^2 - z^2)^{1/2}}{z - t}\,(z/h)^k\, dz.$$

The complex integral $L_k(t)$ around the branch cut can be intergrated by means of the Cauchy integral theorem described in §110 of [6], i.e.

$$L_k(t_0) = 2\pi i\left\{(a^2 - t_0^2)^{1/2}\left(\frac{t_0}{h}\right)^k + i\left(\frac{t_0}{h}\right)^{k+1}\cdot h\left[1 - \frac{1}{2}\left(\frac{a}{t}\right)^2 \cdots - \frac{1\cdot 1\cdot 3 \cdots (2k-3)}{2\cdot 4\cdot 6 \cdots 2k}\left(\frac{a}{t}\right)^{k'}\right]\right\},$$

where $k' = k + 1$ for k = odd integer or $k' = k$ for k = even integer. Substituting the expression for $L_k(t)$ into Eq. (A6) and rearranging the terms, we obtain

$$I_2 = -\frac{2\pi}{t}\left[(a^2 - t^2)^{1/2}\sin\left(\xi\frac{t}{h}\right) + t\sum_{k=0,1}^{\infty}\frac{(-1)^k}{(2k)!}\left(\xi\frac{a}{h}\right)^{2k}P_k^*(t)\right], \tag{A7}$$

where

$$P_k^* = \left(\frac{t}{a}\right)^{2k} - \frac{1}{2}\left(\frac{t}{a}\right)^{2k-2} \cdots - \frac{1\cdot 1\cdot 3 \cdots (2k-3)}{2\cdot 4\cdot 6\cdots (2k)}.$$

Since I_2 is found to be pure real, we have, from Eqs. (A3), (A4) and (A7)

$$I = \tfrac{1}{4}e^{-\xi}I_2 = -\frac{\pi}{Z}\left[\frac{(a^2 - t^2)^{1/2}}{t}\sin\left(\xi\frac{t}{h}\right) + \sum_{k=0,1}^{\infty}\frac{(-1)^k}{(2k)!}\left(\xi\frac{a}{h}\right)^{2k}P_k^*(t)\right]e^{-\xi}. \tag{A8}$$

(B) *Evaluation of the Integral* $H(\xi, t)$.

$$H(\xi, t) = \int_0^{\infty} S(z_0, t)\cos\xi\eta\, d\eta = \operatorname{Im}\left[-h^2\int_0^{\infty}F'(z_0)\cos\xi\eta\, d\eta\right], \qquad \xi \geq 0. \tag{A9}$$

Using Eq. (A2), we have

$$H = -(h/4)\operatorname{Im}[e^{\xi}H_1 + e^{-\xi}H_2], \tag{A10}$$

where

$$H_1 = \int_S F'(z)e^{-i\xi(z/h)}\, dz, \qquad \xi \geq 0,$$

$$H_2 = \int_S F'(z)e^{i\xi(z/h)}\, dz.$$

Again, applying the complex integration along the closed contour formed by S and the semicircle $C_{R'}$, it can be found, similar to Eq. (A4), that $H_1 = 0$. Integrating the integral H_2 by parts and using $F(z) \to 0$ as $|z| \to \infty$, yields

$$H_2 = -i\frac{\xi}{h}\int_S F(z)e^{i\xi(z/h)}\, dz = -\frac{i\xi}{h}I_2\,. \tag{A11}$$

Finally, by combining Eqs. (A9), (A3) and (A11), we obtain

$$H = -\frac{h}{4}e^{\xi}\operatorname{Im}H_2 = \frac{\xi e^{-\xi}}{4}I_2 = \xi I. \tag{A12}$$

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