# TWO DIMENSIONAL SINGULAR SOLUTIONS IN INFINITE REGIONS WITH COUPLE-STRESSES*

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**Introduction.** In this paper the singular solutions for a concentrated force and a concentrated couple are obtained for the two dimensional couple-stress theory. These solutions are generated from the corresponding known singularities in three dimensional couple-stress theory.

Since the usual statements of the uniqueness theorem in linear elasticity do not apply to singular problems, it is possible to find many other pseudo-solutions which yield the same resultant tractions as the properly constructed singularities. These pseudo-solutions are shown to omit some of the characteristic behavior which is due to couple-stress effects.

**1. The concentrated-couple paradox in two dimensional couple-stress theory.** In the couple-stress theory (alternately called "Cosserat theory with constrained rotations") the potential energy of an elastic material is assumed to depend on strains and rotation gradients. Within such materials each surface element is subjected not only to normal and tangential forces but also to moments per unit area, called "couple stresses".

It has been shown by Mindlin [1][1] that, in the case of plane-strain, it is possible to express stresses and couple-stresses in terms of two Airy-type stress functions $\Phi$ and $\psi$. In cylindrical coordinates $\rho$, $\theta$, and $z$ these expressions read

$$
\begin{aligned}
\sigma_\rho &= \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{1}{\rho}\frac{\partial^2\psi}{\partial\rho\,\partial\theta} + \frac{1}{\rho^2}\frac{\partial\psi}{\partial\theta},\\
\sigma_\theta &= \frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial^2\psi}{\partial\rho\,\partial\theta} - \frac{1}{\rho^2}\frac{\partial\psi}{\partial\theta},\\
\tau_{\rho\theta} &= -\frac{1}{\rho}\frac{\partial^2\Phi}{\partial\rho\,\partial\theta} + \frac{1}{\rho^2}\frac{\partial\Phi}{\partial\theta} - \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} - \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2},\\
\tau_{\theta\rho} &= -\frac{1}{\rho}\frac{\partial^2\Phi}{\partial\rho\,\partial\theta} + \frac{1}{\rho^2}\frac{\partial\Phi}{\partial\theta} + \frac{\partial^2\psi}{\partial\rho^2},\\
\mu_\rho &= \frac{\partial\psi}{\partial\rho},\\
\mu_\theta &= \frac{1}{\rho}\frac{\partial\psi}{\partial\theta},
\end{aligned}
\tag{1}
$$

where, in view of compatibility conditions

$$
\begin{aligned}
\frac{\partial}{\partial\rho}(\psi - l^2\nabla^2\psi) &= -2(1-\nu)l^2\frac{1}{\rho}\frac{\partial}{\partial\theta}\nabla^2\Phi,\\
\frac{1}{\rho}\frac{\partial}{\partial\theta}(\psi - l^2\nabla^2\psi) &= 2(1-\nu)l^2\frac{\partial}{\partial\rho}\nabla^2\Phi
\end{aligned}
\tag{2}
$$

*Received February 22, 1967. Work supported under U. S. Army Research Office-Durham, Contract DA-ARO(D)-416.

[1]Numbers in square brackets indicate references at end of paper.

and

$$\nabla^4\Phi = 0, \qquad \nabla^2(1 - l^2\nabla^2)\psi = 0. \tag{3}$$

Consider a two dimensional concentrated moment $M$ acting at the origin $\rho = 0$. The classical solution is given by the Airy stress function $\Phi = (M/2\pi)\theta$. Since this function $\Phi$ is harmonic, one may be led to assume from (2) that $\psi = 0$. Substitution in (1) yields that on surfaces $\rho = \rho_0$ the only stress is

$$\tau_{\rho\theta} = \frac{M}{2\pi}\frac{1}{\rho_0^2}$$

so that over any such surface the resultant traction vanishes and the resultant moment is

$$\tau_{\rho\theta}\cdot\rho_0\cdot 2\pi\rho_0 = M.$$

On the other hand one may consider $\psi \neq 0$ but $(1 - l^2\nabla^2)\psi =$ Const. or, without loss of generality, $(1 - l^2\nabla^2)\psi = 0$. There are infinitely many such functions $\psi$, all of which satisfy (2) with $\Phi = (M/2\pi)\theta$, that yield no resultant traction or couple on surfaces $\rho = \rho_0$. For instance, let $\psi = AK_0(\rho/l)$. Then, corresponding to this function $\psi$ one obtains at $\rho = \rho_0$

$$\tau_{\rho\theta} = -\frac{A}{\rho_0}\frac{\partial K_0(\rho/l)}{\partial\rho}\bigg|_{\rho=\rho_0}, \qquad \mu_\rho = A\,\frac{\partial K_0(\rho/l)}{\partial\rho}\bigg|_{\rho=\rho_0}.$$

Again, the resultant traction vanishes at $\rho = \rho_0$ and the resultant moment is

$$2\pi\rho_0^2\tau_{\rho\theta} + 2\pi\rho_0\mu_\rho = 0.$$

This ambiguity, which also exists in the case of the concentrated force, is eliminated when the singular solution is constructed "properly". One such "proper" method is to obtain the singular solution as a suitable limit of the solution to the corresponding non-singular problem. However, rather than establish such solutions directly it has been found advantageous, for the present cases, to construct the two dimensional singularities by superposition of the known three dimensional singular solutions, which in themselves were constructed "properly".

**2. The singular solution due to a concentrated couple.** Taking account of couple-stresses, Mindlin and Tiersten [2] gave a Boussinesq-Papkovitch type formulation for the displacement field $\mathbf{u}$ in terms of displacement functions $\mathbf{B}$ and $B_0$. The expression for $\mathbf{u}$ reads

$$\mathbf{u} = \mathbf{B} - l^2\nabla\nabla\cdot\mathbf{B} - \alpha'\nabla[\mathbf{r}\cdot(1 - l^2\nabla^2)\mathbf{B} + B_0] \tag{4}$$

in which

$$G(1 - l^2\nabla^2)\mathbf{B} = -\rho\mathbf{f} - \tfrac{1}{2}\rho\nabla\times\mathbf{c}, \qquad G\nabla^2B_0 = \mathbf{r}\cdot(\rho\mathbf{f} + \tfrac{1}{2}\rho\nabla\times\mathbf{c}). \tag{5}$$

In (4) and (5) $r$ is the radial distance from the origin, $\alpha' = 1/(4(1 - \nu))$, $\mathbf{f}$ and $\mathbf{c}$ are body force and body couple, respectively.

The solution for a concentrated couple of magnitude $M$ acting at the origin and oriented about the $z$ axis is given by [2]

$$\mathbf{B} = -(M/8\pi G)\mathbf{e}_z \times \nabla\varphi, \qquad B_0 = 0 \tag{6}$$

where, in (6)

$$\varphi = 1/r - f = (1 - e^{-r/l})/r. \tag{7}$$

Substituting (6) and (7) into (4) and expressing results in terms of the cylindrical coordinates $\rho$, $\theta$, $z$ one obtains

$$\mathbf{u} = u_\theta\mathbf{e}_\theta, \qquad u_\theta = -\frac{M}{8\pi G}\frac{\partial\varphi}{\partial\rho}. \tag{8}$$

Since $r^2 = \rho^2 + z^2$, it is clear from (7) that $\varphi = \varphi(\rho, z)$. To obtain a state of plane strain consider couples $\mathbf{M} = M\mathbf{e}_z$ strung uniformly along the entire $z$ axis. In view of (8), such distributed couples yield

$$\begin{aligned} u_\theta &= -\frac{M}{8\pi G}\int_{-\infty}^{\infty}\frac{\partial}{\partial\rho}\left[\frac{1-\exp(-((\rho^2+(z-\xi)^2)^{1/2})/l)}{(\rho^2+(z-\xi)^2)^{1/2}}\right]d\xi, \\ &= -\frac{M}{8\pi G}\int_{-\infty}^{\infty}\frac{\partial\varphi(\rho, z-\xi)}{\partial\rho}\,d\xi. \end{aligned} \tag{9}$$

Now, introducing the change of variables $(z - \xi)/\rho = \sinh\theta$ one arrives at

$$\begin{aligned} \int_{-\infty}^{\infty} f(\rho, z-\xi)\,d\xi &= \int_{-\infty}^{\infty}\frac{\exp(-((\rho^2+(z-\xi)^2)^{1/2})/l)}{(\rho^2+(z-\xi)^2)^{1/2}}\,d\xi \\ &= \int_{-\infty}^{\infty}\exp\left(-\frac{\rho}{l}\cosh\theta\right)d\theta = 2K_0\left(\frac{\rho}{l}\right). \end{aligned} \tag{10}$$

Furthermore, since $K_0(\rho/l)$ exists for all $\rho > \epsilon \geq 0$ and since it can be shown that

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial\rho} f(\rho, z-\xi)\,d\xi$$

converges uniformly for all $t = z - \xi$, it is permissible to interchange the order of differentiation and integration of the exponential portion of (9), whereby the computation of $u_\theta$ is greatly simplified. Altogether one obtains

$$u_\theta = \frac{M}{4\pi G}\left[\frac{1}{\rho} + \frac{d}{d\rho}K_0\left(\frac{\rho}{l}\right)\right]. \tag{11}$$

This displacement can be attributed to the functions

$$\Phi = \frac{M}{2\pi}\theta, \qquad \psi = \frac{M}{2\pi}K_0\left(\frac{\rho}{l}\right). \tag{12}$$

**3. The singular solution due to a concentrated force.** The three dimensional displacement field that corresponds to a concentrated force [2] $\mathbf{P} = P\mathbf{e}_x$, acting in the origin, is given by (4) with

$$\mathbf{B} = \frac{P}{4\pi G r}\varphi\mathbf{e}_x\,, \qquad B_0 = 0. \tag{13}$$

The function $\varphi$ is again given by (7).

Substituting (13) into (4) and expressing in terms of cylindrical coordinates $\rho$, $\theta$, $z$ one obtains

$$\begin{aligned} u_\rho &= \frac{P}{4\pi G}\left[\varphi - l^2\frac{\partial^2\varphi}{\partial\rho^2} - \alpha'\frac{\partial}{\partial\rho}\frac{\rho}{(\rho^2+z^2)^{1/2}}\right]\cos\theta, \\ u_\theta &= \frac{P}{4\pi G}\left[-\varphi + l^2\frac{1}{\rho}\frac{\partial\varphi}{\partial\rho} + \frac{\alpha'}{(\rho^2+z^2)^{1/2}}\right]\sin\theta, \\ u_z &= -\frac{P}{4\pi G}\left[l^2\frac{\partial^2\varphi}{\partial z\,\partial\rho} + \alpha'\frac{\partial}{\partial z}\frac{\rho}{(\rho^2+z^2)^{1/2}}\right]\cos\theta. \end{aligned} \tag{14}$$

The determination of the two dimensional solution for a concentrated force cannot proceed on the basis of the two dimensionalized displacement field which corresponds to (14). The difficulty is due to the presence of the term $1/r$ (that is the classical part of the function $\varphi$) in the displacements $u_\rho$ and $u_\theta$, which leads to the divergent integral

$$\int_{-\infty}^{\infty} \frac{d\xi}{(\rho^2 + (z - \xi)^2)^{1/2}}.$$

However, the term $1/r$ does not appear in the expressions for the strain and rotation fields which correspond to (14), so that these quantities lead to convergent integrals.

When the strains and rotations are computed, and the effect of concentrated forces $P\mathbf{e}_z$ strung uniformly along the entire $z$ axis is considered, one obtains a two dimensional field which can be attributed to the Airy-type functions

$$\begin{aligned} \Phi &= \frac{1 - 2\nu}{2(1 - \nu)} \frac{P}{\pi} \rho \log \rho \cos \theta - \frac{P}{2\pi} \rho\theta \sin \theta, \\ \psi &= \frac{P}{\pi} l^2 \frac{\sin \theta}{\rho} - \frac{P}{\pi} l K_1\left(\frac{\rho}{l}\right) \sin \theta. \end{aligned} \tag{15}$$

The function $K_1(\rho/l) \sin \theta$ does not contribute to the resultant traction and moment over any surface $\rho = \rho_0$. An analysis based solely on resultant considerations may lead to the omission of this function.

**Conclusions.** It is well known that pseudo singular-solutions exist in classical elasticity. Thus, it is not at all surprising to discover their presence in couple-stress theory. However, since the field equations of couple-stress theory are higher in degree than those of classical theory, these more complicated equations can accommodate a larger variety of erroneous solutions.

It has been shown in this paper that the basic two-dimensional singularities of couple-stress theory are distinctly different from their classical counterparts. The couple-stress singularities contain transcendental functions, which are traceable to the operator $1 - l^2\nabla^2$. This characteristic distinction cannot be omitted even though it does not affect resultants.

Several other pseudo singular-solutions in couple-stress theory, for the half-plane and for a flat crack were discussed in a recent paper by Sternberg and Muki [3]. A similar situation, regarding singularities, undoubtedly exists in the so-called strain-gradient theory.

## Nomenclature

| | |
|---|---|
| $\Phi$, $\psi$ | Stress functions |
| $\rho$, $\theta$, $z$ | Cylindrical coordinate |
| $r$ | Radial distance in spherical coordinates |
| $\sigma_\rho$, $\sigma_\theta$, $\tau_{\rho\theta}$, $\tau_{\theta\rho}$ | Stresses in cylindrical coordinates |
| $\mu_\rho$, $\mu_\theta$ | Couple-stresses in cylindrical coordinates |
| $l$ | Characteristic length of couple-stress theory |
| $\nabla^2$ | Laplacian operator |
| $K_0$, $K_1$ | Modified Bessel functions of second kind |
| $\nabla$ | Gradient operator |
| $\mathbf{B}$, $B_0$ | Boussinesq-Papkovitch type displacement functions |
| $\nu$ | Poisson's ratio |
| $G$ | Shear modulus |

## References

[1] R. D. Mindlin, *Influence of couple-stresses on stress concentrations*, Experimental Mechanics, January, 1963, pp. 1–7.

[2] R. D. Mindlin, and H. F. Tiersten, *Effects of couple-stresses in linear elasticity*, Arch. Rational Mech. Anal., No. (5) **11**, pp. 415–448 (1962) (Especially Secs. 3, 5, 11 and 15)

[3] E. Sternberg, and R. Muki, *The effects of couple-stresses on the stress concentration around a crack*, ONR, Contract Nonr 220(58) NR-064-431, Technical Report No. 6, California Institute of Technology, Pasadena, Calif., April, 1966