SERIES REPRESENTATIONS OF FOURIER INTEGRALS*

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Summary. General series representations, valid at least for small values of x, are obtained for the representative Fourier integral $\int_0^\infty A(k) \exp(ikx)\, dk$ for a variety of asymptotic forms of behaviour of the function $A(k)$, assumed bounded and integrable in any finite range. The results obtained should be of value in the numerical evaluation of such integrals as well as in the determination of their analytic properties.

1. Introduction. The solutions of many problems in applied mathematics can be obtained in terms of Fourier integrals for which no closed inverted form is available. In such cases recourse is often made to the use of asymptotic approximations and numerical methods in order to obtain useful answers. This paper is concerned with the derivations of series approximations to such integrals in cases where the standard numerical methods are of doubtful value and the usual asymptotic methods are not applicable. It suffices to consider the integral,

$$I(x) = \int_0^\infty A(k) \exp(ikx)\, dk \tag{1}$$

where $A(k)$ is a bounded, integrable function in any finite range and x is of unrestricted sign, as being representative of all Fourier integrals as all others can be compounded simply from integrals of this type. When $|x|$ is large, suitable approximate representations are readily obtainable by asymptotic methods such as steepest descents or stationary phase. For other values of $|x|$, Filon's method of numerical integration can be applied but when $|x|$ is small and $A(k)$ does not decay rapidly at infinity significant contributions to I may arise from a large range of integration so that numerical integration techniques become much less attractive and accurate.

Some approximate analytic techniques are available for dealing with the case $|x|$ small but these are far from comprehensive. Thus if $A(k)$ is exponentially small for large values of k, then a formal power series representation can be obtained for $I(x)$ by replacing $\exp(ikx)$ by its Taylor series and integrating term by term. The resulting series,

$$I(x) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \int_0^\infty A(k) k^n\, dk, \tag{2}$$

will be at least usefully asymptotic since all the integrals are convergent. If $A(k)$ is

*Received February 27, 1967.

[1]This paper is based in part on work done by the first-named author shortly before his death, in connection with a difficult Fourier integral occurring in reference [1]. Examination of his unpublished notes on the derivation of the dominant term for small times suggested to the second author that the method could be extended to yield far more general results. This was done while the latter was on leave at Harvard University and he wishes to acknowledge gratefully the hospitality of that institution and support under N.S.F. Grant GP-5792 and O.N.R. Contract Nonr-1866(34).

algebraically small, with a large enough index, for large values of k then the integrals in the early terms of the series in Eq. 2 will be convergent. In such cases one may expect that the early, well-defined terms provide an asymptotic representation for $I(x)$ but the number of such terms obtainable in this way is limited by the power of the dominant term in the asymptotic behaviour of $A(k)$. When $A(k)$ is merely bounded or algebraically large at infinity the integral, when interpreted in terms of generalized functions, is dominated for small values of $|x|$ by the contributions from the neighbourhood of $k = \infty$. In such cases it therefore suffices to find some standard function $f(x)$, whose Fourier transform $F(k)$ shares the same asymptotic behaviour for large k as $A(k)$, at least to some desired order. Then one merely writes

$$I(x) = f(x) + \int_0^\infty \left[A(k) - \frac{1}{2\pi} F(k) \right] e^{ikx} \, dk \tag{3}$$

and the dominant behaviour for small values of $|x|$ can be determined from that of f. Since for any singularity in f there are standard procedures and results (e.g. Lighthill [2]) for determining the asymptotic behaviour of $F(k)$ it is normally a matter of no great difficulty to determine the general nature of the singularities of $I(x)$ by inspection of the asymptotic behaviour of $A(k)$. Again this method is not applicable to the determination of any regular portion of $I(x)$ since the first approach clearly indicates that there will be contributions from the finite portion of the range.

In this paper series approximations to $I(x)$ are obtained, for small values of $|x|$, for a variety of asymptotic behaviours of $A(k)$. The methods used do not rely on the fact that the contributions come solely from infinity or the finite portion of the range nor do they place any restriction on the number of terms in the series which can be obtained. In view of the previously noted possibility of extracting singular behaviour by the use of standard functions attention will be confined to those cases where $I(x)$ is defined as an ordinary function, at least for $x \neq 0$. Thus it will be assumed that $A(k)$ tends to zero as k tends to infinity. The analysis which follows can be justified by careful use of limit processes but its presentation in a manner which makes this clear involves excessive computation, none of which contributes to the form of the final result. Thus, in order to justify the method of derivation used in the later work, a simple example is considered in Sec. 2 and the analysis is presented in a way which permits the rigorous demonstration of asymptotic nature of the series containing only the first few terms. Moreover, certain features of the structure of the computation are recognized, which enable one to demonstrate a rationale whereby much of the labour involved in obtaining the general term can be avoided. This rationale is applied in the remainder of the paper to derive a range of general results.

2. Motivating example. Consider the case when x is strictly positive and, for large values of k, $A(k)$ has the asymptotic behaviour $k^{-1} + O(\exp -k)$. Here and in what follows the notation $O(\exp -k)$ is loosely used to denote any quantity whose limit behaviour is essentially, that is neglecting algebraic factors, that of the exponential. Let R be any suitably large parameter, obviously dependent on x, chosen to satisfy the relations

$$R \gg 1 \qquad \text{and} \qquad Rx \ll 1. \tag{4}$$

Then one divides the range of integration at $k = R$ and considers the two subranges separately. In the finite range $(\exp ikx)$ can be replaced by a finite Taylor approximation together with error estimate. In the infinite range $A(k)$ can be replaced by its asymptotic representation together with the appropriate error estimate. Thus, for example, the following estimate[2]

$$I(x) = \int_0^R A(k)(1 + ikx - \tfrac{1}{2}k^2x^2)\,dk + O(R^4x^3) + \int_R^\infty e^{ikx}\,dk/k + O(e^{-R})$$

can be obtained. The first integral becomes large with R but the manner in which this occurs can be extracted explicitly by writing

$$\int_0^R A(k)\,dk = \operatorname{Ln} R + \int_0^R \left\{A(k) - \frac{H(k-1)}{k}\right\} dk,$$

$$= \operatorname{Ln} R + \int_0^\infty \left\{A(k) - \frac{H(k-1)}{k}\right\} dk + O(e^{-R})$$

where $H(k)$ denotes the Heaviside unit step function. Similarly the other terms can be arranged as

$$\int_0^R kA(k)\,dk = R + \int_0^\infty \{kA(k) - 1\}\,dk + O(e^{-R})$$

and

$$\int_0^R k^2A(k)\,dk = \tfrac{1}{2}R^2 + \int_0^\infty \{k^2A(k) - k\}\,dk + O(e^{-R})$$

which take the form, appropriate to this simple example, of asymptotic series in descending powers of R. Further the infinite integral can be rearranged as

$$\int_R^\infty e^{ikx}\,dk/k = \int_{Rx}^\infty e^{iu}\,du/u,$$

$$= \int_{Rx}^\infty \frac{e^{iu} - H(1-u)}{u}\,du - \operatorname{Ln} Rx,$$

$$= -\operatorname{Ln} Rx + \int_0^\infty \frac{e^{iu} - H(1-u)}{u}\,du - \int_0^{Rx} \frac{e^{iu} - 1}{u}\,du,$$

$$= -\operatorname{Ln} Rx + \int_0^\infty \frac{e^{iu} - H(1-u)}{u}\,du - iRx + \tfrac{1}{4}R^2x^2 + O(R^3x^3)$$

which is in the form of series in ascending powers of Rx. When the contributions from the two subranges are added together there is considerable cancellation of terms involving R and the resulting estimate

$$I(x) = -\operatorname{Ln} x + \int_0^\infty \left\{A(k) - \frac{H(k-1)}{k}\right\} dk + \int_0^\infty \frac{e^{iu} - H(1-u)}{u}\,du$$
$$+ ix\int_0^\infty \{kA(k) - 1\}\,dk + -\tfrac{1}{2}x^2\int_0^\infty \{k^2A(k) - k\}\,dk + O(R^4x^3)$$

is obtained. It is now a trivial matter to show, by appropriate choice of $R(x)$ that the largest error is $o(x^{2+\alpha})$ for any α in the range $0 \le \alpha < 1$.

[2]This is overgenerous in that it permits $A(k)$ to be its maximum over the whole range despite the fact that $A(k)$ is $O(k^{-1})$ for large k.

Because of its implications in later work the important observation about the above analysis is the cancellation of the terms such as Ln R, ixR and $x^2R^2/4$ which appear as large terms in the finite range integral. If this cancellation can be shown to be a general phenomenon and not merely specific to this particular problem then it will become a straightforward matter to calculate higher terms in such series even when $A(k)$ is not merely limited to a single algebraic term. There are heuristic arguments which suggest that in fact the cancellation must occur. Thus the integral $I(x)$ must be independent of the particular value of R chosen and hence so must be the coefficients in the asymptotic representation for small x. In order to complete this line of argument however it would be necessary to show that the integral does have an asymptotic expansion of the form which develops in the analysis. It is just conceivable that other powers of x arise through some eventual relationship forced between R and x by the behaviour of the remainder term together with the failure of cancellation to occur. Alternatively, it is noteworthy that the cancellation of the terms large in R is of exactly the same type which occurs in the method of matched asymptotic expansions, which is unproved but highly successful.

However, for the case when positive powers of R only occur in the coefficients, it is possible to give a rigorous inductive demonstration that this cancellation must occur, at least when $A(k)$ is dominantly algebraic in its behaviour at infinity. Other cases will be discussed in Sec. 4 where they arise. First the cancellation is shown to occur in the leading term and the error estimate obtained, as above, for this leading term. Next, let it be assumed that the coefficient of x^m is the first term in which there is some uncancelled power of R so then its coefficient $g_m(R)$ tends to infinity with R. Let $I_{m-1}(x)$ denote the approximation to I obtained by stopping at the x^{m-1} term. Then the straightforward estimate, as above, based on a remainder estimate $R^{m+1}x^m$ shows that $[I(x) - I_{m-1}(x)]$ is $O(x^{m-1+\alpha})$. But

$$I(x) - I_{m-1}(x) = g_m(R)x^m + O(R^{m+2}x^{m+1})$$

and by choice of $R = x^{-1/(m+2)}$, when the remainder is still small in comparison with $g_m(R)$, a contradiction is constructed. It therefore follows that once a first algebraic error estimate can be established, cancellation of all subsequent algebraically large terms must occur. We are thus led to the following procedure for handling such Fourier integrals.

1. For the contribution from the finite integral, pick out the coefficient, independent of R, in the asymptotic representation for large R.
2. For the contribution from the infinite integral, pick out the coefficient, independent of Rx, in the asymptotic representation for small Rx save that if a Ln Rx term should occur, the Ln x part should be kept.
3. All other terms may be ignored.

These procedure rules will be applied in the next sections to a variety of functions $A(k)$ and it will be shown that they permit the determination of the approximation series in a simple manner. Extensions of the justification to other than positive powers of R will be given as the need is shown to arise. In each case the procedure rules will be applied first and then justified.

3. Algebraic asymptotic behaviour. Consider now the case of $A(k)$ bounded, integrable and described, for large values of k, by the representation

$$A(k) = \sum_{1}^{\infty} a_n k^{-n} \tag{5}$$

which may be either convergent or merely asymptotic. Again the range of integration is divided at $k = R$ where R is chosen so that relations (4) apply. Then one writes

$$\int_0^R A(k)e^{ikx}\,dk = \sum_0^\infty \frac{(ix)^n}{n!}\int_0^R A(k)k^n\,dk$$

and, in a manner quite analogous to the working of the previous section, the large portion of the integrals are extracted by writing

$$\int_0^R A(k)k^n\,dk = \int_0^\infty \left\{A(k)k^n - \sum_{r=1}^n a_r k^{n-r} - a_{n+1}H(k-1)/k\right\}dk$$
$$+ a_{n+1}\operatorname{Ln} R + \sum_{r=1}^n \frac{a_r}{n-r+1}R^{n-r+1} - \sum_{r=n+2}^\infty \frac{a_r}{r-n-1}R^{-(r-n-1)}$$

and it is to be noted that this case differs from the previous example in that negative powers of R also occur. The infinite integral is transformed, in an analogous manner to that in the previous section, as follows

$$\int_R^\infty e^{ikx}k^{-n}\,dx = |x|^{n-1}\int_{R|x|}^\infty u^{-n}e^{ius}\,du \qquad (s = \operatorname{sign} x),$$
$$= \frac{|x|^{n-1}}{(n-1)!}\left\{\exp(iR\,|x|\,s)\sum_{r=1}^{n-1}(R\,|x|)^{-(n-r)}(is)^{r-1}(n-r-1)!\right\}$$
$$+ (is)^{n-1}\int_{R|x|}^\infty e^{ius}\,du/u$$

where the last result has been obtained by repeated integration by parts. If one applies the procedure rules, deferring for the present the question of their validity, then $\exp(iR\,|x|\,s)$ will be expanded in series form and the terms independent of R selected. The integral is treated as in the previous section. Thus the only terms which would be retained from the value of the infinite integral are

$$\frac{(ix)^{n-1}}{(n-1)!}\left\{\sum_{r=1}^{n-1}\frac{1}{r} + \int_0^\infty \frac{e^{ius} - H(1-u)}{u}\,du - \operatorname{Ln} R\,|x|\right\}.$$

The value of the integral in this expression can be obtained from the behaviour of the exponential integral for small values of its argument (e.g. Jahnke and Emde [3]). Thus when the two ranges are combined, and the terms dependent on R are ignored, one obtains the series

$$I_n(x) = \sum_{n=0}^\infty \frac{(ix)^n}{n!}\left\{\int_0^\infty \left[A(k)k^n - \sum_{r=1}^n a_r k^{n-r} - a_{n+1}H(k-1)/k\right]dk\right.$$
$$\left.+ a_{n+1}\left[-\operatorname{Ln}|x| + \operatorname{Ln}\gamma + si\pi/2 + \sum_{r=1}^n \frac{1}{r}\right]\right\} \tag{6}$$

where γ is Euler's constant. The convergence properties of the series cannot be discussed without an extensive knowledge of the behaviour of $A(k)$ but the occurrence of the $n!$ in the denominator gives reason to hope that a curtailment of the above series will prove useful for not too small a range of x for quite a wide variety of functions $A(k)$. Moreover, the potential oscillation in sign, implied by the presence of the factor i^n, offers scope for the extension of the useful range by the use of rate of convergence improvement techniques.

The question which remains to be answered is to whether in fact the postulated cancellation does take place. It is a simple matter to observe that all the Ln R terms do cancel so that it is only the integral powers of R which need to be considered. It does not take extensive calculations to show that all the negative powers of R in the coefficient of x^0 also cancel; there are no positive powers. It therefore follows that the limit arguments can be applied to show that the error term for the series in Eq. 6 must be $O(x^\alpha)$ for any $\alpha < 1$. The previous argument, that the terms involving positive powers of R in the coefficient of x must also cancel, now applies for, as before, their presence would permit the refutation of the error estimate by suitably choosing R. However, the negative powers of R in the coefficient, if they do not cancel, prevent one from inferring that the error involved, in dropping all higher powers of x, is $O(x^{1+\alpha})$ for any $\alpha < 1$. If one chooses $R(x)$ small enough that the R^3x^2 is $O(x^{1+\alpha})$ then the x/R^m term will not be small enough. All one can infer, if there is no cancellation, is that there exists a $\beta > 0$ such that the error in curtailing the series at the x term is $O(x^{1+\beta})$. However, this suffices to enable the demonstration that the negative powers of R must not occur in the coefficient of x. For, if there is any term of the form x/R^m, choose the least value of m and then by choosing R small enough it is possible to refute the error estimate $O(x^{1+\beta})$. The demonstration hinges on showing that if the coefficients are R dependent inconsistent error estimates can be obtained by making suitable choices of $R(x)$. Moreover the above argument is obviously capable of inductive extension so that the presence of negative powers does not affect the validity of the procedural rules stated at the end of the previous section. Thus (6) can be shown to be at least asymptotic for small x.

The above analysis has been based on the assumption that $A(k)$ only involves integral powers of k in its asymptotic behaviour but the analysis can be extended to fractional powers without much difficulty. Thus if, for large values of k,

$$A(k) = k^{-\nu} \sum_{r=0}^{\infty} b_r k^{-r}, \tag{7}$$

where $0 < \nu < 1$, then the discussion of the finite range integrals is very similar and the significant contribution from that range is merely

$$\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \int_0^{\infty} \left\{ k^n A(k) - \sum_{r=0}^{n} b_r k^{n-r-\nu} \right\} dk.$$

The calculation of the contribution from the large values of k takes the slightly different form

$$\int_R^{\infty} k^{-(r+\nu)} e^{ikx}\, dk = |x|^{r+\nu-1} \left[\frac{e^{iRx}(Rx)^{-(r+\nu-1)}}{r+\nu-1} + \frac{is}{r+\nu-1} \int_{R|x|}^{\infty} u^{-(r+\nu-1)} e^{iu}\, du \right]$$

and it can be seen immediately that in the integration of parts none of the contributions from the terminal give terms imdependent of R when expanded for small Rx. The reader can easily verify that the previous proof that the other terms must cancel so that the contribution from the infinite integral is

$$\sum_{r=0}^{\infty} b_r\, |x|^{r+\nu-1} \frac{(is)^r \Gamma(\nu)}{\Gamma(r+\nu)} \int_0^{\infty} u^{-\nu} e^{iu}\, du$$

or the more compact form

$$\pi \operatorname{cosec} \nu\pi \, |x|^{\nu-1} \sum_{r=0}^{\infty} \frac{b_r (ix)^r}{\Gamma(\nu + r)} \exp [i(\nu + 1)\pi/2]$$

obtained by identifying the integral with the Γ function and using the properties of that function (Whittaker and Watson). The resulting expression for $I(x)$ is thus

$$I(x) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \int_0^{\infty} \left\{ k^n A(k) - \sum_{r=0}^{n} b_r k^{n-r-\nu} \right\} dk$$

$$+ \frac{\pi}{\operatorname{Sin} \nu\pi} \exp [i(\nu + 1)\pi/2] \, |x|^{\nu-1} \sum_{r=0}^{\infty} b_r \frac{(ix)^r}{\Gamma(\nu + r)}. \tag{8}$$

It is apparent that by combining results (6) and (8) one can deal with any combination of fractional powers in the asymptotic behaviour of k. As far as the finite range contribution is concerned one merely takes large terms in the integrand until the integral just converges. In addition, there is a contribution from infinity, which can be immediately written down by examination of the appropriate coefficient of a_r and b_r on the series not under the integral sign in Eqs. 6 and 8.

4. Oscillatory type asymptotic behaviour. A few examples of this type, the first of which is

$$A(k) = e^{ik\alpha} \sum_{r=1}^{\infty} C_r k^{-r}, \tag{9}$$

will be considered to illustrate the type of problems which arise and how they may be overcome. The same technique of subdivision of the range of integration is used and the consideration of the finite portion leads to integrals of the form

$$\int_0^R A(k) k^n \, dk = \int_0^{\infty} \left\{ k^n A(k) - \sum_{r=1}^{n} C_r k^{n-r} e^{ik\alpha} \right\} dk$$

$$+ \sum_{r=1}^{n} C_r \int_0^R k^{n-r} e^{ik\alpha} \, dk + O(e^{iR\alpha}/R).$$

The finite integral can be evaluated by repeated integration by parts and contributions from the terminal R of the form $R^{\pm m} e^{iR\alpha}$ can be ruled out on grounds similar to the error discussion in Sec. 3. The finite terms with a factor $e^{iR\alpha}$ would then leave the coefficients in the resulting series dependent on R so these two must cancel. Thus all contributions from the upper terminal must be neglected but not those from the lower terminal so that one can show that the contribution form the finite range is

$$I(x) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \left\{ \int_0^{\infty} \left[k^n A(k) - \sum_{r=1}^{n} C_r k^{n-r} e^{ik\alpha} \right] dk + \sum_{r=1}^{n} C_r (n - r)! \left(\frac{i}{\alpha} \right)^{n-r+1} \right\}. \tag{10}$$

It can be shown that the infinite range integral contributes nothing to the accuracy of the result which almost certainly has an exponentially small error. It suffices to examine the contribution from the largest term since the other integrals can be treated similarly. The behaviour of the integral

$$\int_R^{\infty} k^{-1} e^{ik\alpha} e^{ikx} \, dk$$

is dominated, for small x, by the factor $e^{ik\alpha}$ in the integral so that it is most readily estimated by deforming to a path of steepest descent from R to imaginary infinity, the direction being dependent on the sign of α. For the case α positive, introduce the variable change $k = R + it$ and the integral becomes

$$iR^{-1} \exp [iR(\alpha + x)] \int_0^\infty \exp [-(\alpha + x)t]\left(1 + \frac{it}{R}\right)^{-1} dt$$

and standard asymptotic techniques may be applied. Thus the whole expression contains a factor $e^{iR\alpha}$ with sums of various powers of R and x but all these are of the type which must cancel. Thus Eq. (10) is the complete asymptotic series for $I(x)$. From the results of the previous section, it is known that this $I(x)$ has a logarithmic singularity of some type at $x = \alpha$ so that the series in 10 certainly does not converge in $|x| > |\alpha|$ and if $|\alpha|$ is small it is clear that a better approximation in a reasonable neighbourhood of $x = 0$ may well be obtained by using the results of Sec. 3 on the small variable $x + \alpha$.

Similar arguments may be used to discuss cases where $A(k)$ has the asymptotic representation

$$A(k) = \exp (i\beta k^\lambda) \sum_{r=1}^\infty d_r k^{r-1} \qquad \lambda > 0.$$

These examples divide naturally into the cases $\lambda > 1$ and $\lambda < 1$. When $\lambda > 1$ there is only a trivial difference from the case $\lambda = 1$ discussed above. The infinite range integral is dominated throughout by exp $i\beta k^\lambda$ and so the previous demonstration that it makes no significant contribution applies with only minor modification. However, because $\int_0^\infty \exp (i\beta k^\lambda)\, dk$ is convergent it is necessary to make a minor modification to the form of Eq. (10) by subtracting one less term so that

$$I(x) = \sum_{n=0}^\infty \frac{(ix)^n}{n!}\left\{\int_0^\infty \left[k^n A(k) - \sum_{r=1}^{n-1} d_r k^{n-r} \exp (ik^\lambda \beta)\right] dk + \sum_{r=1}^{n-1} \frac{d_r}{\lambda} \int_0^{\infty *} u^{(n-r+1-\lambda)/\lambda} e^{iu\beta}\, du\right\}$$

where the star * denotes the appropriate finite part of this integral the evaluation of which depends on the parameter λ and which will not be completed here although it can be seen to be related to the Γ function.

The case $\lambda < 1$ was the one which was of concern to Levey [1] in his work on the initial stages of the solution obtained in reference. When $\lambda < 1$ two subcases can be recognized by considering the question of estimating the integral whose integrand is dominated by exp $i\{\beta k^\lambda + xk\}$. It is apparent that if one used the technique of deforming the path of integration off the real axis one will be concerned with the relative dominance of the two terms near $k = R$ and k near infinity. If x and β have the same sign the path of steepest descent from R will reach infinity in the appropriate part of the plane but if they have opposite signs there will also arise a contribution from the saddlepoint at $k = (-\beta\lambda/x)^{1/(1-\lambda)}$. Similar conclusions can be reached by the use of stationary phase arguments. As before, the contribution from near the terminal point of integration $k = R$ will have an oscillatory exponential factor and hence must be of the type which cancels. The contribution from the infinite range of integration thus comes solely from the saddlepoint but the nature of the methods for determining such contributions prevent the general term from being evaluated in this case. In these circumstances only the evaluation of the leading useful term, arising from $\int_R^\infty k^{-1} \exp i\{kx - \beta k^\lambda\}\, dk$ when x is positive, will be given explicitly. It follows from standard analysis that the infinite range contributes a dominant term, for small positive x,

$$(2\pi)^{1/2}[\lambda(1-\lambda)\beta]^{-1/2}(x/\lambda\beta)^{\lambda/(2(1-\lambda))} \exp\left(-i\{(1-\lambda)\beta(\lambda\beta/x)^{-\lambda/(1-\lambda)}-\pi/4\}\right).$$

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