ON LOCAL STABILITY OF A FINITELY DEFORMED SOLID SUBJECTED TO FOLLOWER TYPE LOADS*

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Abstract. In this study the problem of the local stability (stability in the small) of a finitely deformed solid subjected to a set of follower type surface loads is analyzed, and a necessary, and a sufficient condition for asymptotic stability is established. Certain implications of the commonly used modal analysis are also investigated, and necessary and sufficient conditions for stability are formulated.

1. Introduction. Although the problem of the stability of an equilibrium configuration of a finitely deformed elastic solid has received considerable attention in recent years, most writers on the subject consider the special case of dead loading[1] [1]–[5].[2] Since such a system of loads constitutes a potential force field, a static analysis proves adequate provided a liberal attitude is adopted in viewing several exceptions [5]–[7].

The static approach ceases to be valid when the solid is subjected to a more general type of surface loading, for example, a system of follower type loads. This fact has been vividly demonstrated in a number of studies [8]–[11], where a dynamic concept of stability had to be adopted instead. The majority of these studies, however, are based on approximations and therefore have limited applicability [10], [11]. The deformations which precede the loss of stability are, in general, neglected, and Hooke's Law is used to relate the additional stresses to the small strains that are superposed on the equilibrium state of the initial strain. Moreover, the description of the follower type loads employed [10] appears to be dubious and involves approximations which are questionable.

In this study the problem of the local stability (stability in the small) of a finitely deformed solid subjected to a set of follower type surface loads is analyzed. The actual motion rather than a "virtual" displacement from the equilibrium configuration is considered, and a necessary, and a sufficient condition for asymptotic stability is established. Certain implications of the commonly used modal analysis are also investigated, and necessary and sufficient conditions for stability are formulated.

To avoid unnecessary complications, body forces are neglected, and it is assumed that on part of the boundary of the solid the displacements are prescribed so as to preclude a rigid-body motion. Moreover, although dissipation of energy is by necessity included in the analysis (by postulating the existence of a "damping stress"), no encroachment is made upon the territory of the thermodynamics of deformations. The

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[1]In addition to dead loading, Pearson [1] considered pressure loading which, however, constitutes a conservative force field in most practical cases (see Sec. 3 of this paper).

[2]Numbers in brackets refer to references listed at end of the paper.

investigation of thermoelastic stability and other related thermal phenomena would require an extensive study [12], [13] and is avoided here. It may, therefore, be assumed that the considered deformations take place under isothermal conditions.

2. Statement of problem and basic equations. Let a body B with material volume V_0 and material surface S_0 be deformed from its unstrained configuration C_0 to a strained configuration C by the application of surface tractions **T** on part of its boundary S_0^T. On the remaining part of the boundary $(S_0 - S_0^T)$ of the body, the displacements are prescribed so as to preclude all possible rigid-body motions. Let V denote the volume and S the surface of the body B in configuration C which is assumed to constitute an equilibrium state. Using a fixed Cartesian coordinate system, the Lagrangian strain tensor η_{ij} is

$$\eta_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i} + \frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right) \tag{2.1}$$

where u_i denotes the displacement of a material point which is carried from position a_i in configuration C_0 into position $x_i = a_i + u_i$ in configuration C. The mapping a_i into x_i must of course be one-to-one so that the Jacobian $J = \det |\partial x_i/\partial a_j|$ exists and is finite.

In the absence of body forces, the equations of equilibrium are

$$\partial \sigma_{ij}/\partial x_i = 0; \qquad \text{in} \quad V \tag{2.2a}$$

or

$$\partial S_{ij}/\partial a_i = 0; \qquad \text{in} \quad V_0 \tag{2.2b}$$

where σ_{ij} is the Eulerian and S_{ij} the Lagrangian stress tensor, respectively [14]. Referred to configuration C, the boundary conditions are

$$\begin{aligned} \sigma_{ij}\nu_i &= t_j \qquad \text{on} \quad S^T, \\ u_i &= 0 \qquad \text{on} \quad S - S^T \end{aligned} \tag{2.3}$$

where ν_i is the exterior unit normal to S^T and t_i the components of surface tractions **T**. If E defines the strain-energy function, then σ_{ij} and S_{ij} are given as, [14],

$$\sigma_{ij} = \frac{\rho}{\rho_0}\frac{\partial x_i}{\partial a_k} S_{kj} = \frac{\rho}{\rho_0}\frac{\partial x_i}{\partial a_k}\frac{\partial x_j}{\partial a_l}\frac{\partial E}{\partial \eta_{kl}} \tag{2.4}$$

where ρ is the mass-density in C and ρ_0 that in C_0.

To assure the stability of the configuration C, it is customary to consider an arbitrary virtual[3] displacement v_i^* from configuration C to a virtual configuration C^*, and then require that the work of the surface tractions on these displacements does not exceed the corresponding increase in the total internal energy of the body [1]–[4]. Such a stability criterion, though not limited to small deformations from C, places restrictions on the type of loading. Since it considers a virtual rather than the actual motion of the body from C to C^*, the work of the applied surface tractions cannot, in general, be uniquely defined unless these tractions are derivable from a potential (for example, when they constitute a system of dead loads). In this regard, the assertion made by Beatty [4]

[3]Such a virtual displacement must of course be compatible with the prescribed displacement boundary conditions.

that such a criterion is not restricted by the type of loading does not appear convincing. Indeed, if the surface tractions are of "follower" type, that is if they follow the deformations of the surface elements upon which they are acting (see Sec. 3), then Beatty's general criterion (his Eq. (1.2.2)), in general, yields no information as to the stability of the configuration C. To study the stability of the configuration C when surface tractions are of follower type, the actual motions of the body in the vicinity of C must be considered [8]–[11], [15]. This immediately raises the question of energy dissipation which is an inseparable companion to any actual motion and which can be avoided, when the body is subjected to a system of dead loads, by considering quasi-static deviations of B from C. Lacking a universally accepted method for the description of energy dissipation, it appears desirable to sacrifice generality for the sake of simplicity and postulate the existence of "damping stresses", which develop within the body as it deforms from the equilibrium configuration C to a nonequilibrium configuration C^*_t , and which are responsible for energy dissipation within the body. Here, the subscript t emphasizes that configuration C^*_t is changing with time, since the actual motions are being considered.

Let v_i define the displacement of a material point of B from C to C^*_t . To the first order of accuracy in the derivatives of v_i with respect to x_j , the following equations are obtained:[4]

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{2.5a}$$

$$\tau_{ij} = \sigma_{ik}\frac{\partial v_j}{\partial x_k} + \gamma_{ijkl}e_{kl} + \tau_{ij}^{(d)}, \tag{2.5b}$$

$$\gamma_{ijkl} = \frac{\rho}{\rho_0}\frac{\partial^2 E}{\partial \eta_{pq}\,\partial \eta_{rs}}\frac{\partial x_i}{\partial a_p}\frac{\partial x_j}{\partial a_q}\frac{\partial x_k}{\partial a_r}\frac{\partial x_l}{\partial a_s} \tag{2.5c}$$

where e_{ij} is the additional strain and τ_{ij} the Lagrangian stress tensor referred to the stressed configuration C. The tensor γ_{ijkl} is symmetric with respect to exchange of i and j, k and l, and ij and kl [2]. In Eq. (2.5b), $\tau_{ij}^{(d)} = \tau_{ji}^{(d)}$ denotes the "damping stress" which is included to account for energy dissipation within the body. The basic assumption here is that the stress tensor can be written as the sum of two terms, (1) the elastic term which, to the first order of accuracy in $\partial v_i/\partial x_j$, is given by the first two terms in the right side of (2.5b), and (2) the "damping" (or viscous) term denoted by $\tau_{ij}^{(d)}$. Coleman and Noll [16] have assumed that the damping (or viscous) stresses depend linearly on the velocity gradient.[5] This assumption will be employed in Sec. 5 where, for the analysis, $\tau_{ij}^{(d)}$ needs to be defined explicitly. In the remaining part of this paper it will be assumed that the stress tensor can be written as the sum of the elastic and the damping parts, and that the rate of the energy dissipation $\tau_{ij}^{(d)}\dot{v}_{i,j}$ per unit of volume in configuration C is a positive-definite quantity. (The superposed dot denotes partial differentiation with respect to time t, and a comma followed by the index j indicates partial differentiation with respect to x_j , t and x_j being the independent variables.) Moreover, $\tau_{ij}^{(d)} \equiv 0$ if $\dot{v}_{i,j} = 0$, that is if quasi-static deformations from C are considered.

Since τ_{ij} in Eq. (2.5b) denotes the Lagrangian stress tensor referred to the stressed configuration C, the conservation of linear momentum can be stated as

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[4]See Appendix A for the derivation of Eqs. (2.5b, c).

[5]Note that $\tau_{ij}^{(d)}$, in general, depends also on the strain and the specific entropy [16].

$$\tau_{ij,i} = \rho v_j^{\cdot\cdot}; \qquad \text{in} \quad V \tag{2.6a}$$

which must now be coupled with the following boundary conditions:

$$\begin{aligned} \tau_{ij}\nu_i &= t_j' ; \qquad \text{on} \quad S^T \\ v_i &= 0; \qquad \text{on} \quad S - S^T \end{aligned} \tag{2.6b}$$

where t_j' denotes the change in the surface tractions per unit area in C due to deformation from configuration C to C_i^* . t_j' is identically zero only if the surface tractions constitute a system of dead loads [4], but in general it is nonzero. Its explicit form depends on the manner in which the surface tractions change because of the deformations of the element upon which they are acting. In the following section, t_j' will be defined for various follower type surface loadings.

3. Description of follower type loads. In the equilibrium configuration C, a surface element dS with the exterior unit normal ν_i is subjected to a surface load $dP_i = t_i dS$ which may be expressed as

$$dP_i = t_i \, dS = (\tau \, dS)\pi_i \tag{3.1}$$

where π_i denotes a unit vector along t_i chosen in such a manner that $\pi_i\nu_i > 0$, and τ is the load-intensity which is to be taken negative if $t_i\nu_i < 0$ and positive otherwise. Let μ_i define a unit vector tangent to a line element dL_i in dS and in the plane which contains ν_i and π_i . The surface load dP_i on dS may then be expressed as

$$dP_i = (\tau \, dS)[(\pi_j\nu_j)\nu_i + (\pi_j\mu_j)\mu_i]. \tag{3.2}$$

The deformation which carries the body B from configuration C into configuration C_i^* , maps dS into dS^* with an exterior unit normal ν_i^* , and dL_i into dL_i^* with a unit tangent vector μ_i^* such that

$$dS^* = dS + dS', \tag{3.3a}$$

$$\nu_i^* = \nu_i + \nu_i' , \tag{3.3b}$$

$$dL_i^* = dL_i + dL_i' , \tag{3.3c}$$

$$\mu_i^* = \mu_i + \mu_i' \tag{3.3d}$$

where the primed quantities denote the changes due to the deformation. To the first order of accuracy in the derivatives of v_i with respect to x_j , these changes are given by the following equations:

$$dS' = dS(e_{jj} - e^{(\nu)}), \tag{3.4a}$$

$$\nu_i' = \nu_i e^{(\nu)} - \nu_j v_{j,i} , \tag{3.4b}$$

$$dL_i' = v_{i,j} \, dL_j , \tag{3.4c}$$

$$\mu_i' = \mu_j v_{i,j} - \mu_i e^{(\mu)}, \tag{3.4d}$$

where $e^{(\nu)} = \nu_j v_{j,k}\nu_k$, and $e^{(\mu)} = \mu_j v_{j,k}\mu_k$ in accord with equation (2.5a).

Let dP_i^* define the surface force which is acting on dS^* in configuration C_i^* . When $dP_i^* \equiv dP_i$ everywhere on S^T, the surface loads are said to constitute a system of dead loads [4]. If, on the other hand, dP_i^* remains in the plane which contains ν_i^* and μ_i^* such that

$$\mu_i \pi_i = \mu_i^* \pi_i^* \tag{3.5}$$

everywhere on S^T, then the surface loads are said to constitute a follower type system of forces. Here π_i^* denotes a unit vector directed along $dP_i^* = (\tau^* dS^*)\pi_i^*$ such that $\pi_i^* \nu_i^* \geq 0$, where τ^* designates the load-intensity in configuration C_t^*. The surface loads are said to constitute a system of "follower tractions" if $\tau^* = \tau$, and a system of "follower loads" if $(\tau^* dS^*) = (\tau dS)$. After equations (3.3), dP_i^* may be written as

$$dP_i^* = dP_i + dP_i' \tag{3.3e}$$

where dP_i' is given by

$$dP_i' = dP_i(e_{jj} - e^{(\nu)}) + dP_l[\nu_l \nu_i e^{(\nu)} - \mu_l \mu_i e^{(\mu)} - \nu_l \nu_j v_{j,i} + \mu_l \mu_j v_{i,j}] \tag{3.6a}$$

for follower tractions, and by

$$dP_i' = dP_l[\nu_l \nu_i e^{(\nu)} - \mu_l \mu_i e^{(\mu)} - \nu_l \nu_j v_{j,i} + \mu_l \mu_j v_{i,j}] \tag{3.6b}$$

for follower loads. Note that in either case, dP_i' is a linear and homogeneous function of both the surface tractions t_i and the displacement gradient $v_{i,j}$. Several special cases follow from Eqs. (3.6), namely:

(a) normal traction of constant intensity $-p$:

$$dP_i' = (p\, dS)(\nu_j v_{j,i} - \nu_i e_{jj}) \tag{3.6c}$$

which, if extended over the entire S^T, constitutes a conservative force field. This may be seen by considering the rate of the work $W^{\cdot}$ done by such a load on the body as it deforms from configuration C;

$$W^{\cdot} = p \int_{S^T} (\nu_j v_{j,i} - \nu_i e_{jj}) v_i^{\cdot}\, dS = \tfrac{1}{2} p \frac{d}{dt} \int_V (v_{i,j} v_{j,i} - e_{ii} e_{jj})\, dv.$$

(b) normal load (pdS = constant):

$$dP_i' = (p\, dS)(\nu_i e^{(\nu)} - \nu_j v_{j,i}) \tag{3.6d}$$

which does not constitute a conservative force field even if it is extended over the entire S^T. It, however, contains a conservative component which is identified by considering $W^{\cdot}$,

$$\begin{aligned} W^{\cdot} &= p \int_{S^T} (\nu_i e^{(\nu)} - \nu_j v_{j,i}) v_i^{\cdot}\, dS \\ &= -\tfrac{1}{2} p \frac{d}{dt} \int_V (v_{i,j} v_{j,i})\, dv - p \int_V e_{jj,i} v_i^{\cdot}\, dv + p \int_S e^{(\nu)} \nu_j v_j^{\cdot}\, dS, \end{aligned}$$

(c) tangential traction of constant intensity τ:

$$dP_i' = (\tau\, dS)[\mu_i(e_{jj} - e^{(\nu)} - e^{(\mu)}) + \mu_j v_{i,j}] \tag{3.6e}$$

which also contains a conservative part, since $W^{\cdot}$ can be written as

$$W^{\cdot} = \frac{1}{2} \frac{d}{dt} \int_V (\sigma_{kj} v_{i,k} v_{i,j})\, dv + \int_V \sigma_{kj}[v_{i,jk} v_i^{\cdot} + (e_{ii} v_j^{\cdot})_{,k}]\, dv - \int_S t_i (e^{(\nu)} + e^{(\mu)}) v_i^{\cdot}\, dS$$

where $\tau \mu_j = t_j = \sigma_{kj} \nu_k$ on S^T.

(d) tangential load ($\tau\, dS$ = constant):

$$dP_i' = (\tau\, dS)(\mu_j v_{i,j} - \mu_i e^{(\mu)}) \tag{3.6f}$$

the conservative part of which can be identified by considering $W^{\cdot}$.

$$W^{\cdot} = \frac{1}{2}\frac{d}{dt}\int_V (\sigma_{kj}v_{i,k}v_{i,j})\,dv + \int_V \sigma_{kj}v_{i,jk}v_i^{\cdot}\,dv - \int_S t_i e^{(\mu)}v_i^{\cdot}\,dS.$$

From Eqs. (3.6a, b), the change in surface tractions due to deformations from configuration C is given by

$$t_i' = \{t_i(e_{jj} - e^{(\nu)} - e^{(\mu)}) + t_l\nu_l[\nu_i(e^{(\nu)} + e^{(\mu)}) - 2\nu_j e_{ij}]\} + t_j v_{i,j} \tag{3.7a}$$

for follower tractions, and by

$$t_i' = \{t_l\nu_l[\nu_i(e^{(\nu)} + e^{(\mu)}) - 2\nu_j e_{ij}] - t_i e^{(\mu)}\} + t_j v_{i,j} \tag{3.7b}$$

for follower loads. In either case, these surface tractions do not constitute a conservative force field. For instance, the rate at which the additional tractions t_i' defined by Eq. (3.7b) do work is given by

$$W^{\cdot} = \frac{1}{2}\frac{d}{dt}\int_V \sigma_{ki}v_{j,k}v_{j,i}\,dv + \int_V \sigma_{ki}v_{j,ki}v_j^{\cdot}\,dv + \int_S w^{\cdot}\,dS$$

where

$$w^{\cdot} = \{\sigma^{(\nu)}[\nu_i(e^{(\nu)} + e^{(\mu)}) - 2\nu_j e_{ij}] - t_i e^{(\mu)}\}v_i^{\cdot}$$

and $\sigma^{(\nu)} = \nu_i\sigma_{ij}\nu_j$.

For small deformations of the body from configuration C, Eqs. (3.7) precisely define the changes of surface tractions for follower type loadings. These equations are not reported in Bolotin's book [10]. Bolotin uses an approximate procedure to develop t_i' in terms of t_j and the displacement gradient. His expression does not contain the quantity in the braces in Eqs. (3.7) and, therefore, appears questionable.

4. Stability criterion, follower type loads. Substitution from Eq. (2.5b) into Eqs. (2.6) yields

$$(\gamma_{ijkl}v_{k,l})_{,i} + \sigma_{ik}v_{j,ki} + \tau_{ij,i}^{(d)} = \rho v_j^{\cdot\cdot}; \qquad \text{in} \quad V, \tag{4.1a}$$

$$(\gamma_{ijkl}v_{k,l} + \sigma_{ik}v_{j,k} + \tau_{ij}^{(d)})\nu_i = t_j' ; \qquad \text{on} \quad S^T \tag{4.1b}$$

where t_j' is defined by equation (3.6c–f) or (3.7), depending on the nature of loading. Multiplying both sides of equation (4.1a) by $v_j^{\cdot}$ and integrating over V results in

$$\frac{1}{2}\frac{d}{dt}\int_V [\gamma_{ijkl}v_{k,l}v_{j,i} + \sigma_{ik}v_{j,i}v_{j,k} + \rho v_j^{\cdot}v_j^{\cdot}]\,dv + \int_V \tau_{ij}^{(d)}v_{j,i}^{\cdot}\,dv - \int_S t_j'v_j^{\cdot}\,dS = 0. \tag{4.2}$$

The surface integral in this equation can readily be identified with $W^{\cdot}$ which was calculated in the preceding section for several cases of surface loadings. For the sake of explicitness, the tangential loads will be considered first. This way, not only will all the relevant features of these types of problems be vividly demonstrated, but also lengthy equations will be avoided. The results will then be extended to include more general types of surface loadings.

Substitution from Eq. (3.6f) into (4.2) now yields

$$\left\{\frac{1}{2}\frac{d}{dt}\int_V [\gamma_{ijkl}v_{k,l}v_{j,i} + \rho v_j^{\cdot}v_j^{\cdot}]\,dv\right\} + \left\{\int_V [\tau_{ij}^{(d)}v_{j,i}^{\cdot} - \sigma_{ki}(v_{j,ik}v_j^{\cdot})]\,dv + \int_S t_i e^{(\mu)}v_i^{\cdot}\,dS\right\} = 0. \tag{4.3}$$

In Eq. (4.3), the expression in the first set of braces represents the rate of change of the internal and the kinetic energies of the body, and the expression in the second set denotes the rate of energy dissipation plus the rate at which work is done by the nonconservative part of the surface loads. The potential and the kinetic energies H_t of the body in configuration C_t^* relative to that in C may be written as

$$H_t = \frac{1}{2} \int_V [\gamma_{ijkl} v_{j,i} v_{k,l} + \rho \dot{v}_j \dot{v}_j] \, dv. \tag{4.4}$$

If the configuration C is to be locally stable, H_t must stay arbitrarily small for all $t > 0$ when it is sufficiently small at $t = 0$. For asymptotic stability, moreover, H_t must also approach zero for $t \to \infty$. This definition of stability, although in accord with the usual energy requirements, does not account for concentration of energy and other related focusing phenomena [5]–[7], [17]. Such problems are not considered here and the reader is referred to a paper by Shield [7] for an interesting discussion of the subject.

Since H_t is a positive-definite functional, vanishing identically only at the equilibrium state, a necessary condition for asymptotic stability is

$$\int_{t_0}^{t_0+t} \left\{ \int_V [\tau_{ij}^{(d)} \dot{v}_{j,i} - \sigma_{ki} v_{j,ik} \dot{v}_j] \, dv + \int_S t_i e^{(\mu)} \dot{v}_i \, dS \right\} dt > 0 \tag{4.5a}$$

for all finite and positive t_0, and all sufficiently large t. Condition (4.5a) is obviously assured if the following inequality holds:

$$\int_V [\tau_{ij}^{(d)} \dot{v}_{j,i} - \sigma_{ki} v_{j,ik} \dot{v}_j] \, dv + \int_S t_i e^{(\mu)} \dot{v}_i \, dS > 0 \tag{4.5b}$$

which constitutes a sufficient condition for asymptotic stability.[6]

It is worth noting that requirements (4.5) do not contain elastic properties of the body directly. They do, however, include the damping stresses which are responsible for energy dissipation within the body. Since it is the difference between the energy dissipation and the work done on the body by the nonconservative part of surface loads that may render the equilibrium configuration C unstable, a stability criterion which is to account for a general loading but does not involve the energy dissipation must be viewed with skepticism [4], [15]. When, in contrast to the case of nonconservative loading the applied tractions are conservative, the energy dissipation does not enter directly into the analysis, a fact which partly accounts for the great success that the usual energy method has enjoyed in the past.

Requirements similar to (4.5) are not, however, sufficient for stability when other loading conditions are considered, since the loss of stability may occur statically. In the present case, such a possibility is *a priori* ruled out due to the fact that H_t in Eq. (4.4) is a positive-definite functional. Had there been a contribution from the work of surface loads to this quantity, it would have been also necessary to require that H_t be positively definite for all sufficiently small compatible deformations of the body from configuration C. As an example, consider a body that is subjected to follower loads defined by Eq. (3.6d). Equation (4.3) must then be replaced by

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[6]Note that if (4.5b) holds, then H_t is a monotonically decreasing function of time and, since it is positive-definite and vanishes only at the equilibrium state, it must approach zero as $t \to \infty$ (see Secs. 3 and 4 of [5]). This latter assertion follows from the fact that, infinitely adjacent to configuration C, no equilibrium state with positive H_t can exist.

$$
\begin{aligned}
H_t^{\cdot} &= \left\{\frac{1}{2}\frac{d}{dt}\int_V [\gamma_{ijkl}v_{j,i}v_{k,l} + \sigma_{ik}v_{j,i}v_{j,k} + pv_{i,j}v_{j,i} + \rho v_j^{\cdot}v_j^{\cdot}]\,dv\right\} \\
&= -\left\{\int_V [\tau_{ij}^{(d)}v_{j,i}^{\cdot} + pe_{jj,i}v_i^{\cdot}]\,dv - p\int_S e^{(\nu)}\nu_j v_j^{\cdot}\,dS\right\}.
\end{aligned} \tag{4.6}
$$

For asymptotic stability, it is now sufficient that

$$
\int_V [\gamma_{ijkl}v_{j,i}v_{k,l} + \sigma_{ik}v_{j,i}v_{j,k} + pv_{i,j}v_{j,i} + \rho v_j^{\cdot}v_j^{\cdot}]\,dv > 0
$$

and

$$
\left\{\int_V [\tau_{ij}^{(d)}v_{j,i}^{\cdot} + pe_{jj,i}v_i^{\cdot}]\,dv - p\int_S e^{(\nu)}\nu_j v_j^{\cdot}\,dS\right\} > 0.
$$

It should be noted that for surface loads, which constitute a conservative force field, conditions similar to (4.5) are trivially satisfied and the requirement of H_t being positive-definite (with contributions from the work of surface loads included) suffices to assure stability. To illustrate this, consider the case of normal tractions (Eq. (3.6c)). Equation (4.2) now becomes

$$
\begin{aligned}
H_t^{\cdot} &= \frac{1}{2}\frac{d}{dt}\int_V [\gamma_{ijkl}v_{j,i}v_{k,l} + \sigma_{ik}v_{j,i}v_{j,k} - p(v_{i,j}v_{j,i} - e_{ii}e_{jj}) + \rho v_j^{\cdot}v_j^{\cdot}]\,dv \\
&= -\int_V \tau_{ij}^{(d)}v_{j,i}^{\cdot}\,dv.
\end{aligned} \tag{4.7}
$$

Since the right side of Eq. (4.7) is a negative-definite functional, vanishing only when $v_{j,i}^{\cdot} = 0$ everywhere in V, the left side of this equation is at least a nonincreasing functional. It therefore follows [5], [15] that, in this case of conservative loading, the configuration C is stable if

$$
\int_V [\gamma_{ijkl}v_{k,l}v_{j,i} + \sigma_{ik}v_{j,i}v_{j,k} - p(v_{i,j}v_{j,i} - e_{ii}e_{jj}) + \rho v_j^{\cdot}v_j^{\cdot}]\,dv > 0 \tag{4.8a}
$$

for all compatible deformations of B from C. Now, since the last term in the brackets in (4.8a) by its very nature is always a positive-definite quantity, a sufficient condition for the local stability is

$$
\int_V [\gamma_{ijkl}v_{k,l}v_{j,i} + \sigma_{ik}v_{j,i}v_{j,k} - p(v_{i,j}v_{j,i} - e_{ii}e_{jj})]\,dv > 0 \tag{4.8b}
$$

for all sufficiently small compatible quasi-static deformations of the body from configuration C. Here v_i may be viewed as a virtual displacement, since condition (4.8b) is static in character. Similar results can be obtained for dead loading and other conservative cases. For further discussion regarding stability under dead loads, the reader is referred to excellent articles by Pearson [1], Hill [2], Beatty [4], and Koiter [5].

The following section will be devoted to the investigation of stability by modal analysis which is commonly used in aeroelasticity where follower type forces occur frequently. This kind of analysis, however, requires that the damping stresses $\tau_{ij}^{(d)}$ be defined explicitly. Following Coleman and Noll [16], it will be assumed that $\tau_{ij}^{(d)}$ depends linearly on the velocity gradient. In addition, it will be assumed that the damping stresses are very small and are defined by

$$\tau_{ij}^{(d)} = \epsilon\gamma'_{ijkl}\dot{v}_{k,l} \tag{4.9}$$

where the tensor γ'_{ijkl} is symmetric with respect to exchange of i and j, k and l, and ij and kl in analogy with γ_{ijkl} (Eq. (2.5c)), and ϵ is a magnitude parameter which is assumed to be very small; $\epsilon \ll 1$. Moreover, to avoid lengthy equations, the examples of the tangential loads, normal tractions, and dead loads will be used for the detailed analysis. The extension of the results to other cases of loadings should of course entail no difficulties.

5. Modal analysis. Substitution of relation (4.9) into Eqs. (4.1) results in a system of linear, autonomous partial differential equations which admits a solution of the form

$$v_r = \chi_r e^{i\lambda t}, \qquad i = (-1)^{1/2}, \qquad r = 1, 2, 3. \tag{5.1}$$

Such a solution is bounded if the eigenvalue λ possesses a nonzero real part and a nonnegative imaginary part. In this section the implications of such requirements will be investigated for sufficiently small damping stresses ($\epsilon \ll 1$). In this case, λ and χ_r may be written as

$$\begin{aligned} \lambda &= \omega + i\epsilon\beta + O(\epsilon^2), \\ \chi_r &= \varphi_r + i\epsilon\psi_r + O(\epsilon^2), \qquad r = 1, 2, 3. \end{aligned} \tag{5.2}$$

Substitution of equations (5.2) and (3.6f) into equations (4.1) now yields

$$(\gamma_{rjkl}\varphi_{k,l})_{,r} + \sigma_{rk}\varphi_{j,kr} + \rho\omega^2\varphi_j = 0; \quad \text{in } V \tag{5.3a}$$

$$(\gamma_{rjkl}\psi_{k,l})_{,r} + \sigma_{rk}\psi_{j,kr} + \rho\omega^2\psi_j + 2\rho\omega\beta\varphi_j + \omega(\gamma'_{rjkl}\varphi_{k,l})_{,r} = 0; \quad \text{in } V \tag{5.3b}$$

$$(\gamma_{rjkl}\varphi_{k,l} + \sigma_{rj}e^{(\mu\varphi)})\nu_r = 0; \quad \text{on } S^T \tag{5.3c}$$

$$(\gamma_{rjkl}\psi_{k,l} + \sigma_{rj}e^{(\mu\psi)} + \omega\gamma'_{rjkl}\varphi_{k,l})\nu_r = 0; \quad \text{on } S^T \tag{5.3d}$$

where terms of $O(\epsilon^2)$ and higher are neglected, $e^{(\mu\varphi)} = \mu_i\varphi_{i,j}\mu_j$, and $e^{(\mu\psi)} = \mu_i\psi_{i,j}\mu_j$.

An explicit expression for β can be obtained by multiplying both sides of (5.3a) by ψ_j and (5.3b) by φ_j, integrating over V, and then subtracting one of the resulting equations from the other to arrive at

$$\beta = \frac{1}{2}\left\{\int_V [\omega\gamma'_{rjkl}\varphi_{j,r}\varphi_{k,l} - \sigma_{rk}(\psi_{j,kr}\varphi_j - \varphi_{j,kr}\psi_j)]\, dv + \int_S \sigma_{rk}(e^{(\mu\psi)}\varphi_r - e^{(\mu\varphi)}\psi_r)\nu_k\, dS\right\} \Big/ \left\{\omega\int_V \rho\varphi_j\varphi_j\, dv\right\} \tag{5.4}$$

where in addition to the Gauss theorem, the boundary conditions (5.3c, d) have been used. Similarly, multiplying both sides of equations (5.3a, c) by φ_j, integrating over V and S respectively, and then subtracting one of the resulting equations from the other yields

$$\omega^2 = \frac{\left\{\int_V [\gamma_{rjkl}\varphi_{j,r}\varphi_{k,l} - \sigma_{rk}\varphi_{j,kr}\varphi_j]\, dv + \int_S \sigma_{rk}e^{(\mu\varphi)}\varphi_r\nu_k\, dS\right\}}{\left\{\int_V \rho\varphi_j\varphi_j\, dv\right\}}. \tag{5.5}$$

For stability, both β and ω^2 must be real and positive. Since the volume integral in the denominator of Eqs. (5.4) and (5.5) is positive-definite, for stability the following inequalities must hold:

$$\int_V [\gamma_{rjkl}\varphi_{j,r}\varphi_{k,l} - \sigma_{rk}\varphi_{j,kr}\varphi_j]\, dv + \int_S \sigma_{rk} e^{(\mu\varphi)}\varphi_r \nu_k \, dS > 0, \tag{5.6}$$

$$\int_V [\omega\gamma'_{rjkl}\varphi_{j,r}\varphi_{k,l} - \sigma_{rk}(\psi_{j,kr}\varphi_j - \varphi_{j,kr}\psi_j)]\, dv + \int_S \sigma_{rk}(e^{(\mu\psi)}\varphi_r - e^{(\mu\varphi)}\psi_r)\nu_k \, dS > 0 \tag{5.7}$$

which constitute necessary and sufficient conditions for asymptotic stability of configuration C for the considered loading condition. Similar results can readily be obtained for other loading conditions that were discussed in Sec. 3. Note that, in inequality (5.6), φ_i cannot be considered as an arbitrary but compatible, virtual displacement field, since it must satisfy the boundary value problem defined by Eqs. (5.3a, c). This is in contrast to the case of dead loading where a condition similar to (5.7) is trivially satisfied and (5.6) is replaced by

$$\int_V [\gamma_{rjkl}\varphi_{j,r}\varphi_{k,l} + \sigma_{ik}\varphi_{j,i}\varphi_{j,k}]\, dv > 0 \tag{5.8}$$

in which φ_i may be viewed as a virtual displacement field. Similarly, for pressure loading (normal tractions) the stability criterion becomes

$$\int_V [\gamma_{rjkl}\varphi_{j,r}\varphi_{k,l} + \sigma_{ik}\varphi_{j,i}\varphi_{j,k} - p(\varphi_{i,j}\varphi_{j,i} - \varphi_{i,i}\varphi_{j,j})]\, dv > 0 \tag{5.9}$$

which was also considered by Pearson [1] and Masur [18], but in somewhat different forms.

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Appendix. When the body B is in the stressed configuration C, the Lagrangian stress tensor S_{lj} (referred to C_0) is, Eq. (2.4),

$$S_{lj} = \frac{\partial x_j}{\partial a_k} \frac{\partial E}{\partial \eta_{lk}}. \tag{A.1}$$

Denoting the small change in S_{lj}, due to deformation of the body from configuration C to C_t^*, by S'_{lj}, to the first order of accuracy in $\partial v_j/\partial a_k$ and $\partial v_j/\partial x_k$, Eq. (A.1) yields

$$S'_{lj} = \frac{\partial v_j}{\partial a_k} \frac{\partial E}{\partial \eta_{lk}} + \frac{\partial x_j}{\partial a_s} \frac{\partial^2 E}{\partial \eta_{ls}\, \partial \eta_{pq}} \eta'_{pq} \tag{A.2}$$

where $\eta'_{pq} = \frac{1}{2}[\partial x_r/\partial a_p\, \partial v_r/\partial a_q + \partial x_r/\partial a_q\, \partial v_r/\partial a_p]$. Multiplying both sides of Eq. (A.2) by $(\rho/\rho_0)(\partial x_i/\partial a_l)$ now results in the following equation:

$$\frac{\rho}{\rho_0} \frac{\partial x_i}{\partial a_l} S'_{lj} = \sigma_{ik} \frac{\partial v_j}{\partial x_k} + \gamma_{ijkl} e_{kl} \tag{A.3}$$

where

$$\sigma_{ik} \frac{\partial v_j}{\partial x_k} = \left(\frac{\rho}{\rho_0} \frac{\partial x_i}{\partial a_l} \frac{\partial x_k}{\partial a_m} \frac{\partial E}{\partial \eta_{lm}}\right) \frac{\partial v_j}{\partial x_k} \quad \text{by Eq. (2.4)},$$

$$\frac{\rho}{\rho_0} \frac{\partial x_i}{\partial a_p} \frac{\partial x_j}{\partial a_q} \frac{\partial^2 E}{\partial \eta_{pq}\, \partial \eta_{rs}} \eta'_{rs} = \frac{\rho}{\rho_0} \frac{\partial x_i}{\partial a_p} \frac{\partial x_j}{\partial a_q} \frac{\partial^2 E}{\partial \eta_{pq}\, \partial \eta_{rs}} \frac{\partial x_k}{\partial a_r} \frac{\partial x_l}{\partial a_s} e_{kl} = \gamma_{ijkl} e_{kl},$$

and e_{kl} is defined by Eq. (2.5a). Now, noting that the left side of Eq. (A.3) defines the Lagrangian stress tensor referred to the stressed configuration C, (A.3) may be written as (2.5b) by adding the "damping stress" $\tau_{ij}^{(d)}$ to its right side.

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