CONTROLLABLE VISCOMETRIC FLOWS*

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1. Introduction. In a viscometric flow, each infinitesimal fluid element is in a state of steady simple shearing motion. These flows are of particular interest in the study of incompressible viscoelastic fluids because of their remarkable simplicity. However complex the general behavior of such a fluid may be, it is governed in viscometric flow by the same relation between stress and strain-rate that holds for steady parallel shearing. This relation involves only three functions of the shear rate, the viscometric functions, which represent the shearing stress and two independent normal stress differences.

In spite of the fact that a good many examples of viscometric flow are known, there presently exists no global description of all such motions, or at least none which is known to be generally valid. It would be desirable to have such a global definition. In the present paper, however, we take only a first step in this direction, by determining all of the viscometric flows within a special category which includes all previously known exact solutions.

Most of the viscometric flows which have been discussed in the literature fall into a category which we call *partially controllable* flows. The geometry of these motions is such that the normal stresses can be equilibrated by a hydrostatic pressure, no matter what forms the normal stress functions may take. In other words, the normal stress functions do not influence the velocity field in a partially controllable flow, although the viscosity function does. Such motions include Poiseuille and Couette flows, and a general class of rectilinear and helical motions between coaxial circular cylinders (Rivlin [1];[1] see also the books by Truesdell and Noll [2] and Coleman, Markovitz, and Noll [3]). Rectilinear motions between parallel walls also belong to this class (Coleman and Noll [4]). Admitting the fiction of fluids with zero density in order to allow low Reynolds number approximations into the partially controllable category, flow in circles between rotating disks is also partially controllable (Rivlin [1]).

We reserve the name *controllable* for a velocity field which satisfies the equations of motion no matter what the density and the three viscometric functions may be. Only one controllable flow is cited in the literature, steady simple shearing motion itself. Indeed, controllability is such a severely restrictive requirement that Truesdell and Noll [2, p. 441] have conjectured that "the only velocity fields that can be produced in every incompressible simple fluid by the action of surface tractions and a pressure field alone are homogeneous." Although the examples given in Secs. 9 and 11 show that this is not the case, it is likely that there cannot be many such flows.

In the present paper we determine all steady viscometric velocity fields which are

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[1]Numbers in square brackets indicate references at the end of the paper.

controllable or partially controllable, and for which the shear rate is not constant throughout the region of flow.

The restriction to time-independent velocity fields would appear to be no restriction at all. It seems natural to assume that if a flow is globally viscometric, it must be steady with respect to some reference frame (although not necessarily an inertial frame). However, I have not been able to prove this.

The restriction to flows with nonuniform shear rate is a matter of mathematical convenience, and constitutes a real gap in the completeness of the analysis. It should not be supposed that a motion with uniform shear rate is necessarily a homogeneous motion. Indeed, as an accidental by-product of our analysis, we have found a class of inhomogeneous motions with constant shear rate (Secs. 9 and 12).

In Sec. 2 we briefly review the kinematics of viscometric flow, and put the definition of steady viscometric flow into a form which will be useful in the analysis to follow. Controllability is explained in Sec. 3. In Sec. 4 we derive the geometric consequences of partial controllability with nonuniform shear rate. We show in particular that surfaces of constant shear rate must be parallel planes or coaxial circular cylinders, and that the shear axes must form an orthogonal system of curves, composed of geodesics on these surfaces and the straight lines in the normal direction. In Secs. 5 to 7 we use these restrictions, together with the kinematic constraints, to determine all admissible forms of the velocity field. Finally, we list the five resulting families of flows in Secs. 8 to 12, and examine them to discover the completely controllable cases.

In all of the flows found, the fluid can be divided into material surfaces which slide over one another as rigid bodies. The five partially controllable families and their associated rigid laminae are the following: 1. Rectilinear motions of parallel planes, generally in skew directions but including uniaxial motion (Sec. 8). 2. Axial translation and rotation of nested circular cylinders (Sec. 9). 3. Screw motions of helicoidal surfaces, including rotating parallel planes as a degenerate case (Sec. 10). 4. Motion of fanned planes parallel to their line of intersection (Sec. 11). 5. Parallel planes with parabolic trajectories (Sec. 12). This final family, which was obtained accidentally, consists entirely of motions with uniform shear rate. There are accordingly only four families for which the shear rate is not constant.

Family 4 is the only completely controllable case with nonuniform shear rate. However, Family 5 is also completely controllable, and there is a special case of Family 2 which has constant shear rate and is completely controllable.

The flows of Family 3 are partially controllable only for zero density. However, if inertia is neglected, they satisfy the momentum equations not only for every choice of the normal stress functions but also for every choice of the viscosity function.

2. Kinematics. Fluid motions can be described, redundantly, by specifying the function $\mathbf{p}(s, t, \mathbf{x})$, which represents the position at time s of the particle which is at $\mathbf{x}$ at time t. According to its definition, the position function must satisfy the identity

$$\mathbf{p}[s, t, \mathbf{p}(t, u, \mathbf{x})] = \mathbf{p}(s, u, \mathbf{x}). \tag{2.1}$$

Satisfaction of this identity is assured if $\mathbf{p}(s, t, \mathbf{x})$ is generated from $\mathbf{p}(s, 0, \mathbf{x})$ and its inverse, $\mathbf{p}(0, t, \mathbf{x})$, by the rule

$$\mathbf{p}(s, t, \mathbf{x}) = \mathbf{p}[s, 0, \mathbf{p}(0, t, \mathbf{x})]. \tag{2.2}$$

Let $\mathbf{F}(s, t, \mathbf{x})$ be the relative deformation gradient defined by

$$F_{ij}(s, t, \mathbf{x}) = p_{i,j}(s, t, \mathbf{x}). \tag{2.3}$$

With restriction to performable deformations, det $\mathbf{F} > 0$. We will be concerned exclusively with incompressible materials, for which det $\mathbf{F} = 1$ in any admissible motion. In order for a function $\mathbf{F}(s, t, \mathbf{x})$ to be a deformation gradient, it must satisfy the chain rule which follows from (2.1):

$$\mathbf{F}[s, t, \mathbf{p}(t, u, \mathbf{x})]\mathbf{F}(t, u, \mathbf{x}) = \mathbf{F}(s, u, \mathbf{x}). \tag{2.4}$$

Since $\mathbf{F}(t, t, \mathbf{x}) = \mathbf{I}$, it follows from the chain rule that

$$\mathbf{F}^{-1}[s, t, \mathbf{p}(t, s, \mathbf{x})] = \mathbf{F}(t, s, \mathbf{x}). \tag{2.5}$$

$\mathbf{F}(s, t, \mathbf{x})$ can be generated from $\mathbf{F}(s, 0, \mathbf{x})$ and its inverse by the rule

$$\mathbf{F}(s, t, \mathbf{x}) = \mathbf{F}[s, 0, \mathbf{p}(0, t, \mathbf{x})]\mathbf{F}(0, t, \mathbf{x}). \tag{2.6}$$

A deformation gradient $\mathbf{F}(s, t, \mathbf{x})$ is *viscometric* if it is of the form (Truesdell and Noll [2])

$$\mathbf{F}(s, t, \mathbf{x}) = \mathbf{R}(s, t, \mathbf{x})[\mathbf{I} + (s - t)\gamma(\mathbf{x}, t)\mathbf{a}(\mathbf{x}, t)\mathbf{b}(\mathbf{x}, t)]. \tag{2.7}$$

Here $\mathbf{a}$ and $\mathbf{b}$ are vector fields from an orthonormal system $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, which we call the *shear axes*, and γ is the *shear rate*. We use the dyadic notation $\mathbf{ab}$ for the tensor with cartesian components $a_i b_j$. $\mathbf{R}$ is a rotation: $\mathbf{RR}^T = \mathbf{I}$, det $\mathbf{R} = 1$. It can be shown that the preceding definition of viscometric flow is equivalent to Ericksen's [5] original definition in terms of the strain history.

In all known examples of viscometric flow, the fluid is divided into rigid surfaces which are in relative motion. The vector $\mathbf{b}(\mathbf{x}, t)$ is normal to the rigid layer at $\mathbf{x}$, at time t. The vector $\mathbf{a}(\mathbf{x}, t)$ is in the direction of the relative sliding motion.

The function (2.7) is a deformation gradient only if it satisfies the identity (2.4). This requirement restricts the manner in which $\mathbf{R}$, γ, $\mathbf{a}$, and $\mathbf{b}$ can vary in the course of time at a given particle. To determine these restrictions, we generate $\mathbf{F}$ from $\mathbf{F}(s, 0, \mathbf{x})$ and its inverse, $\mathbf{F}(0, t, \mathbf{x})$, by the rule (2.6). For abbreviation we introduce the notation

$$\mathbf{R}(t, 0, \mathbf{x}) = \mathbf{R}(t, \mathbf{x}), \qquad \gamma(\mathbf{x}, 0) = \gamma(\mathbf{x}), \qquad \mathbf{a}(\mathbf{x}, 0) = \mathbf{a}(\mathbf{x}), \quad \text{etc.} \tag{2.8}$$

We also write $\mathbf{y} = \mathbf{p}(t, 0, \mathbf{x})$. Then

$$\mathbf{F}(t, 0, \mathbf{x}) = \mathbf{R}(t, \mathbf{x})[\mathbf{I} + t\gamma(\mathbf{x})\mathbf{a}(\mathbf{x})\mathbf{b}(\mathbf{x})], \tag{2.9}$$

and it follows that

$$\mathbf{F}(0, t, \mathbf{y}) = \mathbf{F}^{-1}(t, 0, \mathbf{x}) = [\mathbf{I} - t\gamma(\mathbf{x})\mathbf{a}(\mathbf{x})\mathbf{b}(\mathbf{x})]\mathbf{R}^T(t, \mathbf{x}). \tag{2.10}$$

Hence

$$\begin{aligned}\mathbf{F}(s, t, \mathbf{y}) &= \mathbf{F}(s, 0, \mathbf{x})\mathbf{F}(0, t, \mathbf{y}) \\ &= \mathbf{R}(s, \mathbf{x})[\mathbf{I} + (s - t)\gamma(\mathbf{x})\mathbf{a}(\mathbf{x})\mathbf{b}(\mathbf{x})]\mathbf{R}^T(t, \mathbf{x}).\end{aligned} \tag{2.11}$$

By comparison with (2.7), we obtain

$$\mathbf{R}(s, t, \mathbf{y}) = \mathbf{R}(s, \mathbf{x})\mathbf{R}^T(t, \mathbf{x}), \tag{2.12}$$

$$\gamma(\mathbf{y}, t) = \gamma(\mathbf{x}), \tag{2.13}$$

$$\mathbf{a}(\mathbf{y}, t) = \mathbf{R}(t, \mathbf{x})\mathbf{a}(\mathbf{x}), \tag{2.14}$$

$$\mathbf{b}(\mathbf{y}, t) = \mathbf{R}(t, \mathbf{x})\mathbf{b}(\mathbf{x}), \tag{2.15}$$

and, consistent with (2.14) and (2.15) and the orthonormality of **a**, **b**, and **c**, we may take

$$\mathbf{c}(\mathbf{y}, t) = \mathbf{R}(t, \mathbf{x})\mathbf{c}(\mathbf{x}). \tag{2.16}$$

Recalling that $\mathbf{y} = \mathbf{p}(t, 0, \mathbf{x})$, we see that (2.13) states that the shear rate γ must be constant in time at a given particle. According to (2.14) to (2.16), the shear axes at a given particle undergo the rotation $\mathbf{R}(t, \mathbf{x})$ in the course of time. In a steady motion, for which particle trajectories are streamlines, these relations restrict the manner in which γ, **a**, **b**, and **c** can vary in space.

Although it seems to be a step backward, we will use time-differentiated forms of the relations (2.9) and (2.13) to (2.16). We note first that since **R** is orthogonal and $\mathbf{R}(0, \mathbf{x}) = \mathbf{I}$ (which follows from (2.9) with $\mathbf{F}(0, 0, \mathbf{x}) = \mathbf{I}$), its derivative at time zero is antisymmetric:

$$(D/Dt)\mathbf{R}(t, \mathbf{x})\,|_{t=0} = \mathbf{\Omega}(\mathbf{x}) = -\mathbf{\Omega}^T(\mathbf{x}). \tag{2.17}$$

Hence, on differentiating (2.14) to (2.16) and then setting $t = 0$, we obtain

$$D\mathbf{a}/Dt = \mathbf{\Omega}\mathbf{a}, \qquad D\mathbf{b}/Dt = \mathbf{\Omega}\mathbf{b}, \qquad D\mathbf{c}/Dt = \mathbf{\Omega}\mathbf{c}. \tag{2.18}$$

Since **a**, **b**, and **c** are orthogonal unit vectors, it follows from (2.18) that

$$\mathbf{\Omega} = (D\mathbf{a}/Dt)\mathbf{a} + (D\mathbf{b}/Dt)\mathbf{b} + (D\mathbf{c}/Dt)\mathbf{c}. \tag{2.19}$$

From (2.13) we obtain $D\gamma/Dt = 0$. By differentiating (2.9) and then setting $t = 0$, we obtain the following expression for the gradient of the velocity **u**:

$$u_{i,j} = \Omega_{ij} + \gamma a_i b_j\ . \tag{2.20}$$

With $\mathbf{\Omega}$ given by (2.19), (2.20) shows how the shear axes must be related to the velocity gradient in a viscometric flow.

In a steady flow, $\gamma(\mathbf{x})$, $\mathbf{a}(\mathbf{x})$, $\mathbf{b}(\mathbf{x})$, and $\mathbf{c}(\mathbf{x})$ are the shear rate and shear axes at the place **x** not only at time zero but at all times. The time derivative D/Dt is, of course, $\mathbf{u}(\mathbf{x})\cdot\nabla$ in a steady flow.

It is convenient to write (2.20) in terms of the intrinsic velocity components

$$u = \mathbf{u}\cdot\mathbf{a}, \qquad v = \mathbf{u}\cdot\mathbf{b}, \qquad \text{and} \qquad w = \mathbf{u}\cdot\mathbf{c}. \tag{2.21}$$

By using (2.19) and (2.20), and recalling that $\mathbf{\Omega}$ is antisymmetric, in the case of steady motion we obtain

$$\nabla u = \mathbf{u} \times (\nabla \times \mathbf{a}) + \gamma\mathbf{b}, \tag{2.22}$$

$$\nabla v = \mathbf{u} \times (\nabla \times \mathbf{b}), \tag{2.23}$$

and

$$\nabla w = \mathbf{u} \times (\nabla \times \mathbf{c}). \tag{2.24}$$

These relations, along with

$$\mathbf{u}\cdot\nabla\gamma = 0, \tag{2.25}$$

will be our principal tools in the kinematic analysis of steady viscometric flows.

3. Controllability. The stress in any incompressible simple fluid in viscometric flow is of the form

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{S}, \tag{3.1}$$

where p is the reaction pressure and $\mathbf{S}$ is of the form [6], [7]

$$\mathbf{S} = \gamma\eta(\gamma^2)(\mathbf{ab} + \mathbf{ba}) - \gamma^2\nu(\gamma^2)\mathbf{bb} - \gamma^2\phi(\gamma^2)\mathbf{cc}. \tag{3.2}$$

Here γ, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are the shear rate and shear axes at the time and place at which $\boldsymbol{\sigma}$ is evaluated.

The momentum equations for steady flow are

$$\rho u_j u_{i,j} = -p_{,i} + S_{ij,j} \,. \tag{3.3}$$

A conservative body force field can, of course, be absorbed in the pressure term. With (3.2), we can write (3.3) as

$$p_{,i} = -\rho u_j u_{i,j} + [\gamma\eta(a_i b_j + b_i a_j)]_{,j} - (\gamma^2 \nu b_i b_j)_{,j} - (\gamma^2 \phi c_i c_j)_{,j} \,. \tag{3.4}$$

Given a viscometric velocity field $\mathbf{u}$ and the derived quantities γ, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, there exists a pressure p satisfying (3.4) only if the curl of the right-hand side of (3.4) is zero.

We seek velocity fields which are such that the right-hand side of (3.4) is irrotational for *every* choice of the normal stress functions ν and ϕ and for some choice of the density ρ and viscosity function η. We call such flows *partially controllable.* If a specific velocity field is such that (3.4) is irrotational for every choice of ρ and η as well, we say that this field is *completely* controllable, or simple *controllable.*

If (3.4) is irrotational for every choice of ν and φ (for a given velocity field), then the terms involving ν and ϕ must be irrotational separately. Hence in particular, partial controllability requires that the following expression be symmetric on i and k:

$$(\gamma^2\nu)''[b_i\gamma_{,k}b_j\gamma_{,j}] + (\gamma^2\nu)'[(b_ib_j\gamma_{,j})_{,k} + (b_ib_j)_{,j}\gamma_{,k}] + \gamma^2\nu[(b_ib_j)_{,jk}]. \tag{3.5}$$

Here a prime denotes differentiation with respect to γ. Further, if (3.5) is to be symmetrie for every form of the function ν, it is necessary that each of the bracketed expressions in (3.5) be symmetric.

By considering the terms involving ϕ in (3.4), we draw similar conclusions regarding the expressions obtained by replacing $\mathbf{b}$ by $\mathbf{c}$ in (3.5). Finally, by using the identity

$$\delta_{ij} = a_i a_j + b_i b_j + c_i c_j \,, \tag{3.6}$$

we find that the same conclusions must hold for the similar expressions involving $\mathbf{a}$. Accordingly, partial controllability requires symmetry of each of the following expressions:

$$(a_j\gamma_{,j})a_i\gamma_{,k} \,, \qquad (b_j\gamma_{,j})b_i\gamma_{,k} \,, \qquad (c_j\gamma_{,j})c_i\gamma_{,k} \,, \tag{3.7}$$

$$(a_ia_j\gamma_{,j})_{,k} + (a_ia_j)_{,j}\gamma_{,k} \,, \qquad (b_ib_j\gamma_{,j})_{,k} + (b_ib_j)_{,j}\gamma_{,k} \,, \qquad (c_ic_j\gamma_{,j})_{,k} + (c_ic_j)_{,j}\gamma_{,k} \,, \tag{3.8}$$

$$(a_ia_j)_{,jk} \,, \qquad (b_ib_j)_{,jk} \,, \qquad (c_ic_j)_{,jk} \,. \tag{3.9}$$

If $\nabla\gamma$ is identically zero, the quantities (3.7) and (3.8) are trivially symmetric, and partial controllability requires only the symmetry of the quantities (3.9). We leave aside this case, and restrict our attention to cases in which $\nabla\gamma$ is not zero (except possibly at isolated points, lines, or surfaces).

We will consider the further requirements imposed by complete controllability only after the partially controllable fields have been determined.

4. Analysis of the partial controllability conditions. Analysis of the implications of symmetry of the quantities (3.7) to (3.9) is greatly simplified by the fact that Ericksen [8] has analysed a similar set of equations in his work on controllable elastic deformations. (See also [9].) Ericksen's work provides guidelines in all of the more difficult parts of the analysis.

Symmetry of the quantities (3.7) implies that each of the vectors **a**, **b**, and **c** is either parallel or perpendicular to $\nabla\gamma$. For example, either $\mathbf{a}\cdot\nabla\gamma = 0$ or $a_i\gamma_{,k} = a_k\gamma_{,i}$, and the latter implies that $\mathbf{a} \times \nabla\gamma = 0$.

Since at most one of the orthonormal triad **a**, **b**, **c** can be parallel to $\nabla\gamma$ and they cannot all be perpendicular to it, one is parallel and the other two are perpendicular. For definiteness, let us take **c** parallel to $\nabla\gamma$, with the understanding that all of our conclusions are subject to permutations of **a**, **b**, and **c**. Then,

$$\mathbf{a}\cdot\nabla\gamma = \mathbf{b}\cdot\nabla\gamma = 0, \qquad \nabla\gamma = f\mathbf{c}. \tag{4.1}$$

With (4.1), symmetry of the quantities (3.8) implies that $(a_ia_j)_{,j}$, $(b_ib_j)_{,j}$, and $(c_ic_j)_{,j}$ are all parallel to $\gamma_{,i}$. For **a** and **b** the conclusion is immediate. For **c**, we notice that the first term in (3.8c) is

$$(c_ic_j\gamma_{,j})_{,k} = (f^{-1}\gamma_{,i}c_jfc_j)_{,k} = \gamma_{,ik}\,. \tag{4.2}$$

Since this is symmetric, the second term in (3.8c) must also be symmetric.

Symmetry of the quantities (3.9) requires that $(a_ia_j)_{,j}$, $(b_ib_j)_{,j}$, and $(c_ic_j)_{,j}$ be gradients. Since they are parallel to $\gamma_{,i}$, they must be gradients of functions γ. Hence,

$$(a_ia_j)_{,j} = A(\gamma)\gamma_{,i}\,, \qquad (b_ib_j)_{,j} = B(\gamma)\gamma_{,i}\,, \qquad (c_ic_j)_{,j} = C(\gamma)\gamma_{,i}\,. \tag{4.3}$$

The **c**-lines (trajectories of the field **c**) are straight lines. To prove this, we first use (4.1c) and (4.3c) to obtain

$$\mathbf{c}(\nabla\cdot\mathbf{c}) + (\mathbf{c}\cdot\nabla)\mathbf{c} = C(\gamma)f\mathbf{c}. \tag{4.4}$$

Because **c** is a unit vector, $(\mathbf{c}\cdot\nabla)\mathbf{c}$ is orthogonal to **c**. Hence, on decomposing (4.4) into orthogonal components, we obtain

$$\nabla\cdot\mathbf{c} = C(\gamma)f \qquad \text{and} \qquad (\mathbf{c}\cdot\nabla)\mathbf{c} = 0. \tag{4.5}$$

The latter relation implies that the **c**-lines are straight, as we asserted.

Because the straight **c**-lines are normal to the γ-surfaces (surfaces on which γ is constant), it follows that the γ-surfaces are parallel (curved) surfaces. This, in turn, implies that the magnitude of $\nabla\gamma$ is constant over each γ-surface. Hence, in (4.1), $f = f(\gamma)$.

By decomposing the relations (4.3) into orthogonal parts, and using $\gamma_{,i} = f(\gamma)c_i$, we obtain in particular

$$(\mathbf{a}\cdot\nabla)\mathbf{a} = k_{aa}(\gamma)\mathbf{c} \qquad \text{and} \qquad (\mathbf{b}\cdot\nabla)\mathbf{b} = k_{bb}(\gamma)\mathbf{c}, \tag{4.6}$$

with a change in notation for the scalar functions of γ.

Let us introduce the notation mnp for the quantities $\mathbf{m}\cdot(\mathbf{n}\cdot\nabla)\mathbf{p}$, where **m**, **n**, and **p** are vectors from the set **a**, **b**, **c**. From the orthonormality of this set it follows that $pmn = -nmp$, and in particular $pnp = 0$. Since the field **c** is orthogonal to a family of surfaces, it follows that $\mathbf{c}\cdot\nabla \times \mathbf{c} = 0$ (see (4.1)). Hence, $cab = cba$. By using these relations and (4.6), and introducing the notation $k_{ab} = cba$, we obtain

$$(\mathbf{a}\cdot\nabla)\mathbf{b} = k_{ab}\mathbf{c}, \qquad (\mathbf{b}\cdot\nabla)\mathbf{a} = k_{ab}\mathbf{c}. \tag{4.7}$$

The relations (4.6) and (4.7) specify the geometry of the **a**-lines and **b**-lines on each γ-surface. We next show that k_{ab} is also constant over each γ-surface. To prove this, we use the identity

$$(\mathbf{k}\cdot\nabla)mnp - (\mathbf{n}\cdot\nabla)mkp = [(\mathbf{k}\cdot\nabla)\mathbf{m}]\cdot[(\mathbf{n}\cdot\nabla)\mathbf{p}] - [(\mathbf{n}\cdot\nabla)\mathbf{m}]\cdot[(\mathbf{k}\cdot\nabla)\mathbf{p}] + \mathbf{m}\cdot[(\mathbf{k}\cdot\nabla)\mathbf{n} - (\mathbf{n}\cdot\nabla)\mathbf{k}]\cdot\nabla\mathbf{p}. \tag{4.8}$$

By setting $\mathbf{k} = \mathbf{m} = \mathbf{a}$ and $\mathbf{n} = \mathbf{p} = \mathbf{b}$, and using (4.6) and (4.7), we obtain

$$k_{aa}(\gamma)k_{bb}(\gamma) - k_{ab}^2 = 0. \tag{4.9}$$

Hence $k_{ab} = k_{ab}(\gamma)$, as asserted.

From (4.9) we also observe that the determinant of the matrix of curvatures **k** is zero. Consequently, one of the principal curvatures (the principal values of this matrix) is zero. Indeed, a stronger result can be obtained. Let $\mathbf{d} = k_{ab}(\gamma)\mathbf{a} - k_{aa}(\gamma)\mathbf{b}$, or, if k_{ab} and k_{aa} are both zero, $\mathbf{d} = \mathbf{a}$. Then from (4.6), (4.7), and (4.9) we find that

$$(\mathbf{a}\cdot\nabla)\mathbf{d} = (\mathbf{b}\cdot\nabla)\mathbf{d} = 0. \tag{4.10}$$

Thus, **d** is constant over each γ-surface. These surfaces are therefore general cylinders with generator **d**.

Because the curvatures **k** are constant over each γ-surface, the principal curvature in the direction orthogonal to **d** is constant. Hence, each γ-surface is a plane or a circular cylinder.

Recalling that the γ-surfaces form a parallel family, it follows that they are parallel planes or coaxial circular cylinders.

The **a**-lines and **b**-lines form an orthogonal mesh on each γ-surface. The geodesic curvatures of these curves, *baa* and *abb*, are zero according to (4.6). Hence these curves are geodesics. If the γ-surfaces are parallel planes, the **a**-lines and **b**-lines on each plane are an orthogonal system of straight lines. If the γ-surfaces are coaxial circular cylinders, the **a**-lines on a given cylinder are parallel helices, and the **b**-lines are an orthogonal family of helices. As a degenerate case, the two families may be the generators and the circumferential circles.

These conclusions are subject to permutations of **a**, **b**, and **c**, since we arbitrarily singled out **c** as the field parallel to $\nabla\gamma$.

5. Kinematic constraints. Planar γ-surfaces. We now use the results derived from partial controllability in the kinematic constraints (2.22) to (2.25).

We first show that the field **a** cannot be parallel to $\nabla\gamma$. Assuming that it is, it follows from $\mathbf{u}\cdot\nabla\gamma = 0$ that $u = \mathbf{a}\cdot\mathbf{u} = 0$. Furthermore, $\nabla \times \mathbf{a} = 0$ since the surface-normal field is either constant, in the case of plane γ-surfaces, or radial, in the cylindrical case. Hence, from (2.22) we obtain $\gamma = 0$. Since viscometric flows with $\gamma = 0$ are rigid motions, this case is trivial.

In the remainder of this section we consider cases in which the γ-surfaces are parallel planes. Cylindrical γ-surfaces are considered in Secs. 6 and 7.

Let **i**, **j**, and **k** be an orthonormal system of constant vectors, with **k** parallel to $\nabla\gamma$. Let x, y, z be the associated cartesian coordinates, so that $\gamma = \gamma(z)$. Since the field **a** cannot be parallel to $\nabla\gamma$, there are two cases to consider, $\mathbf{b} = \mathbf{k}$ and $\mathbf{c} = \mathbf{k}$.

First take $\mathbf{c} = \mathbf{k}$. Since the **a**-lines and **b**-lines on each surface z = constant form an orthogonal system of straight lines, we have

$$\mathbf{a} = \mathbf{i} \cos \theta(z) + \mathbf{j} \sin \theta(z), \qquad \mathbf{b} = -\mathbf{i} \sin \theta(z) + \mathbf{j} \cos \theta(z). \tag{5.1}$$

Hence,

$$\nabla \times \mathbf{a} = -\theta'(z)\mathbf{a}, \qquad \nabla \times \mathbf{b} = -\theta'(z)\mathbf{b}, \qquad \nabla \times \mathbf{c} = 0. \tag{5.2}$$

Since $\mathbf{u} \cdot \nabla\gamma = 0$, it follows that $\mathbf{u} = u\mathbf{a} + v\mathbf{b}$, i.e. $w = 0$. Then (2.24) is satisfied trivially, and from (2.22) and (2.23) we obtain

$$\nabla u = \theta' v\mathbf{k} + \gamma\mathbf{b}, \qquad \nabla v = -\theta' u\mathbf{k}. \tag{5.3}$$

The latter relation implies that $v = v(z)$ since its gradient is in the z-direction, and we obtain $v'(z) = -\theta'(z)u$. Then if $\theta' \neq 0$, u is also a function of z. In that case, (5.3a) implies that $\gamma = 0$. If $\theta' = 0$, we take $\theta = 0$ without loss of generality, so that $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$. Then (5.3b) yields $v = v_0$, constant. From (5.3a) we find that $u = u(y)$ since ∇u is in the $\mathbf{j}$-direction. But then $u'(y) = \gamma(z)$, whence $\gamma = \gamma_0$, constant, and $u = \gamma_0 y + u_0$. Thus,

$$\mathbf{u} = u_0\mathbf{i} + v_0\mathbf{j} + \gamma_0 y\mathbf{i}. \tag{5.4}$$

We have accidentally found a flow with constant shear rate. Because $\nabla\gamma = 0$, the condition $\mathbf{u} \cdot \nabla\gamma = 0$ is still satisfied if we add a constant component $w_0\mathbf{k}$ to (5.4). We discuss this flow in further detail in Section 12.

Now consider the case $\mathbf{b} = \mathbf{k}$, for which

$$\mathbf{c} = \mathbf{i} \cos \theta(z) + \mathbf{j} \sin \theta(z), \qquad \mathbf{a} = -\mathbf{i} \sin \theta(z) + \mathbf{j} \cos \theta(z), \tag{5.5}$$

and $\mathbf{u} = u\mathbf{a} + w\mathbf{c}$. Then (2.23) is satisfied trivially, and (2.22) and (2.24) yield

$$\nabla u = (\gamma - w\theta')\mathbf{k}, \qquad \nabla w = u\theta'\mathbf{k}. \tag{5.6}$$

Hence, u and w are functions of z, and

$$u' = \gamma - w\theta', \qquad w' = u\theta'. \tag{5.7}$$

With (5.5), these relations yield $\mathbf{u}' = \gamma\mathbf{a}$, whence

$$\mathbf{u}(z) = A\mathbf{i} + B\mathbf{j} + \int_{z_0}^{z} \gamma(z')[-\mathbf{i} \sin \theta(z') + \mathbf{j} \cos \theta(z')]\, dz'. \tag{5.8}$$

Flows of this type are considered further in Sec. 8.

6. Cylindrical γ-surfaces, b radial. For the cases in which the γ-surfaces are coaxial circular cylinders, we introduce cylindrical coordinates r, θ, and z such that $\gamma = \gamma(r)$, and the associated unit vectors $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_z$.

One of the three fields $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is radial (not $\mathbf{a}$), and the other two are tangential to the cylinders. Each tangential field has parallel helical trajectories. Hence, one of the tangential fields takes the form

$$\mathbf{f} = \mathbf{i}_\theta \cos \alpha(r) + \mathbf{i}_z \sin \alpha(r), \tag{6.1}$$

while the other, orthogonal to the first, is of the form

$$\mathbf{g} = -\mathbf{i}_\theta \sin \alpha(r) + \mathbf{i}_z \cos \alpha(r). \tag{6.2}$$

From (6.1) and (6.2) we compute

$$\nabla \times \mathbf{f} = \mathbf{i}_z r^{-1} \cos \alpha - \alpha' \mathbf{f}, \qquad \nabla \times \mathbf{g} = -\mathbf{i}_z r^{-1} \sin \alpha - \alpha' \mathbf{g}. \tag{6.3}$$

There are again two cases to consider, $\mathbf{b} = \mathbf{i}_r$ and $\mathbf{c} = \mathbf{i}_r$. We defer the latter case to Section 7. In the present section we consider the case

$$\mathbf{b} = \mathbf{i}_r, \qquad \mathbf{c} = \mathbf{f}, \qquad \mathbf{a} = \mathbf{g}, \qquad \mathbf{u} = u\mathbf{g} + w\mathbf{f}. \tag{6.4}$$

For this case, (2.23) is trivial, and (2.22) and (2.24) yield

$$\nabla u = \mathbf{i}_r[-\alpha' w + r^{-1} \sin \alpha(u \sin \alpha - w \cos \alpha) + \gamma] \tag{6.5}$$

and

$$\nabla w = \mathbf{i}_r[\alpha' u - r^{-1} \cos \alpha(u \sin \alpha - w \cos \alpha)]. \tag{6.6}$$

Since the gradients of u and w are radial, they are both functions of r. Then, with $\partial\mathbf{f}/\partial r = \alpha'\mathbf{g}$ and $\partial\mathbf{g}/\partial r = -\alpha'\mathbf{f}$, it follows from (6.5) and (6.6) that

$$\partial\mathbf{u}/\partial r = \gamma\mathbf{g} + r^{-1}(u \sin \alpha - w \cos \alpha)(\mathbf{g} \sin \alpha - \mathbf{f} \cos \alpha). \tag{6.7}$$

Hence, with $u_\theta = \mathbf{u} \cdot \mathbf{i}_\theta$ and $u_z = \mathbf{u} \cdot \mathbf{i}_z$,

$$\mathbf{i}_\theta u_\theta'(r) + \mathbf{i}_z u_z'(r) = \gamma(r)[-\mathbf{i}_\theta \sin \alpha(r) + \mathbf{i}_z \cos \alpha(r)] + \mathbf{i}_\theta r^{-1} u_\theta(r). \tag{6.8}$$

Then

$$u_z'(r) = \gamma(r) \cos \alpha(r) \tag{6.9}$$

and

$$[r^{-1} u_\theta(r)]' = -r^{-1}\gamma(r) \sin \alpha(r). \tag{6.10}$$

Hence,

$$\mathbf{u} = \mathbf{i}_\theta r\Omega + \mathbf{i}_z U + \int_R^r \gamma(r')[\mathbf{i}_z \cos \alpha(r') - \mathbf{i}_\theta (r/r') \sin \alpha(r')]\, dr'. \tag{6.11}$$

We consider these flows further in Sec. 9.

7. Cylindrical γ-surfaces, c radial. If $\mathbf{c}$ is radial, we have

$$\mathbf{a} = \mathbf{f}, \qquad \mathbf{b} = \mathbf{g}, \qquad \mathbf{c} = \mathbf{i}_r, \qquad \mathbf{u} = u\mathbf{f} + v\mathbf{g}, \tag{7.1}$$

where $\mathbf{f}$ and $\mathbf{g}$ are defined in (6.1) and (6.2). In this case (2.24) is trivial, while (2.22) and (2.23) yield

$$\nabla u = \mathbf{i}_r[\alpha' v + r^{-1} \cos \alpha(u \cos \alpha - v \sin \alpha)] + \gamma\mathbf{g} \tag{7.2}$$

and

$$\nabla v = -\mathbf{i}_r[\alpha' u + r^{-1} \sin \alpha(u \cos \alpha - v \sin \alpha)]. \tag{7.3}$$

If follows from (7.3) that $v = v(r)$, and

$$v'(r) = -[\alpha'(r) + r^{-1} \sin \alpha(r) \cos \alpha(r)]u + r^{-1}v(r) \sin^2 \alpha(r). \tag{7.4}$$

If the coefficient of u in this equation is not zero, then u is also a function of r. In this case (7.2) implies that $\gamma = 0$. Leaving aside this trivial case, it follows that

$$\alpha'(r) = -r^{-1} \sin \alpha(r) \cos \alpha(r) \tag{7.5}$$

and

$$v'(r) = r^{-1}v(r) \sin^2 \alpha(r). \tag{7.6}$$

From (7.2) we obtain in particular, with (6.2),

$$r^{-1}\,\partial u/\partial\theta = -\gamma(r)\sin\alpha(r) \qquad\text{and}\qquad \partial u/\partial z = \gamma(r)\cos\alpha(r), \tag{7.7}$$

whence

$$u = \gamma(r)[z\cos\alpha(r) - r\theta\sin\alpha(r)] + h(r). \tag{7.8}$$

By using this in the radial component of (7.2) and differentiating with respect to z, we obtain

$$(\gamma\cos\alpha)' = r^{-1}\gamma\cos^3\alpha, \tag{7.9}$$

while differentiation with respect to θ gives

$$(r\gamma\sin\alpha)' = \gamma\sin\alpha\cos^2\alpha. \tag{7.10}$$

These two equations are seen with the help of (7.5) to be consistent, and equivalent to the single relation

$$r\gamma' = \gamma(\cos^2\alpha - \sin^2\alpha). \tag{7.11}$$

The radial component of (7.2) then reduces to

$$h' = r^{-1}h\cos^2\alpha + v(\alpha' - r^{-1}\sin\alpha\cos\alpha). \tag{7.12}$$

From (7.5), either $\alpha = \pi/2$ or

$$\alpha(r) = \arctan(R/r). \tag{7.13}$$

In the latter case, (7.6) yields

$$v(r) = Ur/(R^2 + r^2)^{1/2}, \tag{7.14}$$

(7.11) gives

$$\gamma(r) = \tau(R^2r^{-1} + r), \tag{7.15}$$

and (7.12) implies that

$$h(r) = U\sin\alpha(r) + \Omega(R^2 + r^2)^{1/2}. \tag{7.16}$$

By using these results in (7.8) we obtain

$$u = \tau(R^2r^{-1} + r)(z\cos\alpha - r\theta\sin\alpha) + U\sin\alpha + \Omega(R^2 + r^2)^{1/2}, \tag{7.17}$$

with $\alpha(r)$ given by (7.13). Further, (6.1) and (6.2) yield

$$\mathbf{f} = (r\mathbf{i}_\theta + R\mathbf{i}_z)/(R^2 + r^2)^{1/2}, \qquad \mathbf{g} = (-R\mathbf{i}_\theta + r\mathbf{i}_z)/(R^2 + r^2)^{1/2}. \tag{7.18}$$

By using (7.14), (7.17), and (7.18) in (7.1), we obtain

$$\mathbf{u} = r\Omega\mathbf{i}_\theta + U\mathbf{i}_z + \tau(z - R\theta)(r\mathbf{i}_\theta + R\mathbf{i}_z). \tag{7.19}$$

This flow will be discussed in Sec. 10.

In the case $\alpha = \pi/2$, for which $\mathbf{f} = \mathbf{i}_z$ and $\mathbf{g} = -\mathbf{i}_\theta$, (7.6) gives $v = -\Omega r$, (7.11) gives $\gamma = -A/r$, and (7.12) gives $h = U$, constant. Then (7.8) yields $u = A\theta + U$. Thus,

$$\mathbf{u} = r\Omega\mathbf{i}_\theta + U\mathbf{i}_z + A\theta\mathbf{i}_z\,. \tag{7.20}$$

This flow is considered in further detail in Sec. 11.

8. Rectilinear motions between plane walls. In Sec. 5 we found that if the surfaces of constant shear rate are planes z = constant, and the field $\mathbf{b}$ is normal to these planes, then the velocity field must be of the form (5.8). In these flows the particles on each plane z = constant move together as a rigid surface sliding tangentially. In the special cases in which θ is constant, so that all particles can move in the same direction (if $A = B = 0$), these motions represent a combination of plane Poiseuille and Couette flows (Coleman and Noll [4]), i.e. flow between parallel walls in relative motion, with an applied pressure gradient in the direction of relative motion. Application of a pressure gradient oblique or normal to the direction of motion of the walls produces cases with $\theta(z)$ not constant, which have not been discussed previously so far as I know.

We now consider the further restrictions imposed on such flows by the condition that the inertial and viscous terms in the momentum Eq. (3.4) must be equivalent to a pressure gradient. The inertial term vanishes since the velocity of each particle is constant. The normal stress terms are equivalent to the gradient of a scalar function P since the flow satisfies the partial controllability conditions. With $\mathbf{b} = \mathbf{k}$ and $\mathbf{a}$ given by (5.5), the momentum equation then takes the form

$$\nabla(p - P) = (\gamma\eta\mathbf{a})' = (\gamma\eta)'\mathbf{a} - \gamma\eta\theta'\mathbf{c}, \tag{8.1}$$

where a prime denotes differentiation with respect to z. Since the right-hand side has no z-component, $p - P$ must be independent of z. Its gradient, $(\gamma\eta\mathbf{a})'$, is then also independent of z. But since $(\gamma\eta\mathbf{a})'$ can depend only on z, it is therefore a constant, say $g_0\mathbf{i}$ (defining the direction of $\mathbf{i}$). Thus,

$$\gamma\eta\mathbf{a} = g_0\mathbf{i}(z - z_0) + \sigma_0\mathbf{j}, \tag{8.2}$$

and

$$p - P = g_0x + p_0\,. \tag{8.3}$$

By using (5.5) in (8.2), and separating components, we obtain

$$-\gamma\eta \sin\theta = g_0(z - z_0), \qquad \gamma\eta\cos\theta = \sigma_0\,. \tag{8.4}$$

Hence,

$$(\gamma\eta)^2 = \sigma_0^2 + g_0^2(z - z_0)^2 \qquad \text{and} \qquad \tan\theta = -g_0(z - z_0)/\sigma_0\,, \tag{8.5}$$

with $\theta = \pi/2$ in the special case $\sigma_0 = 0$. The velocity field (5.8) then takes the form

$$\mathbf{u}(z) = A\mathbf{i} + B\mathbf{j} + \int_{z_0}^{z} \eta^{-1}(z')[\mathbf{i}g_0(z' - z_0) + \mathbf{j}\sigma_0]\,dz'. \tag{8.6}$$

When the form of the viscosity function $\eta(\gamma^2)$ is specified, $\gamma(z)$ is determined by (8.5a), and the form of η as a function of z can be obtained. The velocity field is then given in terms of the parameters A, B, σ_0, g_0, and z_0 by (8.6). For one specific velocity field to be possible no matter what the form of $\eta(\gamma^2)$ may be, it is necessary that η be a constant function of z, whence γ is constant, and thus, from (8.5a), $g_0 = 0$. Hence, the flow is completely controllable only in the case of rigid motion superposed on steady simple shearing.

9. Flows in tubes and annuli. In Sec. 6 we found that if the surfaces of constant shear rate are coaxial circular cylinders and the field $\mathbf{b}$ is radial, then the velocity field must be of the form (6.11). This velocity field describes a motion in which the particles on each cylinder move together rigidly, rotating about the z-axis and translating along it. These flows include Poiseuille and Couette flows, and the helical motions described by Rivlin [1]. They also include some motions requiring an azimuthal pressure gradient which have not previously been discussed, presumably because they would be difficult to produce experimentally.

When the velocity field (6.11) and the shear axis fields (6.4) are used in the momentum Eqs. (3.4), it is found that the inertial term and the normal stress terms are purely radial, with magnitudes which depend only on r. These terms are accordingly equivalent to the gradient of a function $P(r)$. The momentum equations then reduce to the form

$$\nabla(p - P) = -\mathbf{i}_\theta r^{-2}(r^2\gamma\eta \sin\alpha)' + \mathbf{i}_z r^{-1}(r\gamma\eta \cos\alpha)' \tag{9.1}$$

where a prime denotes differentiation with respect to r. Since the gradient of $p - P$ has no radial component, $p - P$ can depend only on θ and z. Consequently, its gradient must be of the form

$$\nabla(p - P) = -\mathbf{i}_\theta r^{-1}C + \mathbf{i}_z D, \tag{9.2}$$

where C and D are functions of θ and z. However, comparison with (9.1) shows that C and D can depend only on r. Hence, C and D are constants. Consequently, we obtain

$$p - P = -C\theta + Dz + p_0 \tag{9.3}$$

and

$$(r^2\gamma\eta \sin\alpha)' = Cr, \qquad (r\gamma\eta \cos\alpha)' = Dr. \tag{9.4}$$

We note that if $C \neq 0$, the pressure is multiple-valued in a full annular region, so the region of motion must be restricted to a sector.

From (9.4) we obtain

$$2r^2\gamma\eta \sin\alpha = Cr^2 + E \qquad \text{and} \qquad 2r\gamma\eta \cos\alpha = Dr^2 + F, \tag{9.5}$$

whence

$$4(\gamma\eta)^2 = (C + Er^{-2})^2 + (Dr + Fr^{-1})^2 \tag{9.6}$$

and

$$\tan\alpha = (C + Er^{-2})/(Dr + Fr^{-1}). \tag{9.7}$$

Given a specific form of the viscosity function $\eta(\gamma^2)$, (9.6) determines $\gamma(r)$, and the velocity field (6.11) can be evaluated explicitly.

For a specific field of the form (6.11) to be completely controllable, it must be possible to specify the shear rate without reference to the form of $\eta(\gamma^2)$. From (9.6) we see that this is possible only if $\gamma = \gamma_0$, constant, and $D = E = F = 0$. In this case (9.6) becomes $\gamma_0\eta(\gamma_0^2) = C/2$, and (9.5) yields $\alpha = \pi/2$. The pressure gradient is seen from (9.2) to be purely azimuthal. The velocity field (6.11) takes the form

$$\mathbf{u} = \mathbf{i}_\theta r\Omega + \mathbf{i}_z U - \mathbf{i}_\theta \gamma_0 r \ln(r/R). \tag{9.8}$$

We have again been led to a case in which $\nabla\gamma = 0$, in spite of our use of conditions arising from the assumption $\nabla\gamma \neq 0$. Aside from its mathematical interest as a completely controllable inhomogeneous motion, this flow is also of interest as an example of an inhomogeneous motion at constant shear rate.

10. Generalized torsional flows. In Sec. 7 we found that if the surfaces of constant shear rate are coaxial circular cylinders and the field **c** is radial, then the velocity field must be of the form (7.19) or (7.20). In the present section we consider the fields (7.19). The first two terms correspond to a rigid motion. The terms of interest, multiplied by τ, represent a motion in which the helicoidal surfaces $z - R\theta$ = constant move as rigid bodies, sliding over one another in a screw motion. In the degenerate case $R = 0$, the planes z = constant rotate rigidly with angular velocity τz. This is the torsional flow between rotating plates discussed by Rivlin [1].

None of these flows are even partially controllable if $\tau \neq 0$, except in a fluid of zero density, because the inertial term of the momentum equation cannot be equilibrated by a pressure. The normal stress terms in (3.4) are, of course, equivalent to the gradient of a scalar function P, because we have used this requirement in determining the field (7.19). Hence, neglecting inertia, the momentum equation takes the form

$$\nabla(p - P) = \mathbf{i}_r 2r^{-1}\gamma\eta \sin\alpha \cos\alpha. \tag{10.1}$$

Here we have used the expressions for **a** and **b** given by (7.1), (6.1), and (6.2), and we have used the fact that γ and α depend only on r. Since the right-hand side of (10.1) has the form of the gradient of a function of r, the momentum equation puts no restriction on the field (7.19). Thus, the field (7.19) would be fully controllable if inertia could be neglected.

11. Simple shearing between nonparallel plates. The first two terms of (7.20) correspond to a rigid motion. The third term is a motion in which each plane θ = constant moves rigidly in the axial direction. The momentum equation (3.4) takes the form

$$\nabla p = -\rho(-r\Omega^2\mathbf{i}_r + \Omega A\mathbf{i}_z) - \mathbf{i}_r r^{-1}[(r\gamma^2\phi)' - \gamma^2\nu], \tag{11.1}$$

where a prime denotes differentiation with respect to r. Since the right-hand side of (11.1) is irrotational, the velocity field (7.20) is completely controllable. It is the only completely controllable flow with nonconstant shear rate.

In discussing (7.19) and (7.20) we have left aside the first two terms in either case, saying that they correspond to rigid motions. By themselves, they would correspond to rigid motions. However, in (7.20) the θ-component of velocity carries a given particle through varying values of the z-component, and contrary to what one might at first suppose, the particle trajectories are not helices. In (7.19) the situation is even more complicated.

12. Parabolic motion. Although it is not within the scope of the present investigation, we accidentally turned up the velocity field

$$\mathbf{u} = u_0\mathbf{i} + v_0\mathbf{j} + w_0\mathbf{k} + \gamma_0 y\mathbf{i} \tag{12.1}$$

in Sec. 5. If $v_0 = 0$, this is merely steady simple shearing motion. However if $v_0 \neq 0$, this is a motion with parabolic streamlines, and a pressure gradient,

$$\nabla p = -\rho\gamma_0 v_0\mathbf{i}, \tag{12.2}$$

is required in order to maintain the flow. The flow is, of course, completely controllable.

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