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LINEAR TIME-DEPENDENT FLUID FLOW PROBLEMS*

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Summary. A general method for solving the linear unsteady fluid flow problems through closed conduits has been given. The method is suitable for problems with or without initial conditions. The cases of flow through a circular pipe and a circular annular channel with arbitrary time-dependent pressure gradient have been solved to illustrate the method.

Introduction. In problems of unsteady flow in closed conduits one distinguishes between two general types—one with prescribed initial conditions and the other without. Both types have been solved previously by assuming specific forms for the pressure gradient. Thus the flow through a circular pipe due to an impulsively applied pressure gradient was considered by Szymanski [1]. Later Müller [2] considered the flow through a circular pipe and through an annular channel when the prescribed pressure gradient is an arbitrary function of time. An example of the flow problem without initial conditions was given by Sanyal [3] who considered the flow through a circular pipe with the pressure gradient an exponential function of time.

Recently Ojalvo [4] outlined a method of dealing with boundary value problems with initial conditions. His method is applicable to more general space differentiation and time differentiation operators than are involved in the fluid flow problems mentioned above.

The aim of this paper is to extend Ojalvo's method to boundary value problems without initial conditions, and to indicate application to fluid flow problems both with and without initial conditions when the pressure gradient is an arbitrary function of time.

Ojalvo's method—with initial conditions. Consider the following problem.

$$Mu(x, t) = TNu(x, t) + F(x, t) \quad \text{in} \quad R, \tag{2.1}$$

$$\alpha_0 u + \alpha_1 D_1 u \cdots = G(x, t) \quad \text{on} \quad B, \quad t > 0, \tag{2.2}$$

$$u(x, 0) = H_1(x), \qquad \partial u(x, 0)/\partial t = H_2(x), \tag{2.3}$$

where

R is a continuum domain with boundaries B,
x is space variable vector,
t is time variable,

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M, N are linear differential space operators such that M is of order greater than N,
D_i is linear differential homogeneous space operator of the ith order,
$T \equiv a\partial^2/\partial t^2 + b\partial/\partial t$ is the time operator with a and b as some given constants,
$F(x, t)$, $G(x, t)$, $H_1(x)$, $H_2(x)$ are prescribed functions.

Assume that $F(x, t)$, $G(x, t)$ can be expressed as

$$F(x, t) = \sum_{i=1}^{r} f_i(x)F_i(t), \tag{2.4}$$

$$G(x, t) = \sum_{j=1}^{s} g_j(x)G_j(t), \tag{2.5}$$

where r and s may be finite or infinite.

We look for a solution of the given boundary value problem in the separable form such as

$$u(x, t) = \sum_{k=1}^{\infty} \phi_k(x)\Psi_k(t) + \sum_{i=1}^{r} v_i(x)F_i(t) + \sum_{j=1}^{s} w_j(x)G_j(t). \tag{2.6}$$

Substituting (2.6) into (2.1) and (2.2) one obtains

$$\sum_{k=1}^{\infty} \Psi_k(t)M\phi_k(x) + \sum_{i=1}^{r} F_i(t)[Mv_i(x) - f_i(x)] + \sum_{j=1}^{s} G_j(t)Mw_j(x)$$
$$= \sum_{k=1}^{\infty} T\Psi_k(t)N\phi_k(x) + \sum_{i=1}^{r} TF_i(t)Nv_i(x) + \sum_{j=1}^{\infty} TG_j(t)Nw_j(x), \tag{2.7}$$

and

$$\sum_{k=1}^{\infty} \Psi_k(t)(\alpha_0 u_k + \alpha_1 D_1 u_k + \cdots) + \sum_{i=1}^{r} F_i(t)(\alpha_0 v_i + \alpha_1 D_1 v_i + \cdots)$$
$$+ \sum_{j=1}^{s} G_j(t)(\alpha_0 w_j + \alpha_1 D_1 w_j + \cdots - g_j) = 0. \tag{2.8}$$

The separation of variables can be achieved by assuming that

$$Mv_i(x) = f_i(x) \qquad \text{in} \quad R, \tag{2.9a}$$

$$\alpha_0 v_i + \alpha_1 D_1 v_i \cdots = 0 \qquad \text{on} \quad B; \tag{2.9b}$$

and

$$Mw_j(x) = 0 \qquad \text{in} \quad R, \tag{2.10a}$$

$$\alpha_0 w_j + \alpha_1 D_1 w_j + \cdots = g_j(x) \qquad \text{on} \quad B. \tag{2.10b}$$

Thus (2.9) and (2.10) constitute boundary value problems from which $v_i(x)$, $w_j(x)$ can be determined. If $\{\phi_k(x)\}$ is a complete set of orthogonal functions, we may assume that $v_i(x)$ and $w_j(x)$ can be expressed in terms of $\{\phi_k(x)\}$. Thus, let

$$v_i(x) = \sum_{k=1}^{\infty} a_{ik}\phi_k(x), \tag{2.11}$$

$$w_j(x) = \sum_{k=1}^{\infty} b_{jk}\phi_k(x). \tag{2.12}$$

Equation (2.7) can now be written as

$$\sum_{k=1}^{\infty} \Psi_k(t) M\phi_k(x) = \sum_{k=1}^{\infty} [T\Psi_k(t) + \sum_{i=1}^{r} a_{ik} TF_i(t) + \sum_{j=1}^{s} b_{jk} TG_j(t)] N\phi_k(\bar{x}). \tag{2.13}$$

This equation can be separated for each k into the form

$$\frac{M\phi_k(x)}{N\phi_k(x)} = \frac{T\Psi_k(t) + \sum_{i=1}^{r} a_{ik} TF_i(t) + \sum_{j=1}^{s} b_{jk} TG_j(t)}{\Psi_k(t)} = \lambda_k , \tag{2.14}$$

where λ_k is a separation constant. Thus $\phi_k(x)$ are the eigenfunctions and λ_k the eigenvalues of the problem

$$M\phi(x) = \lambda N\phi(x) \qquad \text{in} \quad R, \tag{2.15a}$$

$$\alpha_0 \phi + \alpha_1 D_1 \phi + \cdots = 0 \qquad \text{on} \quad B. \tag{2.15b}$$

Also, the functions $\Psi_k(t)$ are the solutions of the problem

$$T\Psi_k(t) + \sum_{i=1}^{r} a_{ik} TF_i(t) + \sum_{j=1}^{s} b_{jk} TG_j(t) - \lambda_k \Psi_k(t) = 0. \tag{2.16}$$

The conditions that the functions $\Psi_k(t)$ ought to satisfy are obtained by substituting (2.6) into (2.3). Thus, if

$$H_i(x) = \sum_{k=1}^{\infty} h_{ik} \phi_k(x) \qquad (i = 1, 2) \tag{2.17}$$

then we require that

$$\begin{aligned} \Psi_k(0) &= h_{1k} - \sum_{i=1}^{r} a_{ik} F_i(0) - \sum_{j=1}^{s} b_{jk} G_j(0), \\ \Psi_k'(0) &= h_{2k} - \sum_{i=1}^{r} a_{ik} F_i'(0) - \sum_{j=1}^{s} b_{jk} G_j'(0), \end{aligned} \tag{2.18}$$

where primes mean differentiation.

The coefficients in (2.11), (2.12) and (2.17) can be obtained as follows: Define M^*, the adjoint of M by the relation

$$(M\xi, \eta) = (\xi, M^*\eta),$$

where the inner product (ξ, η) is defined as

$$(\xi, \eta) = (\eta, \xi) = \int_R \xi\eta \, dR.$$

Consider the adjoint problem of (2.15) viz

$$(M^* - \alpha N^*)\xi = 0 \qquad \text{in} \quad R, \tag{2.19a}$$

$$\alpha_0 \xi + \alpha_1 D_1 \xi + \cdots = 0 \qquad \text{on} \quad B. \tag{2.19b}$$

Let ϕ_n and ξ_m be the eigenfunctions of the problems (2.15) and (2.19) respectively. Forming the inner product of (2.15a) by ξ_m and of (2.19a) by ϕ_n and subtracting one obtains

$$(\xi_m , M\phi_n) - (\phi_n , M^*\xi_m) = \lambda_n(\xi_m , N\phi_n) - \lambda_m(\phi_n , N^*\xi_m),$$

or

$$0 = (\lambda_n - \lambda_m)(\xi_m , N\phi_n). \text{ Hence}$$

$$\int_R \xi_m N\phi_n \, dR = \int_R \phi_n N^* \xi_m \, dR = 0, \qquad \text{if} \quad m \neq n.$$

It follows that the coefficients in (2.11), (2.12) and (2.17) are given by

$$a_{ik} = \frac{\int_R v_i N^* \xi_k \, dR}{\int_R \phi_k N^* \xi_k \, dR}, \qquad b_{ik} = \frac{\int_R w_i N^* \xi_k \, dR}{\int_R \phi_k N^* \xi_k \, dR}, \qquad h_{ik} = \frac{\int_R H_i N^* \xi_k \, dR}{\int_R \phi_k N^* \xi_k \, dR},$$

Extension—without initial conditions. In this section we consider the boundary value problem (2.1) and (2.2) but without the initial condition (2.3). The following two cases are considered.

I. $F(x, t)$ and $G(x, t)$ are periodic in t of period $2p$ and $2q$ respectively and are defined over $-p < t < p$ and $-q < t < q$. Assume that F and G satisfy Dirichlet's conditions and are equal to the mean of their right and left limits at the points of discontinuity.

II. $F(x, t)$ and $G(x, t)$ are not periodic but satisfy Dirichlet's conditions and are absolutely integrable.

In the first case, it is possible to write

$$F(x, t) = \sum_{n=-\infty}^{\infty} f_n(x) \exp \frac{in\pi t}{p}, \tag{3.1a}$$

$$G(x, t) = \sum_{n=-\infty}^{\infty} g_n(x) \exp \frac{in\pi t}{q}, \tag{3.2a}$$

where $f_n(x)$, $g_n(x)$ are Fourier coefficients given by

$$f_n(x) = \frac{1}{2p} \int_{-p}^{p} F(x, s) \exp \frac{-in\pi s}{p} \, ds, \tag{3.1b}$$

$$g_n(x) = \frac{1}{2q} \int_{-q}^{q} F(x, s) \exp \frac{-in\pi s}{q} \, ds. \tag{3.2b}$$

Assume a solution to (2.1) and (2.2) in the form

$$u(x, t) = \sum_{n=-\infty}^{\infty} v_n(x) \exp \frac{in\pi t}{p} + \sum_{n=-\infty}^{\infty} w_n(x) \exp \frac{in\pi t}{q}. \tag{3.3}$$

Inserting into the equation (2.1) and the condition (2.2) one obtains

$$\sum_{n=-\infty}^{\infty} \left[Mv_n \exp \frac{in\pi t}{p} + Mw_n \exp \frac{in\pi t}{q} \right]$$

$$= \sum_{n=-\infty}^{\infty} \left[\xi N v_n \exp \frac{in\pi t}{p} + \eta N w_n \exp \frac{in\pi t}{q} + f_n \exp \frac{in\pi t}{p} \right], \tag{3.4}$$

where

$$\xi = -a\left(\frac{n\pi}{p}\right)^2 + \frac{ibn\pi}{p}, \qquad \eta = -a\left(\frac{n\pi}{q}\right)^2 + \frac{ibn\pi}{q},$$

and

$$\sum_{n=-\infty}^{\infty} (\alpha_0 v_n + \alpha_1 D_1 v_n + \cdots) \exp \frac{in\pi t}{p} + \sum_{n=-\infty}^{\infty} (\alpha_0 w_n + \alpha_1 D_1 w_n + \cdots) \exp \frac{in\pi t}{q}$$

$$= \sum_{n=-\infty}^{\infty} g_n(x) \exp \frac{in\pi t}{q} \qquad \text{on} \quad B. \tag{3.5}$$

Equating termwise in n and splitting the differential equation and the boundary conditions into two parts each, the following two boundary value problems are obtained:

$$\begin{aligned} Mv_n(x) &= \xi N v_n(x) + f_n(x) \quad \text{in } R, \\ \alpha_0 v_n + \alpha_1 D_1 v_n + \cdots &= 0 \quad \text{on } B; \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} Mw_n(x) &= \eta N w_n(x) \quad \text{in } R, \\ \alpha_0 w_n + \alpha_1 D_1 w_n + \cdots &= g_n(x) \quad \text{on } B. \end{aligned} \tag{3.7}$$

Solving these two, the solution of the original problem (2.1) and (2.2) is given by (3.3).

In the second case $F(x, t)$, $G(x, t)$ are expressible in the form of Fourier integrals

$$F(x, t) = \int_{-\infty}^{\infty} f(x, \zeta) \exp(i\zeta t)\, d\zeta, \qquad G(x, t) = \int_{-\infty}^{\infty} g(x, \zeta) \exp(i\zeta t)\, d\zeta,$$

where

$$f(x, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x, s) \exp(-i\zeta s)\, ds, \qquad g(x, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x, s) \exp(-i\zeta s)\, ds.$$

By a process very similar to the one outlined above it is not difficult to see that the solution in the present case may be expected in the form

$$u(x, t) = \int_{-\infty}^{\infty} v(x, \zeta) \exp(i\zeta t)\, d\zeta + \int_{-\infty}^{\infty} w(x, \zeta) \exp(i\zeta t)\, d\zeta, \tag{3.8}$$

where $v(x, \zeta)$ and $w(x, \zeta)$ are determinable from the boundary value problems

$$\begin{aligned} Mv(x, \zeta) - rNv(x, \zeta) &= f(x, \zeta) \quad \text{in } R, \\ \alpha_0 v + \alpha_1 D_1 v + \cdots &= 0 \quad \text{on } B; \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} Mw(x, \zeta) - rNw(x, \zeta) &= 0 \quad \text{in } R, \\ \alpha_0 w + \alpha_1 D_1 w + \cdots &= g(x, \zeta) \quad \text{on } B. \end{aligned} \tag{3.10}$$

where $r = a\alpha^2 + ib\alpha$.

Applications: Flow through a circular tube:

I. *With Initial Conditions.* Assume that the fluid is contained in a circular pipe of radius r_0 and is at rest. At the initial instant let the time dependent pressure gradient become operative. One then has to solve the boundary value problem

$$\frac{\partial u^*}{\partial t^*} = T^*(t^*) + \nu\left(\frac{\partial^2 u^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial u^*}{\partial r^*}\right), \tag{4.1a}$$

$$u^* = 0 \quad \text{at} \quad r^* = r_0\,, \quad \text{for} \quad t^* > 0, \tag{4.1b}$$

$$u^* = 0 \quad \text{at} \quad t^* = 0, \quad 0 < r^* < r_0\,. \tag{4.1c}$$

The starred dimensional quantities are nondimensionalized by writing:

$$r = \frac{r^*}{r_0}\,, \qquad u = \frac{u^*}{U}\,, \qquad t = \frac{\nu t^*}{r_0^2}\,, \qquad T = \frac{r_0^{2*}}{U\nu} T^*$$

where U is a constant of the dimensions of velocity. Thus problem (4.1) reads

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} = T(t), \qquad 0 < r < 1, \tag{4.3}$$

$$u(1, t) = 0, \qquad u(r, 0) = 0.$$

The solution may be assumed in the form

$$u(r, t) = \sum_{k=1}^{\infty} \phi_k(r)\Psi_k(t) + v(r)T(t), \tag{4.4}$$

where $\phi_k(r)$ are eigenfunctions of the problem

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\phi = \lambda\phi, \qquad 0 < r < 1, \qquad \phi(1) = 0; \tag{4.5}$$

and $v(r)$ satisfies the problem

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)v(r) = 1, \qquad v(1) = 0. \tag{4.6}$$

If, now, $v(r)$ from (4.6) is expressible as

$$v(r) = \sum_{k=1}^{\infty} a_k\phi_k(r),$$

then $\Psi_k(t)$ should be determined from

$$\frac{d\Psi_k}{dt} - \lambda_k\Psi_k(t) = -a_k\frac{dT}{dt}, \qquad \Psi_k(0) = -a_kT(0). \tag{4.7}$$

Hence

$$\phi_k(r) = J_0(\beta_k r), \qquad \lambda_k = -\beta_k^2,$$

$$v(r) = \tfrac{1}{4}(r^2 - 1),$$

$$a_k = \frac{\int_0^1 rv(r)J_0(\beta_k r)\,dr}{\int_0^1 rJ_0^2(\beta_k r)\,dr} = -\frac{2}{\beta_k^3 J_1(\beta_k)}.$$

Hence the solution to the problem is

$$u(r, t) = \tfrac{1}{4}(r^2 - 1)T(t) + 2\sum_{k=1}^{\infty}\frac{J_0(\beta_k r)}{\beta_k^3 J_1(\beta_k)}\left[T(0) + \int_0^t \frac{dT(s)}{ds}e^{\beta_k^2 s}\,ds\right]e^{-\beta_k^2 t}. \tag{4.8}$$

II. *Without Initial Conditions.* Let $T(t)$ be periodic, and let

$$T(t) = \sum_{k=-\infty}^{\infty} a_k \exp\left(\frac{ik\pi t}{p}\right), \qquad \text{where} \qquad a_k = \frac{1}{2p}\int_{-p}^{p} T(s)\exp\left(\frac{-ik\pi s}{p}\right)ds. \tag{4.9}$$

Assuming the solution as

$$u(r, t) = \sum_{k=-\infty}^{\infty} \phi_k(r)\exp\left(\frac{ik\pi t}{p}\right),$$

one obtains the boundary value problem

$$\frac{d^2\phi_k}{dr^2} + \frac{1}{r}\frac{d\phi_k}{dr} + a_k = \frac{ik\pi}{p}\phi_k, \qquad \phi_k(1) = 0. \tag{4.10}$$

Hence the solution to (4.1) without initial condition (4.1c) is

$$u(r, t) = \sum_{k=-\infty}^{\infty} \left[\frac{J_0(i(ik\pi/p)^{1/2}r)}{J_0(i(ik\pi/p)^{1/2})} - 1\right] \frac{ia_k p}{k\pi} \exp\,(ik\pi t/p). \tag{4.11}$$

If $T(t)$ is expressible in the form of Fourier integral

$$T(t) = \int_{-\infty}^{\infty} F_\alpha e^{i\alpha t}\, d\alpha, \qquad \text{where} \qquad F_\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(s) e^{-i\alpha s}\, ds, \tag{4.12}$$

then the solution to (4.1), without the initial condition, is

$$u(r, t) = \int_{-\infty}^{\infty} \left[\frac{J_0(i(i\alpha r)^{1/2})}{J_0(i(i\alpha)^{1/2})} - 1\right] \frac{iF_\alpha}{\alpha} e^{i\alpha t}\, d\alpha. \tag{4.13}$$

Flow in channels of circular annular cross-section.

I. *With Initial Conditions.* Here one has to solve the non-dimensional equation

$$\frac{\partial u}{\partial t} = -T(t) + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \qquad 1 < r < c \tag{5.1a}$$

subject to the conditions

$$u(1, t) = u(c, t) = 0, \tag{5.1b}$$

$$u(r, 0) = 0, \tag{5.1c}$$

where

$c = r_i/r_0$, and r_i and r_0 are the inner and outer radii of the annulus. The solution to the problem is

$$u(r, t) = \sum_{k=1}^{\infty} \phi_k(r)\Psi_k(t) + v(r)T(t), \tag{5.2}$$

where $\phi_k(r)$ are eigenfunctions of the problem

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\phi = \lambda\phi, \qquad 1 < r < c, \qquad \phi(1) = \phi(c) = 0; \tag{5.3}$$

and $v(r)$ is obtained by solving

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)v = 1, \qquad v(1) = v(c) = 0. \tag{5.4}$$

Thus,

$$\phi_k(r) = Z_0(\lambda_k r), \qquad v(r) = -\frac{1}{4}\left[1 - r^2 + \frac{c^2 - 1}{\log c}\log r\right],$$

where

$$Z_0(\lambda r) = \frac{J_0(\lambda r)}{J_0(\lambda)} - \frac{Y_0(\lambda r)}{Y_0(\lambda)},$$

and λ_k are roots of the equation

$$Z_0(\lambda c) = 0. \tag{5.5}$$

Also, when $v(r)$ is expressed as $\sum_{k=1}^{\infty} a_k\phi_k(r)$, the functions $\Psi_k(t)$ are to be obtained from

$$\frac{d\Psi_k}{dt} + \lambda_k^2\Psi_k = -a_k \frac{dT}{dt}, \qquad \Psi_k(0) + a_kT(0) = 0. \tag{5.6}$$

Thus

$$\Psi_k(t) = -a_k\left[T(0) + \int_0^t \frac{dT(\eta)}{d\eta} \exp(\lambda_k^2\eta)\, d\eta\right] \exp(-\lambda_k^2 t).$$

Hence the solution to the initial condition problem through an annulus in

$$u(r, t) = \sum_{k=1}^{\infty} - a_k \exp(-\lambda_k^2 t)\left[T(0) + \int_0^t \frac{dT(\eta)}{d\eta} \exp(\lambda_k^2\eta)\, d\eta\right]\cdot Z_0(\lambda_k r) - \frac{1}{4}\left[1 - r^2 + \frac{c^2 - 1}{\log c} \log r\right]T(t). \tag{5.7}$$

The coefficients a_k in the expansion of $v(r)$ in $\{\phi_k(r)\}$ are obtained from making use of the relations

$$\int_1^c Z_0(\lambda_i r)Z_0(\lambda_k r)\, dr = 0, \qquad i \neq k,$$

$$\int_1^c Z_0^2(\lambda_k r)\, dr = \tfrac{1}{2}c^2Z_1^2(\lambda_k c) - \tfrac{1}{2}Z_1^2(\lambda_k).$$

Thus

$$a_k = \frac{\int_1^c rv(r)Z_0(\lambda_k r)\, dr}{\int_1^c rZ_0^2(\lambda_k r)\, dr} = -\frac{2}{\lambda_k^3[cZ_1(\lambda_k c) + Z_1(\lambda_k)]}.$$

II. *Without Initial Conditions.* If $T(t)$ is periodic and expressible as (4.9), then the solution to the annular problem without initial conditions is

$$u(r, t) = \sum_{k=-\infty}^{\infty} E_k(r) \exp \frac{ik\pi t}{p}, \tag{5.8}$$

where $E_k(r)$ satisfies

$$\frac{ik\pi}{p} E_k = a_k + \frac{d^2E_k}{dr^2} + \frac{1}{r}\frac{dE_k}{dr}, \qquad E_k(1) = E_k(c) = 0.$$

Hence

$$E_k(r) = \frac{ia_kp}{k\pi}\left[\frac{M(r, 1, k\pi/p) - M(r, c, k\pi/p)}{M(c, 1, k\pi/p)} - 1\right],$$

where

$$M(x, y, \beta) = J_0(i(i\beta x)^{1/2})K_0((i\beta y)^{1/2}) - J_0(i(i\beta y)^{1/2})K_0((i\beta x)^{1/2}).$$

If $T(t)$ is not periodic but is expressible by the Fourier integral formula (4.12) then, the solution to the annular problem without initial conditions is

$$u(r, t) = \int_{-\infty}^{\infty} E_\alpha(r) \exp(i\alpha t)\, d\alpha, \tag{5.9}$$

where $E_\alpha(r)$ satisfies

$$i\alpha E_\alpha = F_\alpha + \frac{d^2E_\alpha}{dr^2} + \frac{1}{r}\frac{dE_\alpha}{dr}, \qquad E_\alpha(1) = E_\alpha(c) = 0.$$

Hence

$$u(r, t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} ds \left[\frac{M(r, 1, \alpha) - M(r, c, \alpha)}{M(c, 1, \alpha)} - 1\right]\frac{T(s)}{\alpha}\exp\{-i\alpha(s - t)\}. \quad (5.10)$$

References

1. P. Szymanski, *Sur l'ecoulement non permanent du fluide visqueux dans le tuyau,* Vol. **1**, p. 249–254, Applied Mech. Congress 3rd., Paris, 1930
2. V. W. Müller, *Zum Problem des Anlaufstromung einer Flussigkeit in geraden Rohr mit Kreising— und Kreisquerschmitt,* Z.A.M.M **16**, 227–238 (1935)
3. L. Sanyal, *The flow of a viscous fluid in a circular tube under pressure gradient varying exponentially with time,* Indian J. Phys. **30**, 57–61 (1956)
4. I. U. Ojalvo, *An extension of 'Separation of Variables' for time dependent excitations.* Quart. Appl. Math. **20**, 390–394 (1962)