ON SOME INVERSE PROBLEMS IN POTENTIAL THEORY*

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Abstract. It is a well known result that the inverse square law has the property that the attraction, at an external point, due to a sphere of uniform density is the same as if the sphere was concentrated at its center. The most general law of attraction having this property for all spheres among the class of central force laws is known to be the inverse square plus linear. For the corresponding internal problem, it is well known that there is no attractive force acting on a particle inside a uniform spherical shell and, inversely, this property characterizes the inverse square law. We establish the first inverse result under the weaker condition that the desired property holds for just two thin spherical shells of any radii. For the corresponding internal problem, we need three incommensurable radii.

"The inverse square law has the property that the attraction, at an external point, due to a sphere of uniform density is the same as if the sphere was concentrated at its center. Are there any other laws of attraction which have this property?"[2]

The above problem was solved in the affirmative,[2] i.e., the most general such central force law being $C(r) = A/r^2 + Br$. For the problem concerning an internal point, Jeans[3] notes that if there is no attractive force acting on a particle inside a uniform spherical shell, then the law of force must be inverse square. He also provides a neat proof and attributes the theorem to Laplace.

In both cases above, the desired property was assumed to hold among the class of central force laws and for spheres or shells of all sizes. In this paper we establish the first result under the weaker condition that the desired property holds for just two thin spherical shells of radii 1 and 2. Our method has some features in common with that given by Jeans. If instead of using shells of radii 1 and 2 we use a pair of arbitrary radii, the problem becomes considerably more difficult and apparently requires a more sophisticated argument. Consequently, this solution is given secondly. Lastly, we extend the internal problem in a similar way. But here it will turn out that the radii of the shells cannot be commensurable.

Let $\Phi(r)$ denote the unknown potential function. Since the desired property is to hold for shells of radii 1 and 2, $\Phi(r)$ must then satisfy

$$\frac{1}{4\pi}\int_{|\zeta|=1} \Phi(|\mathbf{R} - \zeta|)\, dS_\zeta = \Phi(|\mathbf{R}|) + c_1 \qquad \text{for} \quad |\mathbf{R}| \geq 1 \tag{1}$$

and

$$\frac{1}{4\pi}\int_{|\zeta|=2} \Phi(|\mathbf{R} - \zeta|)\, dS_\zeta = \Phi(|\mathbf{R}|) + c_2 \qquad \text{for} \quad |\mathbf{R}| \geq 2 \tag{2}$$

where dS_ζ denotes the surface area element of the spherical surface.

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[2]Problem 4381, proposed by R. J. Walker, Amer. Math. Monthly, (1951), p. 51. This problem also appears in S. L. Loner, *An elementary treatise on statics*, Cambridge University Press, 1942, p. 323.

[3]*The mathematical theory of electricity and magnetism*, Cambridge University Press, Cambridge, 1948, pp. 38–40.

Since a spherical zone has area proportional to its thickness, we can reduce the above double integrals to the following single ones:

$$\frac{1}{2}\int_{-1}^{1} \Phi((r^2 - 2rt + 1)^{1/2})\, dt = \Phi(r) + c_1 \qquad \text{for} \quad r \geq 1 \tag{3}$$

and

$$\frac{1}{4}\int_{-2}^{2} \Phi((r^2 - 2rt + 4)^{1/2})\, dt = \Phi(r) + c_2 \qquad \text{for} \quad r \geq 2. \tag{4}$$

By now letting $\Phi(r) = g'(r)/r$, we obtain the following simplification:

$$\int_{-1}^{1} \Phi((r^2 - 2rt + 1)^{1/2})\, dt = \int_{-1}^{1} \frac{g'((x^2 - 2xt + 1)^{1/2})}{(x^2 - 2xt + 1)^{1/2}}\, dt$$

$$= -\frac{1}{x}\left[g((x^2 - 2xt + 1)^{1/2})\right]_{-1}^{1} = \frac{g(x + 1) - g(x - 1)}{x}.$$

Thus (3) becomes

$$g(x + 1) - g(x - 1) = 2g'(x) + 2c_1 x \qquad \text{for} \quad x \geq 1 \tag{5}$$

and, in a like manner, (4) becomes

$$g(x + 2) - g(x - 2) = 4g'(x) + 4c_2 x \qquad \text{for} \quad x \geq 2. \tag{6}$$

Denoting the differential operator and translation operator by D and E, respectively, (5) and (6) become

$$\{E - E^{-1} - 2\, D\}g = 2c_1 x, \tag{7}$$

$$\{E^2 - E^{-2} - 4\, D\}g = 4c_2 x. \tag{8}$$

By virtue of the identity

$$\{(E + E^{-1})^2 + 2\, D(E - E^{-1}) + 4\, D^2 + 2(E + E^{-1})\}\{E - E^{-1} - 2\, D\}$$

$$+ \{E + 2 + E^{-1}\}\{E^2 - E^{-2} - 4\, D\} \equiv -8\, D^3$$

and (7), (8), we obtain the differential equation

$$g'''(x) = (c_2 - 2c_1)x.$$

Whence,

$$g(x) = ax^4 + bx^2 + cx + d.$$

That the cubic term doesn't appear follows by substituting back in (5).

Since the potential function has the form

$$\Phi(r) = 4ar^2 + 2b + c/r,$$

the force law must have the form

$$F(r) = A/r^2 + Br$$

(a linear combination of the inverse square law and the linear law of forces).

We now establish the above result for a pair of arbitrary radii. Equations (5) and (6) are now of the form

$$(g(x + h) - g(x - h))/2h = g'(x) + c_h x.$$

By letting $g(x) = f(x) + ax^4$, the above is reduced to the homogeneous form

$$(f(x+h) - f(x-h))/2h = f'(x).$$

That the previous result holds here as well, will follow from

THEOREM. *If* $f(x+h) - f(x-h) = 2hf'(x)$ *holds identically in* x *for two distinct positive values of* h, *then* $f(x)$ *is a quadratic polynomial.*

We use the following two lemmas in the proof:

LEMMA 1. If $f(x)$ satisfies the above hypothesis, then it grows at most exponentially (i.e., $|f(x)| \leq Ae^{B|x|}$ for some A, B).

Proof. Let a, b be the aforementioned two values of h with $0 < a < b$. Then

$$f(x+b) = f(x-b) + (b/a)\{f(x+a) - f(x-a)\}.$$

Thus with $M(r) = \text{Max}_{0\leq x\leq r} |f(x)|$, we have for $r > a + b$ that

$$M(r+b-a) \leq M(r-b-a) + (b/a)\{M(r) + M(r-2a)\},$$

$$M(r+b-a) \leq (1 + (2b)/a)M_r\,.$$

Thus,

$$M(r) \leq A(1 + (2b)/a)^{r/(b-a)}.$$

LEMMA 2. *The equation* $\sin z = z$ *does not have two distinct solutions with positive ratio.*

Proof. The equation in real form is

$$\sin x \cosh y = x, \qquad \cos x \sinh y = y.$$

Thus,

$$\frac{x^2}{y^2} = \left\{\frac{1}{y^2} - \frac{1}{\sinh^2 y}\right\} \cosh^2 y.$$

and our lemma will follow immediately once we show that the right hand side increases for $y > 0$.

To show this, note that

$$\left\{\frac{1}{y^2} - \frac{1}{\sinh^2 y}\right\} \cosh^2 y = (\cosh^2 y) \sum_{n=1}^{\infty} \left\{\frac{1}{[2^n \sinh 2^{-n} y]^2} - \frac{1}{[2^{n-1} \sinh 2^{1-n} y]^2}\right\}$$

$$= (\cosh^2 y) \sum_{n=1}^{\infty} 4^{-n} \operatorname{sech}^2 2^{-n} y = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\cosh^2 2^{1-k} y}{4 \cosh^2 2^{-k} y}\,.$$

Now the desired result follows from the simple fact that

$$\frac{\cosh 2t}{\cosh t} = 2 \cosh t - \frac{1}{\cosh t}$$

is clearly increasing.

Proof of Theorem. We will show that $f(x)$ is a quadratic polynomial on $[0, \infty]$; this is sufficient since we could apply the same argument to any translation of $f(x)$.

Lemma 1 allows us to take Laplace transforms and this gives

$$(\sinh sh - sh)F(s) = G_h(s) \qquad \text{for} \quad h = a \quad \text{or} \quad b \tag{9}$$

where

$$F(s) = \int_0^\infty f(x)e^{-sx}\,dx,$$

$$G_h(s) = \frac{e^{sh}}{2}\int_0^h f(x)e^{-sx}\,dx + \frac{e^{-sh}}{2}\int_{-h}^0 f(x)e^{-sx}\,dx - hf(0).$$

Thus $F(s)$ is meromorphic with poles among the common zeros of $\sinh sa - sa$ and $\sinh sb - sb$. By Lemma 2, however, these common zeros occur only at 0 and so we conclude that $H(s) = s^3F(s)$ is entire. By estimating $\sinh z - z$ from below on squares of side $2n\pi$, we conclude that

$$H(s) = \frac{s^3G_h(s)}{\sinh sh - sh}$$

is of exponential type. Next by examining the behavior on the real and imaginary axes, we find that

$$H(s) = o(s^3) \quad \text{on the axes.}$$

Then by a Phragmén–Lindelöff argument, $H(s) = o(s^3)$ everywhere at ∞. By Liouville's theorem, $H(s)$ is a quadratic polynomial which in turn implies that $f(x)$ is also a quadratic polynomial.

For the internal problem, the determining equation for g is

$$g(h + x) - g(h - x) = 2c_hx \qquad \text{for} \quad 0 \le x \le h. \tag{10}$$

By letting $g(x) = f(x) + ax^2 + bx + c$, (10) is reduced to the homogeneous difference equation

$$f(h + x) - f(h - x) = 0. \tag{11}$$

Even if (11) holds for any number of rational values of h, $f(x)$ need not be constant. The same is true if (11) holds for two values of h which are incommensurable. However, if (11) holds for three values, say h_1, h_2, h_3, such that $h_3 - h_2$ and $h_3 - h_1$ are incommensurable, then $f(x)$ must be a constant and the law of force must be inverse square.

Proof. From (11), it follows that

$$f(x) = f(2h_1 - x), \qquad f(x) = f(2h_2 - x), \qquad f(x) = f(2h_3 - x),$$

and also that

$$f(x) = f(2h_3 - 2h_2 + x) = f(2h_3 - 2h_1 + x).$$

Since $f(x)$ has two incommensurable periods and is continuous (we have assumed $g'(x)$ exists), it follows by Jacobi's theorem that $f(x)$ is constant.