ON SOME INVERSE PROBLEMS IN DYNAMICS*

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Abstract. It is known that the time of traverse of a freely falling body through a straight tunnel connecting any two points of the surface of a uniform spherical planet is isochronous. We show here that if the isochronous property is to hold for any planet with a spherical symmetric density, then the density must be constant. Also it is shown that if the isochronous property is to hold for any uniform spherical planet subject to a central force law, then the force law must be inverse square. However, the isochronous property can hold for one uniform spherical planet with a different force law of attraction.

In the January 1966 Amer. J. of Physics, Cooper considered the possibility of traveling between different points on the earth's surface via frictionless subterranean tunnels. He discussed the straight line path, the minimal time two piece linear path and the brachystochrone between any two given points. In the subsequent August issue, there were several notes commenting upon the Cooper note. The first one pointed out that such a mode of transportation through a straight tunnel and the fact that the transit times were isochronous for all straight tunnels were known for quite a long time. The next three notes gave analytic solutions for the brachystochrone problem. Cooper had obtained the pertinent differential equation by variational methods but had only integrated it numerically. The last note is by Cooper who comments upon the previous notes and upon the problem again. In particular, he notes that the universality of equal transit times for all chords between any two points on a spherical planet is based upon the assumption of a uniform mass density. In this paper, we consider a pair of related inverse problems,[2] i.e.:

I. "A spherical planet whose density at any point P is a function only of the distance of P from the center of the planet has the following property: If a straight frictionless tunnel is bored between two points on the planet's surface, the time required for an object to slide from one of these points to the other is independent of the position of the points. Prove the planet has constant density."

II. "Are there any other laws of attraction besides the inverse square law such that the time of descent (from rest) through any straight tunnel through a uniform spherical planet is independent of the path?"

We solve problem I in a simpler manner than it was solved previously (loc. cit.) and then use the solution to solve II. The result is that there are other laws of attraction besides the inverse square law having the desired property. However, if the isochronous property is to hold for all size planets, then the inverse square law is unique.

First we determine the potential function $\Phi(r)$ such that the isochronous property holds for a spherically symmetric planet and a central force law.

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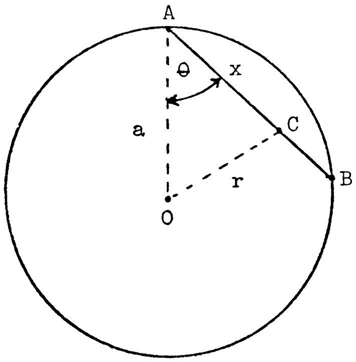
[2]The first problem is problem 4342 in the Amer. Math. Monthly, November, 1950 and it was solved by M. Kac, H. Pollard and R. J. Walker. The second one is problem 4664 in the same journal, December 1955 and was proposed by one of the authors; no solution was published.



Here, $AC = x$, $OC = r$, center $= 0$, radius $= a$, and $r^2 = a^2 + x^2 - 2ax \cos \theta$.

By conservation of energy (using appropriate units), $v^2 + \Phi(r) = \Phi(a)$. Whence the period T is given by

$$T = 4 \int_0^{a \cos \theta} \frac{dx}{(\Phi(a) - \Phi(r))^{1/2}}$$

or by letting $y = a \cos \theta - x$,

$$T = 4 \int_0^{a \cos \theta} \frac{dy}{F(y^2 + a^2 \sin^2 \theta)}$$

where

$$F(\lambda^2) = (\Phi(a) - \Phi(\lambda))^{1/2}.$$

Now let $y^2 = z$ and $t^{1/2} = a \cos \theta$ to give

$$T = \int_0^t \frac{2\, dz}{\sqrt{z}\, F(a^2 - [t - z])}.$$

Since the latter is an Abel type integral equation, we know that its solution is unique and corresponds to the potential function one gets for an inverse square law field, i.e., $\Phi(r) = kr^2$ (here we have taken $\Phi(0)$ to be zero).

The solution of problem I now easily follows. The potential function of a spherical planet for an inverse square force law and a density $\rho(r)$ is also given by

$$\Phi(r) = \int_0^r \frac{4\pi\lambda^2 \rho(\lambda)\, d\lambda}{\lambda} = kr^2.$$

Differentiating, we find that the density must be constant.

To answer problem II, we now show there is, in fact, a central force law for which a uniform unit ball of mass (i.e., a spherical planet) exerts no force anywhere, inside the ball or out! In particular, if this force be added to an inverse square force, there will be no change in the net force acting on any test particle. To produce this "forceless" force, let its potential be denoted by $\Psi(R)$ and consider the integral equation

$$\int_{|\mathbf{X}| \le 1} \Psi(|\mathbf{R} - \mathbf{X}|)\, dV_X = 0. \tag{1}$$

The latter equation expresses the fact that the net potential of the uniform ball of mass ($|\mathbf{X}| \leq 1$) at an arbitrary point $\mathbf{R}$ is zero.

We claim that if $\tan \lambda = \lambda$, then the choice $\Psi(u) = \sin \lambda u/u$ satisfies (1). Indeed, we can evaluate the above integral explicitly, the result being

$$\int_{|\mathbf{X}|\leq 1} \frac{\sin \lambda |\mathbf{R} - \mathbf{X}|}{|\mathbf{R} - \mathbf{X}|} \, dV_X = 4\pi \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \frac{\sin \lambda |\mathbf{R}|}{|\mathbf{R}|}. \tag{2}$$

This of course implies the above assertion and we now establish (2).

It is well known that the area of a zone of sphere is proportional to its width. Consequently,

$$\int_{|\mathbf{S}|=1} F(\mathbf{S}\cdot\mathbf{R}) \, dA_s = 2\pi \int_{-1}^{1} F(S_1 |\mathbf{R}|) \, dS_1 \tag{3}$$

where $\mathbf{S} = (S_1, S_2, S_3)$ and dA_s is the area element on the sphere $|\mathbf{S}| = 1$. In particular,

$$\int_{|\mathbf{S}|=1} \exp (i\lambda \mathbf{S}\cdot\mathbf{R}) \, dA_s = 2\pi \int_{-1}^{1} \exp (i\lambda S_1 |\mathbf{R}|) \, dS_1 = \frac{4\pi}{\lambda} \frac{\sin \lambda |\mathbf{R}|}{|\mathbf{R}|}.$$

Thus the integral in (2) can be rewritten as

$$\frac{\lambda}{4\pi} \int_{|\mathbf{X}|\leq 1} \int_{|\mathbf{S}|=1} \exp (i\lambda \mathbf{S}\cdot(\mathbf{R} - \mathbf{X})) \, dA_s \, dV_X$$

or

$$\frac{\lambda}{4\pi} \int_{|\mathbf{S}|=1} \exp (i\lambda \mathbf{S}\cdot\mathbf{R}) \int_{|\mathbf{X}|\leq 1} \exp (-i\lambda \mathbf{S}\cdot\mathbf{X}) \, dV_X \, dA_s \, .$$

By rotational symmetry, the inner integral is independent of S and, using (3) again, can be rewritten as

$$\frac{\lambda}{4\pi} \int_{|\mathbf{S}|=1} \exp (i\lambda \mathbf{S}\cdot\mathbf{R}) \, dA_s \cdot \int_{|\mathbf{X}|\leq 1} \exp (-i\lambda X_1) \, dV_X$$

$$= \frac{\lambda}{4\pi} \cdot 2\pi \int_{-1}^{1} \exp (i\lambda S_1 |\mathbf{R}|) \, dS_1 \cdot \pi \int_{-1}^{1} \exp (-i\lambda X_1)(1 - X_1^2) \, dX_1$$

$$= 4\pi \frac{\sin \lambda - \lambda \cos \lambda}{\lambda^3} \frac{\sin \lambda |\mathbf{R}|}{|\mathbf{R}|}$$

as required.

If the isochronous property in II is to hold for all size planets, then the inverse square law is unique. To show this, we need only show that the only solution of

$$\int_{|\mathbf{X}|\leq a} \Psi(|\mathbf{R} - \mathbf{X}|) \, dV_X = 0 \qquad (a \text{ arbitrary})$$

is $\Psi(u) = 0$. This follows immediately from

$$\operatorname*{limit}_{a\to 0} \frac{1}{4\pi a^3/3} \int_{|\mathbf{X}|\leq a} \Psi(|\mathbf{R} - \mathbf{X}|) \, dV_X = \Psi(|\mathbf{R}|).$$