THE CONSTRUCTION OF DIFFERENCE APPROXIMATIONS FROM A "SENSITIZED FUNCTIONAL"*

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Abstract. A technique from the calculus of variations is used to derive higher order difference approximations.

1. Introduction. The derivation of higher order difference approximations to partial differential operators has been discussed in several recent papers [1], [2], [3]. This note is written in the spirit of [1] to invite attention to a still different way of obtaining higher order difference equations. The emphasis here is on the symmetry of the entire coefficient matrix rather than the symmetry of the individual difference operators.

The variational formulation of a boundary value problem is often the starting point in the derivation of difference equations [4]. The resulting matrix will be symmetric and if it is nonsingular it will be positive definite. Whether the matrix must be guaranteed to have these properties is a matter of opinion; however, it is quite desirable if it can be accomplished easily. Stiefel shows how to produce the familiar nine point stencil in this manner [4].

2. Courant's sensitized functional. A sequence of papers by Courant [5], [6], [7] contains the basis for the following remarks. His idea was to "sensitize" the energy functional in such a manner that the minimizing function remains the same but the convergence rate of a sequence of approximating functions is enhanced. For example,

$$\iint_R |\nabla u|^2 \, dA = \text{Min} \tag{1}$$

is replaced by

$$\iint_R \{|\nabla u|^2 + k(\Delta u)^2\} \, dA = \text{Min} \tag{2}$$

where k is a function which is nonnegative in the region R. The same constraints on u at the boundary of R apply in both (1) and (2).

The solution will be the same for both minimization problems. However, the associated Euler equations are not the same. The differential equation associated with (1) is second order while that associated with (2) is fourth order. The missing boundary condition needed with the fourth order equation is a natural condition which evolves with the Euler equation derived from (2). It is this condition appearing implicitly in the coefficient matrix which is the essential difference between the algebraic equations derived from the variational problem and those derived directly from the differential equation.

3. An application to $\Delta u = 0$. The sensitized functional is a useful starting point for deriving higher order approximations in a systematic way suitable for computer programming. The analogy with Courant's approach is not complete since the solutions of the algebraic equations derived from (1) and from (2) are not the same. The method proposed here, to mimic the variation of the functional (2), is best illustrated by an example. One of the approximations to Δu given by Bramble and Hubbard seems to be an appropriate choice. The notation for the three classes of mesh points R_h, C_h^{**}, C_h is defined in Fig. 1 and is taken from their paper [3].

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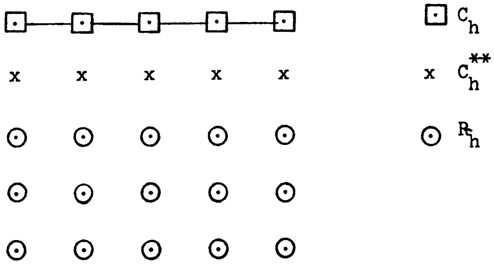

Fig. 1

The approximation to be used for $|\nabla u|^2$ in (2) is that given in [4] for the nine point stencil. The portion of the integral containing $(\Delta u)^2$ is approximated by the simplest five point stencil. With a suitable constant multiplying the functional, we obtain the following difference operator at a general point of R_h .

−1	−4	−1
−4	20	−4
−1	−4	−1

$$+ \frac{k}{h^2}$$

		1		
	2	−8	2	
1	−8	20	−8	1
	2	−8	2	
		1		

By choosing

$$\begin{aligned} k &= h^2/14 \quad \text{in} \quad R_h\,, \\ &= 0 \qquad\qquad\;\; C_h^{**}, \end{aligned}$$

we produce the R_h stencil of [3].

		1		
	−12	−64	−12	
1	−64	300	−64	1
	−12	−64	−12	
		1		

At a point of R_h neighboring C_h^{**}, we obtain

	−13	−60	−13	
1	−64	299	−64	1
	−12	−64	−12	
		1		

and on C_h^{**}

	-14	-56	-14	
	-56	281	-56	
	-13	-60	-13	
		1		

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