THE NONANALYTICITY AT THE SURFACE OF A BODY IN AN OHD OR MHD STREAM*

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It has long been recognized that the extreme stretching of vortex lines in OHD inviscid flow past a stagnation point results in singularities at the surface of an immersed body. A general discussion of this nonanalyticity was initiated by Hayes [1] and amplified by the present authors [2], who were temporarily sidetracked from their original purpose of treating three-dimensional MHD flow [3].

In plane flow there is no stretching, and the nonanalyticity which then occurs in MHD[1] (Ludford & Murray [4]) is due entirely to the special way in which vorticity is generated near a stagnation point (Ludford [5]). In three-dimensional MHD both effects arise, and it helps to understand the stretching first.

The object of the present note is to derive the form of the nonanalyticity by simple arguments. For this purpose it is sufficient to treat the front stagnation point, which is responsible, and an incompressible fluid.

With suitable choice of units the flow on the surface $z = 0$ of the body may be written

$$u_s = (1 - \alpha)x - \gamma y, \qquad v_s = \gamma x + (1 + \alpha)y.$$

Principal axes of deformation have been chosen and the constant α, which may be assumed nonnegative, measures the rate of deformation. The constant γ is half the normal vorticity. The rate of dilatation is 2 so that an incoming flow has been assumed; by continuity,

$$w = -2z \tag{1}$$

since it must vanish for $z = 0$.

The streamline pattern on the surface is characterized by the parameter

$$\Delta = \alpha^2 - \gamma^2 \tag{2}$$

and may be classified as follows:

$$\Delta < 0 \quad \text{spiral,} \qquad 0 \le \Delta < 1 \quad \text{nodal,}$$

$$\Delta = 1 \quad \text{stagnation-line,} \qquad \Delta > 1 \quad \text{saddle-point.}$$

For simplicity assume that the vorticity is in fact singular at the surface. The velocity is bounded near the surface, so that singularities $-G'(z)$ and $F'(z)$ in the x- and y-components of vorticity can only arise from the shears $-\partial v/\partial z$ and $\partial u/\partial z$. [For the same reason, normal vorticity cannot be singular.] Thus

$$u = F(z) + u_s\,, \qquad v = G(z) + v_s\,, \tag{3}$$

where $F(0) = G(0) = 0$. The problem is to determine F and G.

OHD flow is governed by the Helmholtz equation

$$\mathbf{q}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{q} = 0, \quad \text{where} \quad \mathbf{q} = (u, v, w) \quad \text{and} \quad \boldsymbol{\omega} = \nabla \wedge \mathbf{q}. \tag{4}$$

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[1]Without the vorticity itself being singular.

The expressions (1) and (3) give an exact solution when

$$2zG'' + (1 - \alpha)G' = 0, \tag{5a}$$

$$2zF'' + (1 + \alpha)F' = 0, \tag{5b}$$

$$\gamma = 0. \tag{5c}$$

The relevant solution of the first two is

$$F = F_0 z^{(1-\alpha)/2}, \qquad G = G_0 z^{(1+\alpha)/2} \quad (F_0\,,\, G_0 \text{ consts.}). \tag{6}$$

The third is connected with the Proudman-Taylor theorem of rotating fluids. If γ were nonzero, $\mathbf{q}$ would be a solid-body rotation $(-\gamma y, \gamma x, 0)$ plus a small disturbance which varied with z, i.e., in the direction of the axis of rotation. The theorem states that this is not possible.

Equation (2) now reads $\Delta = \alpha^2$, so that the spiral streamline pattern does not occur. For $\alpha > 1$, the velocity F itself is singular and does not satisfy $F(0) = 0$ unless $F_0 = 0$. The loss in generality indicates that the saddle point is special and will not occur in practice. Even if such a flow were realized, the introduction of y-vorticity, however small, into the incident stream would not yield a neighboring flow. In particular, Howarth's irrotational flows [6] with a saddle point are special. The same conclusion can be made for $\alpha = 1$,[2] but it may be wrong: the fuller analysis of [2] is unable to decide whether the stagnation line is also special.

Both components of velocity (6) are bounded and both components of transverse vorticity are singular for $\alpha < 1$. The higher-order analysis in [2] shows that the nodal case retains its full generality, so that it will occur in practice.

The result (6) can be obtained more graphically in terms of the convection and stretching of vortex lines which the Helmholtz equation describes. The z-component of vorticity thereby varies like w and hence must vanish at the surface, i.e., $\gamma = 0$. The shear velocities F and G do not contribute to the stretching parallel to $z = 0$, so that we may take the ends of an element of a vortex line to move on adjacent streamlines

$$x^2 z^{1-\alpha} = \text{const.}, \qquad y^2 z^{1+\alpha} = \text{const.}$$

of the flow $(u_s\,,\, v_s\,,\, w)$. Hence $\omega_x = -G'$ and $\omega_y = F'$ vary as $z^{-(1-\alpha)/2}$ and $z^{-(1+\alpha)/2}$ respectively.

Unfortunately there is no similar argument in MHD and we must rely on the formal approach used first. The right-hand side of the Helmholtz equation (4) is now replaced by the generating term

$$\beta^2 \operatorname{curl} \mathbf{L} \tag{7}$$

where β^2 is the ratio of magnetic and dynamic pressures,

$$\mathbf{L} = R_m[(\mathbf{E} + \mathbf{q} \times \mathbf{H}) \times \mathbf{H}] \tag{8}$$

is the Lorentz force, and R_m is the magnetic Reynolds number.

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[2] $z^{(1-\alpha)/2}$ is then replaced by log z, giving Lighthill's result [7] for weakly sheared flow past a cylinder. However, the coefficient of x in u is not accurately estimated by zero and our whole argument, including Lighthill's result, is invalid for $\alpha = 1$. The axially symmetric case $\alpha = 0$ yields Lighthill's correct result for a sphere.

In a steady state with finite conductivity the electromagnetic field, with one exception, cannot change rapidly going away from the wall. The normal electric field can and does change rapidly, due to charge accumulation,[3] so as just to balance the velocity-induced field. Otherwise there would be a rapidly changing normal current, which would require a similar tangential magnetic field. Hence only the transverse part of the velocity-induced field is important in Eq. (8), i.e., the significant part of the current is

$$nR_m(G, \ -F, \ 0)$$

where n is the normal component of magnetic intensity; which leads to n^2R_mF and n^2R_mG as the transverse components of **L**. It is the rapid change of these normal to the wall which gives the leading terms in the expression (7), so that Eqs. (5a, b) now become

$$2zG'' + (1 - \alpha)G' - \gamma F' = 2\mathfrak{N}G', \tag{9a}$$

$$2zF'' + \gamma G' + (1 + \alpha)F' = 2\mathfrak{N}F', \tag{9b}$$

where $\mathfrak{N} = \beta^2R_mn^2$ is the interaction parameter based on normal magnetic field. Equation (5c) is replaced by one defining γ, which is now, in general, nonzero.

It should be stressed that Eqs. (9a, b) only determine the way in which singular vorticity is created. Only the accompanying singularities in the other state variables have been taken into account in their derivation. The F and G determined by them do not provide an exact solution (1, 3) of the full MHD equations (in contrast to OHD above).

The left-hand sides of Eqs. (9a, b) give the creation of this vorticity by OHD stretching; the right-hand sides by MHD generation. Note that the latter is proportional to the vorticity itself. The relevant solution is a linear combination of z^{ϵ_1} and z^{ϵ_2} whose coefficients are not important; here

$$\epsilon_{1,2} = \tfrac{1}{2}(1 \mp \Delta^{1/2}) + \mathfrak{N}.$$

The spiral pattern ($\Delta < 0$) can again be excluded: the velocity would oscillate infinitely often near $z = 0$. We may also exclude the saddle-point patterns for which Δ is greater than $(2\mathfrak{N} + 1)^2$. Further investigation [3] shows, however, that the velocity becomes singular in higher-order terms[4] whenever Δ is greater than 1. All saddle points are therefore excluded. The stagnation line $\Delta = 1$ may also be special, though only when $n = 0$ (magnetic field parallel to the plane).

Although our discussion has been based on singular vorticity, ϵ_1 and ϵ_2 in fact always determine the nonanalytic character of the flow, with z^{ϵ_1} the most important such term. Note that they depend on two parameters only. Δ characterizes the streamline pattern on the surface. $\mathfrak{N}$ is the interaction parameter based on normal component of magnetic field at the stagnation point, and characterizes the MHD production of vorticity.

Hayes has pointed out that the present results hold equally well for a compressible fluid, when there may also occur entropy gradient effects. What effect these nonanalyticities have on the computation of blunt-body flows has not yet been established.

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[3]The charge density, though singular, is integrable.

[4]Powers of z appear with exponent $m(1 - \Delta^{1/2})/2 + \mathfrak{N}$ and m arbitrarily large. $\mathfrak{N}$ is zero in OHD and no further restriction arises from these terms.

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