QUARTERLY OF APPLIED MATHEMATICS

Vol. XXVI JULY 1968 No. 2

JETS OF RADIATION*

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Abstract. This paper deals with radiation (scalar, electromagnetic, or gravitational) produced by an extended source. The object is to construct a model source the radiation from which is concentrated in a jet of small angle with an assigned target direction. This is done by taking for the source an infinite train of high-frequency plane waves travelling in the target direction with the basic speed of propagation, the amplitude falling off exponentially with distance. The critical number, to be made large in this model, is the ratio a/λ, where a is a typical radius of the source and λ the wave length. However, it is also shown that a jet of scalar radiation may be obtained from a source which possesses no frequency but consists of a single shock wave.

1. Notation. The summation convention is used. Suffixes have the following ranges: Latin 1, 2, 3, 4 and Greek 1, 2, 3. Units are such that the fundamental speed of propagation is unity. Imaginary time is used ($x_4 = it$). The coordinates are rectangular cartesians, and an event is indicated by x_a, (x_σ, it), $(\mathbf{x}, it)$ or simply by x. The notation for vectors is similar. Partial differentiation is indicated by a comma. To permit without confusion exponential forms for trigonometric functions, a second independent imaginary unit j is introduced so that we have $i^2 = -1$, $j^2 = -1$, $ij = ji$. The symbol Re indicates the real part with respect to j.

In general the notation is that used by me elsewhere [1], [2].

2. Scalar radiation. We are concerned with a scalar field $\phi(x)$ generated by a source-function $S(x)$ and the wave-equation

$$\phi_{,aa} = \Box\phi = S. \tag{2.1}$$

It is understood that $S = 0$ outside some finite domain D of space, or that S tends to zero sufficiently fast as we go to spatial infinity at any time.

As a particular case, we have sound-propagation in a gas for which a pressure-density equation is given. The density ρ satisfies

$$\Box\rho = P_{\sigma,\sigma}, \tag{2.2}$$

where P_σ is a body-force per unit volume constituting the source. This differs from (2.1) only in that the right-hand side is the divergence of a vector field. We shall proceed with (2.1) and return to (2.2) later.

The retarded potential solution of (2.1), observed at the event X, is

*Received June 2, 1967.

$$\phi(\mathbf{X}, iT) = -(4\pi)^{-1} \int S(\mathbf{x}, it) \, |\mathbf{X} - \mathbf{x}|^{-1} \, d_3x, \tag{2.3}$$

where t is the retarded time,

$$t = T - |\mathbf{X} - \mathbf{x}|. \tag{2.4}$$

This means in fact that the integral is to be taken over the null cone drawn in spacetime from the event X into the past. This is illustrated in Fig. 1, which shows the time-axis

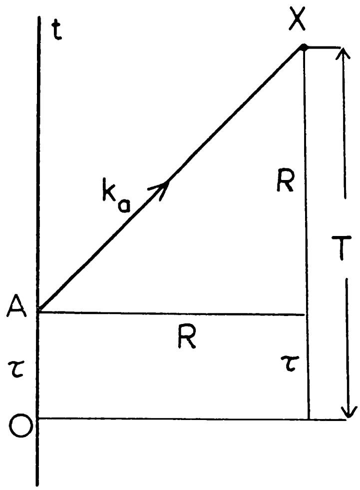

FIG. 1. Space-time diagram.

Ot, the observation event X, and the event $A(0, i\tau)$ at the intersection of the said null cone with the time-axis. We see also the distance R of X from O ($R = |\mathbf{X}|$), and we note that $T = R + \tau$. Fig. 1 also shows an important null vector k_a drawn on the null line AX, the first three components k_σ being direction cosines and $k_4 = i$; we have

$$k_\sigma k_\sigma = 1, \qquad k_4 = i, \;\; k_a k_a = 0. \tag{2.5}$$

The 3-vector $\mathbf{k}$ appears in Fig. 2, which is a space-diagram, the projection of Fig. 1 on $t = 0$.

Now (2.3) is an exact formula for all X, but we are interested only in the principal part of the distant field. This means that we are to push the observation event away towards spatial infinity. It is most convenient to do this by pushing X out along the line AX, keeping τ and k_a unchanged. R and T become very large. When we do this, two things happen to the integral (2.3). First,

$$|\mathbf{X} - \mathbf{x}|^{-1} = R^{-1} + O(R^{-2}). \tag{2.6}$$

Second, we are to replace the null cone by the 3-flat tangent to it along the null vector k_a, that is, we are to replace (2.4) by

$$t = k_\sigma x_\sigma + \tau. \tag{2.7}$$

In fact, with the understanding that X is pushed out along k_a as stated above, we have

$$\lim \left(-4\pi R\phi(\mathbf{X}, iT)\right) = \Phi(\mathbf{k}, \tau), \tag{2.8}$$

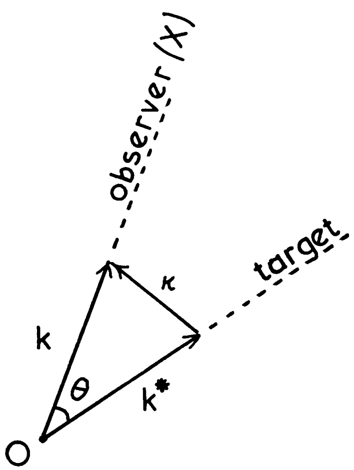

FIG. 2. Space diagram.

where

$$\Phi(\mathbf{k}, \tau) = \int S(\mathbf{x}, it)\, d_3x \tag{2.9}$$

with t as in (2.7). The details of the above argument present no difficulty. Our attention is now directed to the function Φ as the only thing we are interested in.

So far there is no mention of a jet of radiation. To get a jet, we have to choose the source function $S(\mathbf{x}, it)$ in such a way that the outgoing radiation at great distance is concentrated near some selected target direction. Let this direction be indicated by the unit 3-vector $\mathbf{k}^*$ (Fig. 2). It is convenient to take a fourth component $k_4^* = i$, so that we have a null vector k_a^*:

$$k_\sigma^* k_\sigma^* = 1, \qquad k_4^* = i, \qquad k_a^* k_a^* = 0. \tag{2.10}$$

It is also convenient to introduce an auxiliary vector

$$\kappa_a = k_a - k_a^*. \tag{2.11}$$

Then we have

$$\kappa_4 = 0, \qquad \kappa^2 = \kappa_\sigma \kappa_\sigma = 2(1 - \cos\theta), \qquad \kappa = 2 \sin \tfrac{1}{2}\theta \tag{2.12}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{k}^*$.

We now come to the crucial point in the argument, the choice of the source-function $S(\mathbf{x}, it)$. We would prefer to choose a function S vanishing outside some finite domain D, but it is more illuminating to take S vanishing exponentially at spatial infinity, because it makes the calculations much simpler. Accordingly let us take

$$S(\mathbf{x}, it) = \operatorname{Re} C \exp(-r^2/a^2 + j\omega u). \tag{2.13}$$

where C, a and ω are real constants, $r = |\mathbf{x}|$ and

$$u = t - k_\sigma^* x_\sigma - \tau^*, \tag{2.14}$$

k_σ^* being the direction cosines of the target direction and τ^* a real constant. The geometrical significance of u is that $u = 0$ is the equation of the 3-flat tangent along the null vector k_a^* to the null cone with vertex at $(0, i\tau^*)$; in fact this 3-flat is tangent to all null cones with vertices on the null line drawn through that event in the direction of k_a^*.

To obtain Φ, we are to substitute in (2.9) the function S as in (2.13), at the same time substituting for t the value (2.7). This gives

$$\Phi(\mathbf{k}, \tau) = \operatorname{Re} C \int \exp(-r^2/a^2 + j\omega v)\, d_3x, \tag{2.15}$$

where

$$v = \kappa_\sigma x_\sigma + \mu, \qquad \mu = \tau - \tau^*. \tag{2.16}$$

The integral is to be taken over all space. It is easy to evaluate, yielding

$$\Phi(\mathbf{k}, \tau) = C' \cos \omega(\tau - \tau^*), \tag{2.17}$$

the amplitude C' being

$$C' = C(a\pi^{1/2})^3 \exp(-a^2\omega^2\kappa^2/4), \tag{2.18}$$

with $\kappa = 2 \sin \frac{1}{2}\theta$ as in (2.12).

It is clear then that we can concentrate the radiation on the target direction ($\theta = 0$) by making the dimensionless quantity $a\omega$ large. Now a may be regarded as the radius of the source. Once it has been chosen, we can ensure such concentration (i.e., obtain a jet of radiation) by making the circular frequency ω sufficiently large. Equivalently, in terms of wave length $\lambda = 2\pi/\omega$, we are to make the ratio a/λ large.

If we define the directivity as the ratio of the square of the maximum amplitude to the mean value of the square of the amplitude for all directions of $\mathbf{k}$, we find by a simple calculation that the directivity is

$$2p(1 - e^{-2p})^{-1}, \qquad p = a^2\omega^2. \tag{2.19}$$

Let us now turn to the equation for sound waves (2.2) and make a modification in the above argument. Instead of choosing S as in (2.13), let us choose the body force to be

$$P_\sigma = \operatorname{Re} C_\sigma \exp(-r^2/a^2 + j\omega u) \tag{2.20}$$

with C_σ a real constant vector and u as in (2.14). We secure agreement between the equations (2.1) and (2.2) by taking $S = P_{\sigma,\sigma}$:

$$S(\mathbf{x}, it) = \operatorname{Re} C_\sigma(-2x_\sigma/a^2 - j\omega k_\sigma^*) \exp(-r^2/a^2 + j\omega u). \tag{2.21}$$

Thus, if ρ denotes the deviation of the density from its equilibrium value, we have, as in (2.8),

$$\lim(-4\pi R\rho(\mathbf{X}, iT)) = \Phi(\mathbf{k}, \tau) = \operatorname{Re} \int C_\sigma(-2x_\sigma/a^2 - j\omega k_\sigma^*) \exp(-r^2/a^2 + j\omega v)\, d_3x, \tag{2.22}$$

with v as in (2.16). To handle this integral, let us put

$$w = -r^2/a^2 + j\omega v, \tag{2.23}$$

and note that

$$\partial(e^w)/\partial x_\sigma = (-2x_\sigma/a^2 + j\omega\kappa_\sigma)e^w = (-2x_\sigma/a^2 - j\omega k_\sigma^* + j\omega k_\sigma)e^w. \tag{2.24}$$

Thus (2.22) may be written

$$\Phi(\mathbf{k}, \tau) = \operatorname{Re} C_\sigma \int [(e^{w})_{,\sigma} - j\omega k_\sigma e^{w}]\, d_3x. \tag{2.25}$$

The first part of the integrand contributes nothing, while the second part is, to within a constant, what we already had in (2.15). Thus we get

$$\Phi(\mathbf{k}, \tau) = C'' \sin \omega(\tau - \tau^*), \tag{2.26}$$

where the amplitude is

$$C'' = \omega C_\sigma k_\sigma (a\pi^{1/2})^3 \exp(-a^2\omega^2\kappa^2/4). \tag{2.27}$$

This amplitude vanishes in a direction $\mathbf{k}$ perpendicular to $\mathbf{C}$. The vector $\mathbf{C}$ is so far arbitrary. To get the best jet in the direction $\mathbf{k}^*$, we should choose $C_\sigma = Ck^*_\sigma$. Then (2.27) gives

$$C'' = \omega C(a\pi^{1/2})^3 \cos\theta \exp(-a^2\omega^2 \sin^2 \tfrac{1}{2}\theta). \tag{2.28}$$

For large $a\omega$, the directivity is the same as in (2.19), viz., $2a^2\omega^2$.

3. Electromagnetic radiation. Let $F_{ab}(= -F_{ba})$ be the electromagnetic tensor and J_a the 4-current. Maxwell's equations read

$$F_{ab,b} = J_a, \qquad F_{ab,c} + F_{bc,a} + F_{ca,b} = 0. \tag{3.1}$$

Hence it follows that

$$F_{ab,cc} = \Box F_{ab} = J_{a,b} - J_{b,a}. \tag{3.2}$$

The 4-current J_a is the source of the radiation, but these functions cannot be freely chosen: they must satisfy the constraint

$$J_{a,a} = 0. \tag{3.3}$$

It is more convenient to work with a source-function which is free except for the condition of vanishing sufficiently strongly at spatial infinity. Accordingly we shall consider only a 4-current given by

$$J_a = S_{ab,b}, \qquad S_{ab} = -S_{ba}, \tag{3.4}$$

where $S_{ab}(\mathbf{x}, it)$ may be freely chosen except for the skew condition and the condition of vanishing sufficiently fast at spatial infinity. Any J_a given by (3.4) automatically satisfies (3.3).

The equation (3.2) now reads

$$\Box F_{ab} = S_{ac,cb} - S_{bc,ca}, \tag{3.5}$$

and we have the exact retarded potential solution as in (2.3)

$$F_{ab}(\mathbf{X}, iT) = -(4\pi)^{-1} \int (S_{ac,cb} - S_{bc,ca}) \,|\mathbf{X} - \mathbf{x}|^{-1}\, d_3x, \tag{3.6}$$

where, after the indicated differentiations have been performed, we are to substitute $t = T - |\mathbf{X} - \mathbf{x}|$ as in (2.4).

For the distant field, we have as in (2.8)

$$\lim (-4\pi R F_{ab}(\mathbf{X}, iT)) = \Phi_{ab}(\mathbf{k}, \tau) = \int (S_{ac,cb} - S_{bc,ca})\, d_3x, \tag{3.7}$$

where after differentiation we are to substitute, as in (2.7),

$$t = k_\sigma x_\sigma + \tau. \tag{3.8}$$

The notation is as in Figs. 1 and 2.

Although we shall ultimately take a source of the type (2.13) (changed from scalar to tensor), it is better at this point to be more general and take

$$S_{ab}(\mathbf{x}, it) = A_{ab}(\mathbf{x}) f(u), \tag{3.9}$$

where $A_{ab} = -A_{ba}$ and vanishes sufficiently rapidly at spatial infinity, and $f(u)$ is an arbitrary function of

$$u = t - k^*_\sigma x_\sigma - \tau^* = -k^*_a x_a - \tau^*. \tag{3.10}$$

Here $\mathbf{k}^*$ is the unit vector in the target direction, and the notation of (2.10) to (2.12) will be used.

In order to calculate the integral (3.7), we have to differentiate S_{ab} as indicated and then make the substitution (3.8). This substitution means that we are to change u into v, where, as in (2.16),

$$v = \kappa_\sigma x_\sigma + \mu = \kappa_a x_a + \mu, \tag{3.11}$$

since $\kappa_4 = 0$.

To bring out the essential features of the following argument, let us suppress the suffixes in (3.9) and write

$$S(\mathbf{x}, it) = A(\mathbf{x}) f(u). \tag{3.12}$$

By (3.10) we have $u_{,a} = -k^*_a$, and so

$$\begin{aligned} S_{,a} &= A_{,a} f(u) - k^*_a A f'(u), \\ S_{,ab} &= A_{,ab} f(u) - (k^*_a A_{,b} + k^*_b A_{,a}) f'(u) + k^*_a k^*_b f''(u). \end{aligned} \tag{3.13}$$

To calculate an integral of the type (3.7), we are first to write in v instead of u in this last expression. Then we note the following results of integration by parts, the function $A(\mathbf{x})$ vanishing sufficiently fast at spatial infinity (note that $A_{,4} = 0$, and so a number of terms vanish for that reason):

$$\begin{aligned} \int A_{,ab} f(v)\, d_3x &= -\int A_{,a} \kappa_b f'(v)\, d_3x = \int A \kappa_a \kappa_b f''(v)\, d_3x, \\ \int A_{,b} f'(v)\, d_3x &= -\int A \kappa_b f''(v)\, d_3(x). \end{aligned} \tag{3.14}$$

We have used the fact that $v_{,a} = \kappa_a$. It follows then that

$$\begin{aligned} &\int (S_{,ab})_{u\to v}\, d_3x \\ &\quad = (\kappa_a \kappa_b + \kappa_a k^*_b + \kappa_b k^*_a + k^*_a k^*_b) \int A f''(v)\, d_3x = k_a k_b \int A f''(v)\, d_3x, \end{aligned} \tag{3.15}$$

since $\kappa_a + k^*_a = k_a$.

Applying this to (3.7) with S_{ab} as in (3.9), we get

$$\Phi_{ab}(\mathbf{k}, \tau) = k_b k_c \int A_{ac}(x) f''(v)\, d_3x - k_a k_c \int A_{bc} f''(v)\, d_3x. \tag{3.16}$$

We have now the distant field as in (3.7), and we can verify immediately that that field is a null field in the sense that $E = H$ and $\mathbf{E}\cdot\mathbf{H} = 0$. For it follows from (3.16) that the principal part of the field satisfies

$$F_{ab}k_b = 0, \qquad F^*_{ab}k_b = 0, \tag{3.17}$$

F^*_{ab} being the conjugate electromagnetic tensor, and these conditions tell us that the field is null.

Finally, in order to get a jet of radiation in the direction $\mathbf{k}^*$, let us make the following choice of the source-tensor:

$$\begin{gathered} S_{ab}(\mathbf{x}, it) = A_{ab}(\mathbf{x}) f(u), \\ A_{ab}(\mathbf{x}) = B_{ab} \exp(-r^2/a^2), \qquad f(u) = \cos \omega u, \\ u = t - k^*_\sigma x_\sigma - \tau^* = -k^*_a x_a - \tau^*. \end{gathered} \tag{3.18}$$

Here $B_{ab}(= -B_{ba})$ are constants. Then by (3.16) the distant field is given by

$$\lim (-4\pi R F_{ab}(\mathbf{X}, iT)) = \Phi_{ab}(\mathbf{k}, \tau) = C_{ab} \cos \omega(\tau - \tau^*), \tag{3.19}$$

the tensor amplitude being

$$C_{ab} = \omega^2(k_a k_c B_{bc} - k_b k_c B_{ac})(a\pi^{1/2})^3 \exp(-a^2\omega^2 \sin^2 \tfrac{1}{2}\theta). \tag{3.20}$$

The exponential factor suggests that we may obtain a concentrated jet of radiation by making $a\omega$ large. However there remains another factor dependent on the direction of observation, and its effect must be explored. We shall pursue the investigation in terms of flux of energy, which is more interesting than amplitude.

The energy tensor is

$$T_{ab} = F_{ac}F_{bc} - \delta_{ab}F_{cd}F_{cd}/4, \tag{3.21}$$

and the 3-vector representing flux of energy is

$$-iT_{\sigma 4} = -iF_{\sigma c}F_{4c}\,. \tag{3.22}$$

Then by (3.19) the amount of energy passing through solid angle $d\Omega$ on a large sphere in time $dT(= d\tau)$ is

$$-i(16\pi^2)^{-1} k_\sigma \Phi_{\sigma c}\Phi_{4c}\, d\Omega\, d\tau, \tag{3.23}$$

k_σ being the unit normal to the sphere. The time-average of the flux is then

$$-i(32\pi^2)^{-1} k_\sigma C_{\sigma c} C_{4c}\, d\Omega. \tag{3.24}$$

Define

$$D_{ab} = k_a k_c B_{bc} - k_b k_c B_{ac}\,. \tag{3.25}$$

Then, since B_{ab} is skew and k_a is null,

$$D_{ac}D_{bc} = k_a k_b B_{cp} B_{cq} k_p k_q\,. \tag{3.26}$$

Hence, since $k_4 = i$,

$$k_\sigma D_{\sigma c} D_{4c} = iB_{cp}B_{cq}k_pk_q . \tag{3.27}$$

Applying this result to (3.24), we see that the time-average of flux is

$$\omega^4(32\pi^2)^{-1}(\pi a^2)^3 B_{cp}B_{cq}k_pk_q \exp\left(-2a^2\omega^2 \sin^2 \tfrac{1}{2}\theta\right) d\Omega. \tag{3.28}$$

So far we have left the constants B_{ab} arbitrary except for skewness. If they happen to be such that

$$B_{cp}B_{cq}k_p^*k_q^* = 0, \tag{3.29}$$

the flux (3.28) is zero in the target direction, and we do not have the desired jet, but rather a jet with a hole in the middle.

To explore the question of appropriate choice of B_{ab}, let us define

$$G = B_{cp}B_{cq}k_pk_q , \qquad G^* = B_{cp}B_{cq}k_p^*k_q^* , \tag{3.30}$$

so that, to avoid a hole in the middle of the jet, we seek to have $G^* \neq 0$. Rotate the coordinate axes so that Ox_3 points in the target direction; then

$$k_1^* = k_2^* = 0, \qquad k_3^* = 1, \qquad k_4^* = i, \tag{3.31}$$

and (3.30) gives

$$G^* = (B_{13} + iB_{14})^2 + (B_{23} + iB_{24})^2. \tag{3.32}$$

It is clear that we have a wide choice. Let us take a simple one:

$$B_{ab} = 0 \quad \text{except} \quad B_{14} = -B_{41} = -iP, \tag{3.33}$$

where P is some positive constant. This makes $G^* = P^2 \neq 0$, as required. Then (3.30) gives

$$G = B_{1p}B_{1q}k_pk_q + B_{4p}B_{4q}k_pk_q = P^2(1 - k_1^2) = P^2(1 - \sin^2\theta\cos^2\phi) \tag{3.34}$$

in terms of spherical polar angles. By (3.28) we have then for the time-average of energy flux per unit solid angle in the direction (θ, ϕ)

$$\omega^4(32\pi^2)^{-1}(\pi a^2)^3 P^2(1 - \sin^2\theta\cos^2\phi) \exp\left(-2a^2\omega^2\sin^2\tfrac{1}{2}\theta\right). \tag{3.35}$$

If we define the directivity D as the ratio of the target value of this (viz., for $\theta = 0$) to its mean value over the unit sphere, we get, as principal part for large $a\omega$,

$$D = 2a^2\omega^2 \tag{3.36}$$

[cf. (2.19), (2.28)].

To find the 4-current J_a which generates this jet, we have by (3.4) and (3.18)

$$J_a = S_{ab,b} = A_{ab,b}f(u) - A_{ab}k_b^{*\prime}f(u), \tag{3.37}$$

where $A_{ab} = 0$ except, by (3.18) and (3.33),

$$A_{14} = -A_{41} = -iP\exp(-r^2/a^2), \tag{3.38}$$

and k_a^* is as in (3.31). Thus the 3-current is

$$\begin{aligned} J_1 &= -iA_{14}f'(u) = \omega P \exp(-r^2/a^2)\sin\omega u, \\ J_2 &= J_3 = 0, \end{aligned} \tag{3.39}$$

and the charge density is

$$\rho = -iJ_4 = -iA_{41,1}f(u) = -2P(x_1/a^2) \exp(-r^2/a^2) \cos \omega u. \tag{3.40}$$

The total charge is zero, as is in fact implied by (3.4) combined with the vanishing of S_{ab} at spatial infinity. In these formulae, u is as in (3.18).

4. Gravitational radiation. We work in the linearised form of Einstein's general relativity. The field is weak in the sense that the metric tensor may be written

$$g_{ab} = \delta_{ab} + \gamma_{ab} ,$$

where the γ's and their derivatives are small and quadratic terms are in consequence dropped. The validity of this approximation is open to question if very high frequencies are involved, but this matter will not be further discussed here. We shall simply accept the linear approximation as is usually done.

Actually the metric tensor will not appear in the work. Instead, the Riemann tensor will be used. This has two advantages. First, the Riemann tensor is the core of a gravitational field, vanishing if and only if the field vanishes. Second, we do not have to consider infinitesimal coordinate transformations, since any such transformation produces only a second order change in the Riemann tensor, which is already small of the first order.

The Riemann tensor R_{abcd} satisfies the symmetry conditions

$$R_{abcd} = R_{cdab} = -R_{bacd} , \qquad R_{abcd} + R_{acdb} + R_{adbc} = 0, \tag{4.1}$$

and also the Bianchi identities, which read, in the linearised theory,

$$R_{abcd,e} + R_{abde,c} + R_{abec,d} = 0. \tag{4.2}$$

Differentiation of this last equation with respect to x_e gives

$$\Box R_{abcd} = R_{abcd,ee} = -R_{abde,ce} - R_{abec,de} . \tag{4.3}$$

By use of (4.1) and (4.2), it is not hard to reduce this equation to

$$\Box R_{abcd} = R_{ad,bc} + R_{bc,ad} - R_{ac,bd} - R_{bd,ac} , \tag{4.4}$$

where

$$R_{ab} = R_{cabc} , \tag{4.5}$$

the linearised form of the Ricci tensor.

So far there is no mention of the field equations. We have in (4.4) merely an identity in the linearised geometry of any Riemannian space.

The accurate form of Einstein's field equations is

$$R_{ab} - \tfrac{1}{2}g_{ab}R = -8\pi T_{ab} \tag{4.6}$$

in units making the gravitational constant and the speed of light both unity. Here T_{ab} is the energy tensor. In linearised form (4.6) may be written

$$R_{ab} = -8\pi(T_{ab} - \tfrac{1}{2}\delta_{ab}T_{cc}). \tag{4.7}$$

Substituting (4.7) in (4.4), we get our basic differential equation, by which the Riemann tensor is generated from a source represented by the energy tensor:

$$\Box R_{abcd} = -8\pi \Delta_{abcd}^{pqrs} T_{pq,rs} \tag{4.8}$$

where the 8-index symbol is a combination of Kronecker deltas, easy to write out explicitly but rather complicated.

Since the covariant divergence of the left-hand side of (4.6) vanishes identically, T_{ab} satisfies an equation which in general involves the metric tensor. But in the linearised theory this constraint reduces to

$$T_{ab,b} = 0. \tag{4.9}$$

Thus, as in the electromagnetic case, we have a source subject to a constraint. As in that case, it is more convenient to use a free source, and so we shall introduce (this is a well-known device) a tensor S_{abcd} which may be freely chosen except for conditions of vanishing sufficiently rapidly at spatial infinity and the symmetry conditions

$$S_{abcd} = S_{cdab} = -S_{bacd} . \tag{4.10}$$

We shall restrict ourselves to energy tensors of the form

$$T_{ab} = S_{acbd,cd} . \tag{4.11}$$

This is symmetric in (a, b) and automatically satisfies the constraint (4.9).

Substituting from (4.11), we obtain from (4.8) the equation

$$\Box R_{abcd} = -8\pi\Delta^{pqrs}_{abcd}S_{puqv,uvrs} . \tag{4.12}$$

This is analogous to (2.1) and (3.5), and may be treated similarly. Thus instead of (2.8) we now have for the distant field

$$\lim \left(\tfrac{1}{2}\,|\mathbf{X}|\,R_{abcd}(\mathbf{X}, iT)\right) = \Phi_{abcd}(\mathbf{k}, \tau) \tag{4.13}$$

where

$$\Phi_{abcd}(\mathbf{k}, \tau) = \Delta^{pqrs}_{abcd} \int S_{puqv,uvrs}\, d_3x, \tag{4.14}$$

with integration over all space, the substitution

$$t = k_\sigma x_\sigma + \tau \tag{4.15}$$

being made in the integrand after differentiation and before integration.

To get a jet in the direction $\mathbf{k}^*$, we follow the pattern of the preceding section, taking

$$S_{puqv}(\mathbf{x}, it) = A_{puqv}(\mathbf{x})f(u), \qquad u = t - k^*_\sigma x_\sigma - \tau^* = -k^*_a x_a - \tau^*, \tag{4.16}$$

so that $u_{,a} = -k^*_a$. The A-tensor is to have the symmetries (4.10) and vanish sufficiently rapidly at spatial infinity. To prepare the integrand in (4.14), we are to differentiate four times and then substitute v for u, where

$$v = \kappa_\sigma x_\sigma + \mu, \qquad \kappa_a = k_a - k^*_a , \qquad \mu = \tau - \tau^*. \tag{4.17}$$

This situation is an elaboration of that discussed at (3.12) and the same plan may be followed, viz., integration by parts. Thus we obtain, as analogue of (3.16),

$$\Phi_{abcd}(\mathbf{k}, \tau) = \Delta^{pqrs}_{abcd}k_u k_v k_r k_s \int A_{puqv}(\mathbf{x})f^{(4)}(v)\, d_3x. \tag{4.18}$$

It remains only to make special choice of the A-tensor and the function f. Following the precedents, we set

$$A_{abcd}(x) = B_{abcd} \exp(-r^2/a^2), \qquad f(u) = \cos \omega u, \tag{4.19}$$

where the B-tensor is constant with the symmetries (4.10). Then the distant field is given by (4.13) with

$$\Phi_{abcd}(\mathbf{k}, \tau) = \omega^4 (a\pi^{\frac{1}{2}})^3 \Delta^{pqrs}_{abcd} B_{puqv} k_u k_v k_r k_s \exp(-a^2\omega^2 \sin^2 \tfrac{1}{2}\theta) \cos \omega(\tau - \tau^*), \tag{4.20}$$

θ being as always the angle between observation and target. To get a jet we are to take $a\omega$ large.

It will be noted that for all three types of disturbance—scalar, electromagnetic, gravitational—the jet is obtained by essentially the same method: the source consists of simple harmonic plane waves travelling towards the target with the fundamental speed of propagation and with wave length small compared with the size of the domain of the disturbance (a/λ large). The assumption that the amplitude of these waves falls off exponentially like $\exp(-r^2/a^2)$ is probably not essential to the result; it is rather a device to make the calculations easy to carry out.

5. Shock-jets. Let us now see whether it is possible to obtain a jet without the use of a high-frequency source. In this connection I am much indebted to Mr. R. G. Medhurst for correspondence about the theory of supergain in electromagnetic aerial arrays as developed by Bloch, Medhurst and Pool [3], [4] (the second reference contains a bibliography). Their work is oriented towards the practical design of arrays and differs greatly in point of view from the present paper. However, it appears that supergain is attainable without recourse to high frequency, and this raises the question as to whether high frequency is essential to the production of jets of radiation when discussed in terms of field theory in the manner of the present paper. It will now be shown that jets can be obtained from a source which has no frequency at all, but consists of a single shock wave.

For purposes of illustration, only a scalar field will be considered.

Let us return to (2.9) and note that, by introducing the Dirac δ-function, we may write that equation in the form

$$\Phi(\mathbf{k}, \tau) = \int S(\mathbf{x}, it)\, \delta(t - k_\sigma x_\sigma - \tau)\, d_4x, \tag{5.1}$$

the integration being now over all space-time. For source let us take

$$S(\mathbf{x}, it) = \exp(-r^2/a^2)\, \delta(t - k^*_\sigma x_\sigma - \tau^*), \tag{5.2}$$

$\mathbf{k}^*$ being as earlier the unit vector in the target direction and τ^* a constant. We are to substitute (5.2) in (5.1) and carry out a fourfold integration. But, by a known formula for δ-functions,

$$\int_{-\infty}^{\infty} \delta(t - k_\sigma x_\sigma - \tau)\, \delta(t - k^*_\sigma x_\sigma - \tau^*)\, dt = \delta(\kappa_\sigma x_\sigma + \mu), \tag{5.3}$$

in the notation of (2.11) and (2.16). Thus (5.1) becomes

$$\Phi(\mathbf{k}, \tau) = \int \exp(-r^2/a^2)\, \delta(\kappa_\sigma x_\sigma + \mu)\, d_3x. \tag{5.4}$$

Suppose now that $\kappa = 0$, which means that observation is made in the target direction. We get

$$\Phi(\mathbf{k}^*, \tau) = (a\pi^{1/2})^3\, \delta(\tau - \tau^*), \tag{5.5}$$

which means that an infinite disturbance arrives at time T corresponding to the retarded time τ^*.

On the other hand, if $\kappa \neq 0$, we can rotate the axes so that one axis points in the direction of κ_σ, and then we have

$$\Phi(\mathbf{k}, \tau) = (a\pi^{1/2})^2 \int_{-\infty}^{\infty} \exp(-x^2/a^2)\ \delta(\kappa x + \mu)\, dx = (a\pi^{1/2})^2\kappa^{-1} \exp(-\kappa^{-2}a^{-2}\mu^2). \quad (5.6)$$

This expression is finite, and so, comparing it with (5.5), we see that we have a concentrated jet in the target direction.

We may assess directivity here by calculating the integral

$$I(\mathbf{k}) = \int_{-\infty}^{\infty} [\Phi(\mathbf{k}, \tau)]^2\, d\tau. \quad (5.7)$$

We find, since $\mu = \tau - \tau^*$,

$$I(\mathbf{k}) = (a\pi^{1/2})^5\kappa^{-1}/\sqrt{2}, \quad (5.8)$$

which tends to infinity like κ^{-1} as we make the direction of observation $\mathbf{k}$ approach the target direction $\mathbf{k}^*$.

6. Cautionary remarks. No mathematical model represents physical reality perfectly, and reasonable caution is always required in the application of mathematical formulae. However, some special cautionary remarks about the preceding results are advisable.

We have been concerned essentially with *linear* field theory. Thus, although the equations of hydrodynamics are nonlinear, we have supposed those equations linearised in dealing with jets of sound. Such linearisation is a standard procedure. It is based on the assumption that the field variables and their derivatives are all small of the same order. The validity of this procedure becomes dubious when high frequencies are involved, and it is certainly not permissible to imagine the frequency tending to infinity. Thus, while it is possible, by increasing the frequency, to obtain in the mathematical model a jet of any desired degree of concentration, a practical limit is imposed by the requirement of remaining within the linear theory. No attempt is made here to assess what the limit is.

The above difficulty about linearisation does not arise in the case of electromagnetic radiation based on Maxwell's equations, because those equations are linear. But it does arise in the case of gravitational waves because we are using the linearised form of a nonlinear theory. Here also excessively high frequency may carry us out of the domain of validity of the linear theory.

There is also another warning in the case of gravitational waves. This arises from the physical fact that density cannot be negative. In the above work, in which imaginary time is used, density is represented by $-T_{44}$. Now when we combine (4.11) with the assumption that $S_{abcd} = 0$ at spatial infinity, we find that T_{44} cannot have a single sign everywhere. In fact, the source has negative density somewhere, and this is not permissible. The way out of this difficulty is to think of the field in question as superimposed on some given statical field, weak enough to fit into the linearised theory but strong enough to balance the negative density arising from (4.11). Once again, however, high frequencies are dangerous, not merely because they may carry us out of the domain of the linearised theory, but because they may give rise to densities too large and negative to be compensated by the background statical field.

As for Sec. 5, the use of the δ-function in (5.1) is above reproach, for this is merely an equivalent way of writing the retarded potential integral (2.9). But the δ-function in (5.2) is a very different matter, for here we contemplate a source which takes infinite values. Thus the results of Sec. 5 need a commonsense interpretation before they are applied to actual physical situations.

References

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