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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime (′), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{\ }$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t\,(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372 (1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

CONTENTS

—BOOK REVIEW SECTION—

Research papers in statistics. Festschrift for J. Neyman. F. N. David, Editor, assisted by Evelyn Fix. John Wiley & Sons, London, New York, Sydney, 1966. viii + 468 pp. $16.75.

In this Festschrift presented to Jerzy Neyman by twenty-six distinguished mathematicians and statisticians, the contributions range widely over theoretical and applied probability and statistics.

E. S. Pearson in an introductory essay describes his collaborative work with Neyman during the period 1926–1934.

The contributors of technical articles are M. S. Bartlett, D. E. Barton, J. Berkson, D. R. Cox, H. Cramér, F. N. David, D. Dugué, E. Fix, R. Fortet, U. Grenander, J. M. Hammersley, K. Ito, D. G. Kendall, L. LeCam, P. Lévy, P. A. P. Moran, T. Page, G. Pólya, C. R. Rao, A. Rényi, S. Rios, L. Schmetterer, H. Solomon, C. Stein, P. V. Sukhatme, I. Yañez, and H. Wold.

J. L. Hodges, jr. (*Berkeley, Calif.*)

Stochastic stability and control. By Harold J. Kushner. Academic Press, New York and London, 1967. xiv + 161 pp. $7.50.

The approach to stochastic stability in this monograph is similar to the Liapunov function method for stability problems in ordinary differential equations. The problem considered in the first half of the book is the stability (in one of various possible senses) of the trajectories of an autonomous Markov process x_t as $t \to \infty$. The role of the Liapunov function is taken by some positive function V such that the process $V(x_t)$ is a supermartingale. The stability results are then deduced from standard theorems about supermartingales, by judicious choices for V. In the part concerning optimal control it is shown that in various situations a solution V of the dynamic programming principle of optimality leads to an optimal control process.

There are five chapters, as follows. I. Background concerning Markov processes: strong Markov property, Ito processes, Poisson driven processes, martingales, etc. II. Stochastic stability: various notions of stability, the stochastic Liapunov function approach, stability theorems, examples (especially scalar processes and some 2-dimensional Markov processes). In many instances the choice of V is suggested by the deterministic theory (see J. P. LaSalle and S. Lefschetz, Stability by Liapunov's Direct Method, with Applications, Academic Press, 1961). The following example is illustrative. Let the scalar process x_t be governed by $dx = ax\,dt + \sigma(x)\,dz$, where z is a Brownian motion, $a > 0$, $\sigma(0) = 0$, σ' is bounded. By taking $V(x) = \exp \lambda\,|x|^\alpha$ for suitable λ and α, one can show that, with probability 1, $x_t \to 0$ as $t \to +\infty$. III. Finite time stability and first-exit times. Explicit upper bounds are obtained for $P_x\{\sup_{0\le t\le T} V(x_t) \ge m\}$ or for the probability that x_t leaves a given region in time T. These are of interest in applications to tracking and control systems. IV. Optimal stochastic control. Families of continuous-parameter Markov processes are considered, depending on a control function u. The control must be determined to minimize the expected value

$$C^u(x) = E_x^u b(x_{\tau_u}) + E_x^u \int_0^{\tau_u} k(x_s\,, u_s)\,ds$$

where u_s is the control applied at time s, x_s the state of the Markov process at time s, $x_0 = x$, and τ_u the time of arrival at a given target set S. Assuming that the state x_s is completely observable at each time s, a dynamic programming method can be applied. Sufficient conditions for a minimum are given. These amount to the existence of a smooth solution V of the principle of optimality satisfying some additional condition. For instance, when $b \equiv 0$ such a condition is the existence of a monotone sequence τ_i tending to τ_u with $V(x_t) \le i$ for $0 \le t \le \tau_i$ such that $E_x^u V(x_{\tau_i}) \to 0$ as $i \to \infty$. The important example of the fixed time, linear system, quadratic loss problem is worked out, including a treatment of the combined filtering and optimization problem when the states are partially observable. V. Some observations about applying Liapunov function methods to the design of controls.

W. Fleming (*Providence, R.I.*)

Éléments de théorie des matrices carrées et rectangles en analyse numérique. By André Korganoff and Monica Pavel-Parvu. Dunod, Paris, 1967. xx + 441 pp. $20.05.

This book is an excellent survey of two topics: *Normed algebras of matrices* and (pseudo, generalized) *inverses of* (retangular) *matrices* which are central to much of the recent developments in matrix theory and applications and in numerical analysis. It is the first (and at the time of writing the only) comprehensive study of such inverses—which are mostly used like prose by Molière's bourgeois gentilhomme, yet are unavoidable in some applications and lead to considerable economy and simplification in many others.

The book consists of two parts with separate bibliographies.

Part I titled: *Vector spaces and normal algebras of matrices* consists of 3 chapters.

Chapter 1.1 (59 pages) is an adequate review of the prerequisites in functional analysis. Chapter 1.2 (17 pages) deals adequately with Hölder matrix norms. Chapter 1.3 (71 pages) is a study of matrix norms associated with vector norms, spectral theory, localization theorems and extremal properties of eigenvalues, matrix functions and approximations. The treatment is detailed and pedantic, emphasizing algebraic and analytic aspects but omitting geometry and much of the material in chapters 2, 3 of Householder's matrix book.

The bibliography with 105 references is adequate.

Part II titled *Inverses of rectangular matrices* consists of 4 chapters.

Chapter 2.1 (60 pages) studies inverse semigroups and the algebraic theory of pseudo inverses.

Chapter 2.2 (38 pages) applies pseudo inverses to the solution of matrix equations, ill-conditioned systems and discretized boundary-value problems.

Chapter 2.3 (56 pages) deals with applications of pseudo inverses to various approximation problems.

Chapter 2.4 (107 pages) is a thorough study of the computational methods of generalized inversion.

The bibliography of 200 references is complete and up to date.

In summary: this well written book is a valuable sourcebook for (graduate) students, teachers and practicioners of numerical analysis and applied matrix methods. The authors should be complimented for writing an authoritative survey of a continuously developing field.

ADI BEN-ISRAEL (*Evanston, Ill.*)

Commutation properties of Hilbert space operators and related topics. By C. R. Putnam. Springer-Verlag New York Inc., New York, 1967. xi + 167 pp. $7.00.

This short monograph provides an interesting and well-written treatment of the theory of commutators $AB-BA$ of linear operators A, B on a Hilbert space and related topics. This subject originated in quantum mechanics where the question of the equivalence of the formal transformation theory with the Schrödinger wave equation led to the problem of proving the essential uniqueness of self-adjoint operators Q and P satisfying (in some appropriate sense) the relation $QP - PQ = i$. While this problem has never been given a truly satisfactory solution from the viewpoint of physics, a fairly broad theory has developed over the years.

Chapter I is concerned with commutators of bounded operators, for which a reasonably satisfactory theory exists. Chapter II discusses the spectral properties of commutators. Chapter III is concerned with the spectrum of semi-normal operators (a bounded operator T is called semi-normal if $TT^* - T^*T = D$, $D \geq 0$ or $D \leq 0$). Chapter IV is concerned with the canonical commutation and anticommutation relations in quantum mechanics. Mathematical physicists should find this chapter interesting reading since it provides the most complete discussion available of the mathematical aspects of the problem of the representations of the commutation relations for systems of a finite number of degrees of freedom. There is also a brief treatment of systems with infinitely many degrees of freedom. Chapter V is concerned with another quantum mechanical problem, the existence and properties of wave operators and the S matrix in scattering theory. In Chapter VI some applications of the preceding material to Laurent and Toeplitz operators, singular integral operators, and Jacobi matrices is given.

E. J. WOODS (*College Park, Md.*)

Computation: finite and infinite machines. By Marvin L. Minsky. Prentice–Hall, Inc., Englewood Cliffs, N. J., 1967. xvii + 317 pp. $12.00.

This is an excellent book and it fills a gap in the current computer science literature. It provides an introduction to automata theory from the several points of view that historically mark the development of this subject. Moreover, these various approaches are related, and their relation to real (i.e., actual) digital computing machines is stressed. From the beginner's viewpoint this subject often suffers from the lack of contact with computers, with which the usual novice in automata theory is already familiar. One hesitates to use the word "introductory" for any book because the term has come to represent such an extreme range of levels of presentation. Suffice it to say that the book seems to correspond to the author's description of level. That is, knowledge of functional notation, mathematical induction and a modest mathematical sophistication do, indeed, seem to be adequate to read and understand the material.

The book is organized in three sections: Finite State Machines, Infinite Machines, and Symbol-Manipulation Systems and Symbol-Manipulation Systems and Computability. The first of these sections covers the state-transition approach, neural networks (McCulloch and Pitts), and the relation of regular expression to finite automata. The second covers computability, Turing machines, recursive functions, and computer-like "program" machines. The final section deals with Post productions, proof procedures, and the so-called "tag" systems as a base for computability.

Minsky writes in an informal manner with many parenthetical comments (and warnings)—a style which does make the book easily readable. The reader is usually warned when an argument is about to become involved. The proof of Kleene's theorem on the correspondence of finite automata and the recognition of regular expressions was one instance. The final chapter which included the "tag" systems, the unsolvability of Post's "Correspondence Problem" and the tag-system derived universal Turing machines is another instance.

There is little to criticize in this volume. There are many problems with solutions and they are presented in the text where, if worked, they facilitate the further reading at that point. The only minor matter that can be raised is perhaps the one of relative emphasis. It seems that the neural network approach is pursued in more detail than its current importance warrants while the consideration of Markov algorithms is not included.

BRUCE W. ARDEN (*Ann Arbor, Mich.*)

Theory of linear active networks. By Ernest S. Kuh and R. A. Rohrer. Holden-Day, Inc., San Francisco, Cambridge, London, Amsterdam, 1967. xii + 650 pp. $17.50.

This book represents a successful attempt to develop a theory of active networks in a coherent and highly teachable form. Since the classic work of Bode ("Network Analysis and Feedback Amplifier Design") no book has succeeded at developing such a theory. Professors Kuh and Rohrer have now filled this vacancy.

The first five chapters of the book are devoted to presenting a study of network behavior with respect to such properties as passivity, activity, and generativity. Chapter 5 is an examination of these and related concepts for two port networks and is particularly well done. The earlier chapters tend to be somewhat repetitious, mainly because the authors have attempted to make each chapter as self-contained as possible. In the opinion of the reviewer, this goal is a mistake since it makes the first few chapters rather difficult to teach and rather boring to read.

In Chapter 6, the authors give an excellent treatment of the scattering matrix. The scattering parameters are first defined and then physically interpreted. All of the material is clearly presented and is used in subsequent chapters. Chapter 7 develops the classical material of Bode and Fano. This material, which explores fundamental limitations on network behavior, is presented in a highly readable form. The following chapter discussed network limitations and matching theory and is based on the modern work of Youla. This theory both complements the earlier work and presents a far more complete theory than the preceding classical material. Chapters 9 and 10 apply the results of Chapters 7 and 8 to study the network limitations and matching problem for negative resistance amplifiers and parametric circuits, respectively. In Chapter 9, a theory for negative resistance amplifiers is given and applied to tunnel diode amplifiers. This chapter is one of the finest in the book. The general theory developed in the preceding chapters is used and extended. This results in a highly satisfying treatment of a most interesting

and difficult problem. Chapter 10 is not quite as satisfying as Chapter 9. Its main shortcoming is its brevity. The subject of parametric amplifiers deserves a much deeper treatment than the one given here.

The remainder of the book (Chapters 11 and 12) give a brief but excellent introduction to feedback amplifier theory. Return differences in single and multiple loop feedback systems are discussed as well as their relation to stability, sensitivity, and other network criteria. The single loop feedback amplifier is treated in detail in Chapter 11 while some generalizations to multiple loop feedback systems are given in Chapter 12. Chapters 11 and 12 are totally independent of the earlier parts of the book and could easily have been omitted without creating any obvious "incompleteness". These chapters should, by all rights, be in a different book with a title such as "Theory of Linear Feedback Amplifiers". However, much of the material has not appeared in other books and since this material is important, these chapters will have a very useful educational function.

In summary, this reviewer highly recommends the book as a text for any course devoted to studying the fundamental principles of linear active circuits. The book is well written and easily accessible to first year graduate students. It contains a large amount of material never before published in book form. This new material is carefully integrated with classical results to give a unified treatment of active networks.

H. Frank (*Berkeley, Calif.*)

Functional analysis and optimization. Edited by E. R. Caianiello. Academic Press, Inc., New York, London, 1966. xiii + 225 pp. $11.50.

This book is a collection of 16 papers presented at the 7th International School of Ravello, June 1965.

The stated aim of the editor is modest enough, namely, to show that functional analysis is very useful in the study of control problems; and this aim is fulfilled (sometimes brilliantly in certain papers). However, the dust-jacket claim that the book is "an up-to-date review of the most important topics in functional analysis and optimization" is highly arguable. For example, none of the papers in the book mentions applications of the Hahn–Banach Theorem or the Principle of Uniform Boundedness which are central to a discussion of certain elementary but basic optimization and stability problems respectively.

Among the topics treated in the collection are the theory of controllability and observability, control problems involving systems described by partial differential equations, control of diffusion processes, applications of convexity to control, computational aspects of control problems, and aspects of nonlinear system theory.

Additional topics include nondistributive algebras (discussed in a long paper whose connection with control theory is highly tenuous despite the title), decision equations, and suboptimal supervisory control (the connection of the latter two papers to functional analysis being equally tenuous).

The book should prove useful, as a reference, for anyone interested in the mathematical theory of control.

Leonard Weiss (*Berkeley, Calif.*)

Optimal and self-optimizing control. Edited by Rufus Oldenburger. The M. I. T. Press, Cambridge, Mass. and London, 1966. x + 500 pp. $6.00.

The book is a collection of thirty-eight papers on control together with an introductory survey chapter by the editor. There is a bibliography containing 321 references through 1964. Both the collection of papers and the list of references are quite specialized, as considerable attention is paid to "on-off" regulators while many fundamental papers are not even listed in the bibliography. For example, the basic filtering paper of Kalman and Bucy is not mentioned nor is there any inclusion of papers on computational methods. Little reference is made to the work of Bellman and there is almost no mention of existing results. On the whole, it seems that a reader interested in an overview of current control theory might better consult the survey papers by M. Athans, H. J. Kushner and R. W. Brockett in the IEEE International Convention Record (Volume 14).

P. L. Falb (*Providence, R.I.*)

Spectral analysis of time series. Edited by Bernard Harris. Proceedings of an Advanced Seminar conducted by the Mathematics Research Center, U. S. Army and the Statistics Department at the University of Wisconsin, Madison, October 3–5, 1966. John Wiley & Sons, Inc., New York, London, Sydney, 1967. x + 319 pp. $7.95.

It is not easy to review a book of this kind, which consists essentially of a set of 10 papers setting out the results of current research in the field, by leading research workers, together with an introductory article by the editor. Except for the two (joint) papers by Brillinger and Rosenblatt the separate works are not related save for their concern with statistical problems of time series analysis nor do they necessarily emphasise the most important aspects of the subject. Thus the book is of interest primarily to people of the same kind as the authors.

The papers fall into roughly three classes. The first are the review type papers, amongst which will be included Tukey's historical review, with its emphasis reflecting the author's central place and his opinion as to the important problems. Amongst these review papers could also be included Panofsky's on meteorological applications and a very clearly written and attractively presented article by Box, Jenkins and Bacon on the estimation of mixed autoregressive moving average type models (generalized to take account of the differencing operator) and their use for prediction. The second class are those concerned with a (new) precise mathematical investigation of the properties of some statistics. The papers by Brillinger and Rosenblatt deal with the estimation of the higher order spectra (polyspectra) associated with the cumulants of a vector valued, strictly stationary time series. Under certain conditions (principally on the rate of convergence of these cumulants to zero as the time lags increase) they show that estimates constructed in a natural fashion from finite Fourier transforms of the data are asymptotically unbiased and normal with covariances zero for polyspectra of different orders and otherwise with covariances depending only on the polyspectra of order up to that estimated. While one can doubt whether these higher order spectra will be very useful (one is reminded of Karl Pearson and the method of moments), a precise investigation of their properties is necessary and these papers by Brillinger and Rosenblatt are thus important. In this second class also falls a paper by Zaremba dealing with certain quartic statistics and in particular the Cesàro sum of the squares of the autocovariances, which he uses for testing for the departure of the spectral distribution function from absolute continuity. The author introduces these to avoid the introduction of assumptions about the smoothness of the spectrum on the null hypothesis and a consequent use of "prior knowledge of a quantitative nature". It is not quite apparent how much is being achieved in this direction and the statistic looks very unsuitable (from the point of view of power) for the situation where an apparent periodicity has been found from examination of the data. However, the derivations involved in the proof of the asymptotic properties are of considerable interest because of the range of techniques used.

The third class of papers deals with new computational techniques. There are useful papers by Akaike (dealing with the estimation and identification of systems with feedback) and by Tick (stressing the point, made at an earlier time by Akaike, that a rapidly changing phase makes the estimation of coherence difficult, together with suggestions for overcoming the problem). Parzen's paper deals, very interestingly, with the analysis of regression type models when the variable regressed upon is measured with white noise added. There is, finally, a paper by Godfrey on prediction for non-stationary processes.

E. J. Hannan (*Canberra, Australia*)

Mechanik der festen Körper. By H. Parkus. 2nd edition. Springer-Verlag, Wien, New York, 1966. X + 368 pp. $12.25.

The fact that a second edition has become necessary so soon after the publication of the first demonstrates the appeal of this well-known work. Additions or changes made in the present edition concern the dynamics of systems with variable mass, yield conditions, lateral instability of open sections, buckling of compressed circular plates, snap-through of a two-bar truss, and the Krylov–Bogolivbov method in the theory of nonlinear vibrations.

W. Prager (*La Jolla, Calif.*)

An introduction to magnetohydrodynamics. By P. H. Roberts. American Elsevier Publishing Co., New York, 1967. x + 264 pp. $11.00.

Although it is only under extreme conditions that magnetic fields seriously affect the flow of electrically conducting fluids in the laboratory, "magnetohydrodynamic" (or "hydromagnetic") effects are of central importance in the work of the astrophysicist or geophysicist interested in the origin of stellar, interstellar, solar and planetary magnetic fields, and in related dynamical problems.

As early as 1919 Larmor suggested that the Sun's magnetic field may be caused by fluid motions occurring in the body of the Sun, but his specific model of the process was subsequently shown by Cowling (in 1933) to be impossible. More recent investigations, due to Bullard, Elsasser, and other workers and including the "existence proofs" of Backus and Herzenberg, demonstrate that the "homogeneous dynamo mechanism" for generating magnetic fields by fluid motions in the general manner visualized by Larmor is possible if the fluid motions are sufficiently complicated.

Theoretical work in magnetohydrodynamics involves the simultaneous solution of Euler's equations of hydrodynamics, with the ($\mathbf{j} \times \mathbf{B}$) term included in the body force ($\mathbf{j}$ being the electric current density and $\mathbf{B}$ the magnetic field), and Maxwell's equations of electromagnetism, with motional induction term ($\mathbf{u} \times \mathbf{B}$) included in the equation for the electric field, where $\mathbf{u}$ is the Eulerian flow velocity vector. The "dynamo problem" is so difficult that in order to make progress it has usually been necessary to regard $\mathbf{u}$ as given and thus avoid having to solve the hydrodynamical equations. The systematic study of the *full* equations of magnetohydrodynamics began seriously around 1940 with the work of Alfvén, his most famous study being that of magnetohydrodynamic, or Alfvén, waves. These he discovered by considering small-amplitude disturbances about a state of no motion of a perfectly conducting fluid of indefinite extent in all directions, pervaded by a uniform magnetic field, the simplest conceivable theoretical study in the subject which nevertheless has far-reaching results.

Several books and extensive review articles have appeared on magnetohydrodynamics in the past decade. In contrast to these works, Professor Roberts's book (which is based on a post-graduate (M.Sc.) course held in the Department of Mathematics of the University of Newcastle upon Tyne and contains student problems at the end of each chapter) devotes considerable attention to the dynamo problem (Chapter 3). Some of the most up-to-date work is included, but it is unfortunate that the recent work of Braginskii, which gives a mathematical basis for the important earlier work of Parker, was omitted. This omission will doubtless be rectified in future editions.

The first two chapters of the book deal systematically with the equations of magnetohydrodynamics, boundary conditions and the effect on magnetic fields of specified fields of fluid motion. Chapter 4 discusses systems for which either $\mathbf{u} = 0$ (magnetohydrostatics), such as the "linear pinch" and other situations in which a conducting plasma is (theoretically) confined by "magnetic bottles", or $\mathbf{j} \times \mathbf{B} = 0$ (force-free magnetic fields).

Chapter 5 is devoted to magnetohydrodynamic waves and includes several developments, due to various workers, of Alfvén's original work to include effects of viscous and Ohmic dissipation and the presence of boundaries.

Chapter 6 is a brief account of certain simple magnetohydrodynamic boundary-layer flows, including boundary layers that arise in the Hartmann flow along a pipe to which Poiseuille flow gives rise in the presence of a transverse magnetic field. Further details of Hartmann flows are treated at length in the first part of Chapter 7 on "the hydromagnetics of the laboratory." Another long section of Chapter 7 describes theoretical work on the effects of magnetic fields on thermal convection in a horizontal layer of fluid subject to a vertical temperature gradient (Bénard convection).

Chapter 8 discusses the theory of the stability of some of the equilibrium situations treated in Chapter 4, and Chapter 9 describes certain applications of Chapter 8.

Professor Roberts's book is attractively printed and illustrated, and contains a useful index. The claim that the "main consideration of this book is the mathematics of magnetohydrodynamics" seems justified, as is the statement that "reasonable attention has been paid to the physical framework".

RAYMOND HIDE (*Cambridge, Mass.*)