A BOUND FOR ENTIRE HARMONIC FUNCTIONS OF THREE VARIABLES*

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In [6], Professor R. P. Gilbert has obtained an upper bound and a lower bound for an entire GASPT[1] function by the integral operator method. Similar methods can be applied to estimate bounds for harmonic functions of three variables x_1, x_2, x_3.

DEFINITION. Let $g(u, \zeta)$ be an analytic function defined by

$$g(u, \zeta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m} u^{-n-1} \zeta^m$$

in some region in the space C^2 (space of two complex variables). $g(u, \zeta)$ is said to belong to the class K if $g(u, \zeta)$ is entire in the u^{-1} plane and the variable ζ is restricted to some compact subset in the ζ-plane.

Let $g(u, \zeta)$ be defined on $B(R, \delta) \equiv \{|u| \geq R\} \times \{1 - \delta \leq |\zeta| \leq 1 + \delta\}$. Then the three dimensional harmonic functions $u(x_1, x_2, x_3) = U(r, \theta, \phi)$ regular at infinity can be generated by the Whittaker–Bergman operator, written as

$$U(r, \theta, \phi) = \mathbf{B}_3[g] \equiv \frac{1}{2\pi i} \int_{|\zeta|=1} g(u, \zeta) \frac{d\zeta}{\zeta} \tag{1}$$

where

$$U(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{a_{n,m}(n-m)!}{n!} (-i)^m r^{-n-1} P_n^m(\cos \theta) e^{im\phi}$$

and

$$u = r\{\cos \theta + i \sin \theta (e^{i\phi}/\zeta + \zeta/e^{i\phi})\}.$$

If we analytically continue x_1, x_2, x_3 into the complex number space, and introduce the complex spherical variables

$$r = +(x_1^2 + x_2^2 + x_3^2)^{1/2}, \qquad \eta = +(x_1 + ix_2/x_1 - ix_2)^{1/2}, \quad \text{and} \quad \xi = x_3/r$$

then an inverse operator $\mathbf{B}_3^{-1}[U]$ can be constructed as

$$g(s, \zeta) = \mathbf{B}_3^{-1}[U] \equiv \frac{1}{4\pi i} \int_{-1}^{1} d\xi \int_{|\eta|=1} \frac{r(s+u)}{(s-u)^2} U(r, \xi, \eta) \frac{d\eta}{\eta} \tag{2}$$

where $u = r[\xi + (i/2)(1 - \xi^2)^{1/2}(\zeta/\eta + \eta/\zeta)]$ and $U(r, \xi, \eta)$ is analytic in the polycylinder

$$D(\epsilon_1, \epsilon_2, \epsilon_3) \equiv \{(r, \xi, \eta) \mid |r|^2 \geq \frac{1}{2\epsilon_1}; |1 - \xi| + |1 + \xi| \leq 2 + \epsilon_2, 1 - \epsilon_3 \leq |\eta| \leq 1 + \epsilon_3\}.$$

For details of the construction of these operators, please refer to [5].

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[1] Generalized axially symmetric potential theory.

DEFINITION. We shall say $H(r, \xi, \eta)$ belongs to the class E in C^3 (space of three complex variables) if $H(r, \xi, \eta)$ is entire in the r^{-1} plane while the other variables ξ, η are confined to some compact subsets in their respective planes.

REMARK. If x_1, x_2, x_3 are real, then $\xi = \cos\theta$, $\eta = e^{\phi i}$, and $U(r, \xi, \eta) = U(r, \theta, \phi)$. We shall say $U(r, \theta, \phi)$ belongs to the class E_R in the R^3 space if $U(r, \theta, \phi)$ is entire in the variable r^{-1} while $-\pi \le \theta \le \pi$ and $0 \le \phi \le 2\pi$, i.e., $U(r, \theta, \phi)$ is an entire harmonic function regular at infinity in the R^3 space.

By the maximum principle for two complex variables, $|g(u, \zeta)|$ attains its maximum on the Bergman–Silov boundary of $B(R, \delta)$, and is denoted by $M_g(R, \delta)$. By elementary estimates, $|u| \le (1 + \epsilon') |r|$ for sufficiently small ϵ', then we have, from (1),

$$|U(r, \theta, \phi)| \le M_g(R, \delta) \tag{3}$$

which holds for $|r| \ge R/(1 + \epsilon')$.

This result can be extended to the case when $g(u, \zeta)$ is in the class K. Since all the terms of $g(u, \zeta)$ involve only the negative powers of u, we are interested in investigating the asymptotical behavior of $g(u, \zeta)$ when $R \to 0$ by restricting ζ in the compact set $\{\zeta \mid 1 - \delta \le |\zeta| \le 1 + \delta\}$.

Suppose $M_g(R, \delta)$ has the order of growth λ, $0 < \lambda < \infty$, with respect to $1/R$ for any fixed but arbitrary δ, then we have asymptotically,

$$\exp(R^{-\lambda+\epsilon}) < M_g(R, \delta) < \exp(Re^{-\lambda-\epsilon}) \tag{4}$$

for every $\epsilon > 0$.

It follows from (3) that

$$|M_U(R, \epsilon_2, \epsilon_3)| < \exp(R^{-\lambda-\epsilon}) \tag{5}$$

where $M_U(R, \epsilon_2, \epsilon_3) = \sup |U(r, \theta, \phi)|$, $(r, \theta, \phi) \in D(\epsilon_1, \epsilon_2, \epsilon_3)$ with $\epsilon_1 = (1 + \epsilon')/2R^2$.

In order to obtain a lower bound for $M_U(R, \epsilon_2, \epsilon_3)$, we consider the inverse operator $\mathbf{B}_3^{-1}[U]$. From properties concerning Cauchy integrals of analytic function of several complex variables, [4, p. 53], both $U(r, \xi, \eta)$ and $g(s, \zeta)$ are analytic if their kernels $g(u, \zeta)$ and $(r(s + u)/(s - u)^2)U(r, \xi, \eta)$ are analytic in their respective domains of definition, and their paths of integration are piecewise smooth curves. It is therefore easy to observe that if $g(u, \zeta)$ is in the class K, then $U(r, \xi, \eta)$ is in the class E. The converse is also true provided that we can confine the variables s, u in suitable domains such that the function $m(r, s, u) = r(s + u)/(s - u)^2$ remains bounded.

Let $|r| = R$ and $|s| = 2R = R_1$. Then

$$|m(r, s, u)| = \left|\frac{r(s + u)}{(s - u)^2}\right| \le \frac{|r| \{|s| + |u|\}}{\{|s| - |u|\}^2} \le \frac{R\{2R + (1 + \epsilon')R\}}{R^2(2 - 1 - \epsilon')} \le \frac{3 + \epsilon'}{1 - \epsilon'}$$

For sufficiently small ϵ', say $0 < \epsilon' < 1/3$, $|m(r, s, u)| < 5$.

Let x_1, x_2, x_3 be real, then $U(r, \xi, \eta) = U(r, \theta, \phi)$. If $U(r, \theta, \phi)$ is in the class E_R, then $g(s, \zeta)$ is in the class K; an estimate for a lower bound of $U(r, \theta, \phi)$ shows that

$$\exp(R_1^{-\lambda+\epsilon}) < M_g(R_1, \delta) < 5M_U(R, \epsilon_2, \epsilon_3). \tag{6}$$

Let

$$\exp(R^{-\lambda+\bar{\epsilon}}) = \exp(R_1^{-\lambda+\epsilon})/5 < M_U(R, \epsilon_2, \epsilon_3). \tag{7}$$

Hence,

$$R^{-\lambda+\bar{\epsilon}} = R_1^{-\lambda+\epsilon} - \log 5$$

$$R^{-\lambda+\bar{\epsilon}}/R_1^{-\lambda+\epsilon} = 1 - (\log 5)/(R_1^{-\lambda+\epsilon}) = 1 - R_1^{\lambda-\epsilon} \log 5.$$

As $R_1 \to 0$, we have $R^{-\lambda+\bar{\epsilon}}/R_1^{-\lambda+\epsilon} \approx 1$

$$R^{-\lambda+\bar{\epsilon}} \approx R_1^{-\lambda+\epsilon} = 2^{-\lambda+\epsilon}R^{-\lambda+\epsilon}$$

$$R^{\bar{\epsilon}} \approx 2^{-\lambda+\epsilon}R^{\epsilon}$$

$$\bar{\epsilon} \log R \approx (-\lambda + \epsilon) \log 2 + \epsilon \log R$$

$$\bar{\epsilon} \approx \epsilon + (-\lambda + \epsilon) \log 2/\log R.$$

As $R \to 0$, we have $\bar{\epsilon} \approx \epsilon$. From (7), we get

$$\exp (R^{-\lambda+\epsilon}) < M_U(R, \epsilon_2, \epsilon_3). \tag{8}$$

Combining the inequalities (5) and (8) we have proved:

THEOREM 1. *Let $U(r, \theta, \phi)$ be an entire harmonic function, regular at infinity, in R_3-space. Then there exists a number λ (its order) such that for R sufficiently small and $\epsilon > 0$, the maximum modulus $M_U(R, \epsilon_2, \epsilon_3)$ for $U(r, \theta, \phi)$ is bounded by*

$$\exp (R^{-\lambda+\epsilon}) < M_U(R, \epsilon_2, \epsilon_3) < \exp (R^{-\lambda-\epsilon}). \tag{9}$$

Theorem 1 implies the function $U(r, \theta, \phi)$ has the same order as its entire associate $g(u, \zeta)$, and vice versa.

A formula on the type of an entire function $U(r, \theta, \phi)$ of order λ can be derived from the corresponding formula of its associate function $g(u, \zeta)$ from (1).

THEOREM 2. *Let $U(r, \theta, \phi)$ be an entire harmonic function regular at infinity of order λ. Then the type α of*

$$U(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m} r^{-n-1} P_n^m(\cos \theta) e^{im\phi}$$

is given by the formula

$$\frac{e}{(1 - \delta)} (e\alpha\lambda)^{1/\lambda} = \limsup_{n\to\infty} n(n + 1)^{1/\lambda} |a_{nn}|^{1/(n+1)} \tag{10}$$

where $g(u, \zeta)$, the associate of $U(r, \theta, \phi)$ in (1), has $B(R, \delta)$ as its domain of definition.

PROOF. The associate function $g(u, \zeta)$ is defined by

$$g(u, \zeta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m} \frac{n!}{(n - m)!} (-i)^m \zeta^m u^{-n-1} = \sum_{n=0}^{\infty} P_n(\zeta) u^{-n-1}$$

where

$$P_n(\zeta) = \sum_{m=-n}^{n} b_{n,m} \zeta^m = \sum_{m=-n}^{n} a_{n,m} \frac{n!}{(n - m)!} (-i)^m \zeta^m.$$

Let $M_{P_n}(\delta) = \sup_{1-\delta \le |\zeta| \le 1+\delta} |P_n(\zeta)|$. Considering the function $g(u, \zeta)$ as a function of the variable u alone, and applying Cauchy's inequality, we obtain

$$M_{P_n}(\delta) \le M_g(R, \delta) R^{n+1}. \tag{11}$$

By our previous remark, $g(u, \zeta)$ is entire with respect to $1/u$, of order λ and type α, say. Then

$$M_{P_n}(\delta) \leq \exp(\alpha R^{-\lambda})R^{n+1} \quad \text{for all } R \text{ when } R \to 0.$$

The minimum value of R is obtained when $R = (\alpha\lambda/(n+1))^{1/\lambda}$. Hence,

$$M_{P_n}(\delta) \leq (e\alpha\lambda/(n+1))^{(n+1)/\lambda}. \tag{12}$$

$P_n(\zeta)$ is analytic in the annulus $1 - \delta \leq |\zeta| \leq 1 + \delta$. Using the representation formula for the Laurent coefficients for $P_n(\zeta)$, we have

$$b_{n,m} = \frac{1}{2\pi i}\int_{|\zeta|=\rho} \frac{P_n(\zeta)\, d\zeta}{\zeta^{m+1}}, \qquad 1 - \delta \leq \rho \leq 1 + \delta.$$

Then

$$|b_{n,m}| \leq M_{P_n}(\delta)\rho^{-m}.$$

In order to estimate $|b_{n,m}|$ for different m and a fixed n, we consider two cases:

(i) $m \geq 0$, the maximum bound M_1 for $|b_{n,m}|$ occurs at $\rho = 1 - \delta$, $m = n$. Hence $M_1 = M_{P_n}(\delta)/(1 - \delta)^n$.

(ii) $m < 0$, the maximum bound M_2 for $|b_{n,m}|$ attains when $\rho = 1 + \delta$, $m = -n$. Hence, $M_2 = M_{Pn(\delta)}(1 + \delta)$, but

$$\begin{aligned} M_1 - M_2 &= M_{P_n(\delta)}[(1-\delta)^{-n} - (1+\delta)^n] \\ &= M_{P_n(\delta)}\delta^2[1 + (1 - \delta^2) + \cdots](1-\delta)^{-n} \geq 0. \end{aligned}$$

For each n, $m = -n, -n+1, \cdots, n-1, n$, the maximum bound for $b_{n,m}$ occurs when $m = n$ and $R_2 = 1 - \delta$. Hence,

$$|b_{n,n}| = \max |b_{n,m}| \leq M_{P_n(\delta)}(1-\delta)^{-n} \leq (e\alpha\lambda/(n+1))^{(n+1)/\lambda}(1/(1-\delta))^{n+1}. \tag{13}$$

Select a subseries $\mathfrak{g}(u, \zeta) = \sum_{n=0}^{\infty} b_{n,n}\zeta^n u^{-n-1}$ of $g(u, \zeta)$. We have

$$\limsup_{n\to\infty} |b_{n,n}| \geq \limsup_{m+n\to\infty} |b_{n,m}|.$$

We can derive the formula for the type of $\mathfrak{g}(u, \zeta)$, considered as an entire function of one complex variable in $1/u$, as

$$\limsup_{n\to\infty} (n+1)^{1/\lambda} |b_{n,n}|^{1/(n+1)} = \frac{1}{1-\delta}(e\alpha\lambda)^{1/\lambda} \qquad \text{see [3, p.292]}. \tag{14}$$

This implies

$$\begin{aligned} \frac{(e\alpha\lambda)^{1/\lambda}}{(1-\delta)} &= \limsup_{n\to\infty} (n+1)^{1/\lambda} |b_{n,n}|^{1/(n+1)} = \limsup_{n\to\infty} (n+1)^{1/\lambda} |a_{n,n}|^{1/(n+1)} |n!|^{1/(n+1)} \\ &= \limsup_{n\to\infty} \frac{n}{e}(n+1)^{1/\lambda} |a_{n,n}|^{1/(n+1)}. \end{aligned}$$

This completes the proof.

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