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APPLICATION OF THE KERNEL FUNCTION FOR THE COMPUTATION OF FLOWS OF COMPRESSIBLE FLUIDS*

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1. Introduction. The method of particular solutions is one of the procedures used for determining flow patterns of incompressible fluids, as well as for solving more general linear partial differential equations.

If the potential function $\Phi(X)$, $X = (x, y)$, is considered in the x, y-plane (so-called physical plane), Φ is a harmonic function; the stream function ψ is the conjugate of Φ. Obviously, if we pass from the x, y-plane by a conformal transformation to some other plane, Φ remains a harmonic function of the new variables. As a rule, Φ is considered in the x, y-plane. The method of particular solutions for the determination of flow patterns of incompressible fluids proceeds as follows:

One can determine harmonic functions, say $H_\nu(X; X_\mu)$, $\nu = 1, 2, 3$, yielding the source, sink or vortex, respectively, at the point, say X_μ, and a set $\{h_\nu\}$, $\nu = 1, 2, 3, \cdots$, of regular harmonic functions, where $\{h_\nu\}$ is "complete" in the domain (say Ω) where the flow is defined. Then the potential function $\Phi(X)$ of a flow in a channel (see Fig. 1) can be written

$$\Phi(X) = \Phi_0(X) + \Phi_1(X), \quad \Phi_0(X) = \sum_{\nu=1}^{\infty} a_\nu h_\nu(X), \quad \Phi_1(X) = \sum_{\mu=1}^{N} \sum_{\nu=1}^{2} A_{\nu\mu} H_\nu(X - X_\mu), \tag{1.1}$$

where $A_{\nu\mu}$ and a_ν are conveniently chosen constants.

One determines the coefficients $A_{\nu\mu}$, $\nu = 1, 2$, so that the sources, sinks and vortices at points X_μ have the prescribed strength and computes then the coefficients a_ν so that $\Phi(X)$ satisfies the prescribed boundary conditions along $\partial\Omega$. In most cases one requires that $\partial\Phi/\partial n = 0$ along $\partial\Omega$. Obviously, instead of $\Phi_0(X)$ one can consider the stream function $\psi_0(X)$ conjugate to $\Phi_0(X)$. In this case one has to determine the values $b(X)$, $X \in \partial\Omega$, of the function $\psi_1(X)$ conjugate to $\Phi_1(X)$ along the boundary $\partial\Omega$ and $\psi_0(X)$ is that harmonic function regular in Ω which assumes the values $-b(X)$ + const along $\partial\Omega$.

One can represent $\Phi_0(X)$ in terms of the kernel function of the domain Ω for the Laplace equation (see [B.S. 1] or [B. 8]).

In the case of compressible fluids the equation for the potential function Φ, when considered in the physical plane, i.e., $\Phi(x, y)$, is nonlinear (see (2.1)). However, if we consider Φ as a function of conveniently chosen new variables θ and λ, the equations for Φ and ψ become linear (see (2.7a)).

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Using this fact, it is possible to obtain a *linear procedure* which will determine the subsonic flow patterns. It will be described in the present paper. Further, we obtain for potential functions of flows in the domains of certain type explicit formulas for Φ in terms of the kernel function for equation (2.7b) and for domains with a boundary curve, to be described in Sec. 5.

As mentioned, instead of the potential function $\Phi(X)$ one can consider the stream function $\psi(X)$. The considerations proceed analogously.

2. The partial differential equations for the potential function Φ. The potential function Φ and the stream function ψ of compressible fluids when considered in the physical plane satisfy the equations

$$\frac{\partial^2\Phi}{\partial x^2}\left[1-\frac{1}{a^2}\left(\frac{\partial\Phi}{\partial x}\right)^2\right]-2\frac{\partial^2\Phi}{\partial x\,\partial y}\frac{1}{a^2}\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}+\frac{\partial^2\Phi}{\partial y^2}\left[1-\frac{1}{a^2}\left(\frac{\partial\Phi}{\partial y}\right)^2\right]=0 \tag{2.1}$$

and

$$\frac{\partial^2\psi}{\partial x^2}\left[1-\frac{1}{\rho^2a^2}\left(\frac{\partial\psi}{\partial y}\right)^2\right]-2\frac{\partial^2\psi}{\partial x\,\partial y}\frac{1}{a^2\rho^2}\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}+\frac{\partial^2\psi}{\partial y^2}\left[1-\frac{1}{\rho^2a^2}\left(\frac{\partial\psi}{\partial x}\right)^2\right]=0, \tag{2.2}$$

respectively (see [M. 1, pp. 241 and 243]). Here

$$a^2=a_s^2-\frac{\kappa-1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2\right]=a_s^2-\frac{\kappa-1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right], \tag{2.3}$$

where a_s, κ are constants ([M. 1], p. 239). We assume that the pressure density relation is

$$p=\sigma\rho^{\kappa}, \tag{2.4}$$

where σ and κ are constants. Then

$$\rho=\left[1-\frac{\kappa-1}{2a_s^2}\left(\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2\right)\right]^{1/(\kappa-1)}. \tag{2.5}$$

See [B. 5, p. 461], [M. 1, pp. 329, 330].

The equations for ψ and for Φ become linear if we consider the stream function ψ and the potential Φ in the so-called pseudo-logarithmic plane instead of considering them in the physical plane. Here θ is the angle which the velocity vector $\mathbf{v}\equiv qe^{i\theta}$ forms with the positive x-axis, while λ is a function of the speed q.

It is convenient to consider Φ and ψ in the so-called *pseudo-logarithmic plane*, with cartesian coordinates θ and

$$\lambda=\tfrac{1}{2}\log\left[\frac{1-(1-M^2)^{1/2}}{1+(1-M^2)^{1/2}}\left(\frac{1+h(1-M^2)^{1/2}}{1-h(1-M^2)^{1/2}}\right)^{1/h}\right],\quad h=\left(\frac{\kappa-1}{\kappa+1}\right)^{1/2}. \tag{2.6}$$

Here

$$M=q/[a_s^2-(\kappa-1)q^2/2]^{1/2} \tag{2.6a}$$

is the Mach number, q is the speed.

In the θ, λ-plane the relations* between Φ and ψ assume the form

$$\Phi_\theta-\psi_\lambda l^{1/2}=0,\qquad \Phi_\lambda+\psi_\theta l^{1/2}=0,\qquad l\equiv l(\lambda)=(1-M^2)/\rho^2. \tag{2.6b}$$

*In the present paper the potential and the stream function, Φ and ψ, as well as some other quantities, are considered in different planes; that is, they are considered as functions of different pairs of variables. In passing from one plane to another, new symbols should be introduced for Φ and ψ since they are

The equations for Φ and ψ become

$$\tfrac{1}{4}\Delta\Phi - iN\frac{\partial\Phi}{\partial\lambda} = 0, \qquad \Delta\Phi \equiv \frac{\partial^2\Phi}{\partial\lambda^2} + \frac{\partial^2\Phi}{\partial\theta^2} = 4\frac{\partial^2\Phi}{\partial z\,\partial\bar{z}}, \tag{2.7a}$$

$$\tfrac{1}{4}\Delta\psi + iN\frac{\partial\psi}{\partial\lambda} = 0, \tag{2.7b}$$

respectively. Here

$$z = \theta + i\lambda, \qquad \bar{z} = \theta - i\lambda \tag{2.7c}$$

and

$$N = -\frac{\kappa+1}{8}\frac{M^4}{(1-M^2)^{3/2}}.$$

See [B. 2], [B. 3], [B. 4], and [B. 5, pp. 462-3].

Instead of Φ and ψ one can introduce

$$\Phi^* = H\Phi \tag{2.8a}$$

and

$$\psi^* = H^{-1}\psi \tag{2.8b}$$

where

$$H(\lambda) = \exp\left[-\int_{-\infty}^{2\lambda} N(\tau)\,d\tau\right] = (1-M^2)^{-1/4}\left[1+(\kappa-1)\frac{M^2}{2}\right]^{-1/2(\kappa-1)} \tag{2.8c}$$

Φ^* and ψ^* satisfy the equations

$$\tfrac{1}{4}\Delta\Phi^* = -P\Phi^* \tag{2.9a}$$

and

$$\tfrac{1}{4}\Delta\psi^* = -F\psi^*, \tag{2.9b}$$

respectively. We substitute for the constant κ the value $\kappa = 1.4$, i.e. as indicated before, we assume $p = \sigma\rho^{1.4}$ (see (2.4)). Then we obtain

$$P = -\frac{0.3M^4}{8}\left[\frac{1.6M^4 - 0.8M^2 + 16}{(1-M^2)^3}\right]. \tag{2.10}$$

We note that $P(M) < 0$ for $M < 1$ (the subsonic case). Further

$$F = \frac{0.3M^4}{8}\left[\frac{-3.2M^4 - 0.8M^2 + 16}{(1-M^2)^3}\right], \tag{2.11}$$

where $F(M) > 0$ for $M < 1$ (see (2.16a), (2.16b) of [B. 5]).

Our aim is to determine a domain and solutions Φ (or ψ) of the respective equations (2.1) (or (2.2)) satisfying the following boundary conditions: the boundary can be divided

different functions of their respective arguments. For instance, when passing from the physical plane to the pseudo-logarithmic plane, we should write $\Phi^{(2)}(\theta, \lambda) \equiv \Phi[x(\theta, \lambda), y(\theta, \lambda)]$ and so on. However, for the sake of brevity we omit the superscripts and write always in the following Φ, ψ, and so on, no matter in which plane the function is considered.

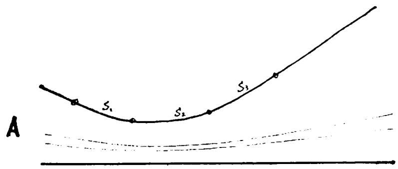

FIG. 1. A flow in a channel.

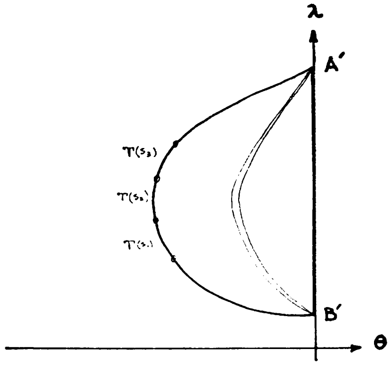

FIG. 2. The image of the flow in Fig. 1 in the pseudo-logarithmic plane. Here λ = −λ.

into finitely many segments s_ν. On each s_ν,

$$\partial\Phi/\partial n = \text{const.} \quad (2.12) \qquad \text{or} \qquad \begin{cases} \psi = \text{const.} & (2.12\text{a}) \\ R_\nu(\partial\psi/\partial x,\ \partial\psi/\partial y) = 0. & (2.13\text{a}) \end{cases}$$

$$P_\nu(\partial\Phi/\partial x,\ \partial\Phi/\partial y) = 0 \quad (2.13)$$

Here P_ν (or R_ν) are some prescribed functions of their arguments. (2.13) (or (2.13a)) represent generalized Poincaré–Levi-Civita conditions.

The formulation of the conditions (2.12) *and* (2.13) *in the θ, λ-plane.* Now we consider the flow in the θ, λ-plane. According to [M. 1, p. 242],

$$\frac{q_x}{q_y} = -\frac{\partial\psi/\partial y}{\partial\psi/\partial x}, \tag{2.14}$$

$$\rho^2 q^2 = (\partial\psi/\partial x)^2 + (\partial\psi/\partial y)^2 \tag{2.15}$$

and

$$\frac{q_x}{q_y} = \frac{\partial\Phi/\partial x}{\partial\Phi/\partial y} = \text{cotan } \theta, \qquad q^2 = \left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 \tag{2.16}$$

(see [M. 1, p. 242 (18′)]). Consequently,

$$q^2 = (\partial\Phi/\partial y)^2[1 + (\text{cotan } \theta)^2]$$

or

$$\left(\frac{\partial\Phi}{\partial y}\right)^2 = \frac{q^2}{[1 + (\text{cotan } \theta)^2]}, \qquad \left(\frac{\partial\Phi}{\partial x}\right)^2 = \frac{q^2(\text{cotan } \theta)^2}{[1 + (\text{cotan } \theta)^2]}.$$

Therefore the relations (2.13) and (2.13a) can be replaced by the relations

$$\tilde{P}_\nu(\lambda, \theta) = 0, \tag{2.13b}$$

$$\tilde{R}_\nu(\lambda, \theta) = 0, \tag{2.13c}$$

respectively. We assume that $T(\Omega)$, the image of Ω in the θ, λ-plane, is a connected domain bounded by segments s_ν , where (2.13a) holds along s_ν . (Hypothesis 1)

We further suppose that the flow in the x, y-plane is defined in a channel, both ends of the channel go to infinity (see Fig. 1). (Hypothesis 2)

3. The analytic representation of singularities at infinity in the pseudo-logarithmic plane. Let $T(\Omega)$ be the image of the domain Ω in the θ,λ-plane. In the case of flows satisfying Hypotheses 1 and 2, two different points of $T(\Omega)$ correspond to infinites in Ω. At these points, say (θ_1 , λ_1) and (θ_2 , λ_2), the potential function has singularities. If at the point $(\theta_\nu , \lambda_\nu)$, $\nu = 1$ or 2, we have a source (or sink), then at this point the potential function Φ has a logarithmic singularity. If at the point $(\theta_\nu , \lambda_\nu)$ the flow has a vortex, then the stream function ψ has a logarithmic singularity. An analytic expression for the singularities of these types has been determined in [B. 5, p. 473 ff]. Once the representation for the potential function is known, using (2.6b) (see also (5.6) of [B. 5, p. 473]), one obtains the corresponding stream function. Analogously, one can determine the potential function once the stream function is known. For the convenience of the reader we reproduce here the above formulas for $\Phi^{(L)}$ and $\psi^{(L)}$. We have

$$\psi^{(L)} = A(\log \zeta + \log \bar{\zeta})/2 + B, \tag{3.1a}$$

$$\Phi^{(L)} = C(\log \zeta + \log \bar{\zeta})/2 + D, \tag{3.1b}$$

where $\zeta = Z - Z_0$, $\bar{\zeta} = \bar{Z} - \bar{Z}_0$, $Z = \theta + i\lambda$, $\bar{Z} = \theta - i\lambda$, $Z_0 = \theta_0 + i\lambda_0$, $-\infty < \lambda_0 < 0$,

$$A = H\left[1 - \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} F\,dZ_1\,d\bar{Z}_1 + \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} F\left(\int_{Z_0}^{Z_1}\int_{\bar{Z}_0}^{\bar{Z}_1} F\,dZ_2\,d\bar{Z}_2\right) dZ_1\,d\bar{Z}_1 \cdots\right], \tag{3.2a}$$

$$B = H\left[\int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} G\,dZ_1\,d\bar{Z}_1 - \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} F\left(\int_{Z_0}^{Z_1}\int_{\bar{Z}_0}^{\bar{Z}_1} G\,dZ_2\,d\bar{Z}_2\right) + \cdots\right], \tag{3.2b}$$

$$G = -\frac{1}{\bar{\zeta}}\frac{\partial(H^{-1}A)}{\partial Z} - \frac{1}{\zeta}\frac{\partial(H^{-1}A)}{\partial \bar{Z}},$$

and

$$C = H^{-1}\left[1 - \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} P\,dZ_1\,d\bar{Z}_1 + \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} P\left(\int_{Z_0}^{Z_1}\int_{\bar{Z}_0}^{\bar{Z}_1} P\,dZ_2\,d\bar{Z}_2\right) dZ_1\,d\bar{Z}_1 - \cdots\right], \tag{3.3a}$$

$$D = H^{-1}\left[\int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} K\,dZ_1\,d\bar{Z}_1 - \int_{Z_0}^{Z}\int_{\bar{Z}_0}^{\bar{Z}} P\left(\int_{Z_0}^{Z_1}\int_{\bar{Z}_0}^{\bar{Z}_1} K\,dZ_2\,d\bar{Z}_2\right) + \cdots\right], \tag{3.3b}$$

$$K = -\frac{1}{\bar{\zeta}}\frac{\partial(HC)}{\partial Z} - \frac{1}{\zeta}\frac{\partial(HC)}{\partial \bar{Z}}.$$

Let $\mathfrak{F}^{(L,1)}$ denote the flow with the stream function having a logarithmic singularity at

Z_0 and let $\mathfrak{F}^{(L,2)}$ be the flow with the potential function having the logarithmic singularity at $Z = Z_0$. By (2b) we obtain for Φ and ψ in the first and second cases, respectively, the expressions

$$\Phi = \Phi^{(L)} \overset{\text{df}}{=} i \int (l^{1/2}\psi_Z^{(L)}\, dZ - l^{1/2}\psi_{\bar{Z}}^{(L)}\, d\bar{Z}), \tag{3.4}$$

$$\psi = \psi^{(L)} \overset{\text{df}}{=} -i \int (l^{-1/2}\Phi_Z^{(L)}\, dZ - l^{-1/2}\Phi_{\bar{Z}}^{(L)}\, d\bar{Z}). \tag{3.5}$$

The formulas for $\psi^{(L)}$ have been indicated in [B. 7, p. 891].

REMARK. The flow with the stream function (4.5) of [B. 7] *does not* represent the pattern indicated in 1a, 1b, 1c and 2a, 2b, 2c of [B. 7]. Since in (3.1) of [B. 7] $\psi^{(L)}(\lambda, \theta, \lambda_A, \theta_A)$ is used (and not $\Phi^{(L)}$), we have at the points A and B (see the above figures) a vortex (and not a source or sink).

The same singularities have been considered by Ludford [L. 1]. Using results by Bers [B. 9–11] and John [J. 1], Finn and Gilbarg [F.G. 1], [F. 1], [G. 1] showed that every flow with single-valued gradient at infinity has a potential which in the pseudo-logarithmic plane has singularity $\Phi^{(L)}$ (or its derivatives). Using other considerations, Finn and Gilbarg showed that the subsonic flow around a profile is uniquely determined by the velocity at infinity and circulation around the profile.

Loewner proved the following result: In the case of an incompressible fluid, when considering a circulation free flow around a simply connected profile, the velocity field represents a conjugate of an analytic function which has at infinity a *branch point of second order*. The only exception is the flow around a straight line segment which is parallel to the velocity at infinity. Loewner* made a conjecture that the same is true in the case of subsonic flows of compressible fluids. L. A. Fine [F. 2] obtained the following result in this direction: the double covering of the hodograph plane at the point corresponding to infinity, $(x, y) = \infty$, is true in the case of a subsonic flow when the obstacle has *only* two points or segments where the tangent to the obstacle is horizontal.

In this connection the problem arises to compute solutions of (2.9a) and (2.9b) which have branch points. Using the integral operator of the first kind (see [B. 1], [B. 6 (4.b), p. 15 and Theorem 6.1.B, p. 21]), one obtains solutions of the above equations possessing branch points of the same order and the same points as the associates.

4. The determination of the solution $\psi_0(\boldsymbol{\vartheta})$ regular in $\mathbf{T}(\boldsymbol{\vartheta})$ with the boundary values $-b(\boldsymbol{\vartheta})$. In Sec. 3 we determined the function $\Phi_1(\boldsymbol{\vartheta})$, $\boldsymbol{\vartheta} = (\theta, \lambda)$, having the necessary singularities and the conjugate solution $\psi_1(\boldsymbol{\vartheta})$ connected with $\Phi_1(\boldsymbol{\vartheta})$ by the relation (2.6b). We now have to determine the regular solution $\psi_0(\boldsymbol{\vartheta})$ of (2.7b) which assumes on $\partial\mathbf{T}(\Omega)$ the values $-\psi_1(\boldsymbol{\vartheta}) + \text{const.}$

Instead of considering the boundary value problem for $\psi_0(\boldsymbol{\vartheta})$, we can consider the boundary value problem for $\psi_0^*(\boldsymbol{\vartheta}) = H^{-1}\psi_0(\boldsymbol{\vartheta})$ (see (2.8a)), which satisfies the equation (2.9b). We wish to stress that in this case, according to (2.11), the coefficient $-F(\boldsymbol{\vartheta}) \equiv -F(\lambda)$ is *negative* in the subsonic region (i.e. for $M < 1$).

The application of the theory of the kernel function to the case of equations (2.9b) with negative coefficients $-F(\lambda)$ is discussed in [B.S. 1, p. 560 ff.], see also [B. 8, p. 122 ff]. Obviously, we can write

$$-F(\lambda) = \tilde{F}(\lambda) - k, \tag{4.1}$$

*Transmitted in an oral communication.

where $\tilde{F}(\lambda) > 0$ in $\mathbf{T}(\Omega)$ and $k > 0$ is a conveniently chosen constant. In this case we can apply the theory of orthogonal functions with unessential modifications as shown in [B.S. 1]. Let $\mathbf{A}_1$ denote the class of square integrable functions in $\mathbf{T}(\Omega)$ satisfying the equation

$$\Delta\psi^*(\boldsymbol{\vartheta}) = -F(\boldsymbol{\vartheta})\psi(\boldsymbol{\vartheta}), \qquad -F(\lambda) = \tilde{F}(\lambda) - k. \tag{4.2}$$

Following [B.S. 1], we associate with (4.2) the metric

$$D_k(\varphi, \psi) = \iint_{\mathbf{T}(\Omega)} [(\operatorname{grad} \varphi, \operatorname{grad} \psi) + (\tilde{F}(\lambda) - k)]\, d\lambda\, d\theta. \tag{4.3}$$

We make the assumption that no solution of (4.2) *in* $T(\Omega)$ *exists which satisfies the conditions*

$$\psi^*(\boldsymbol{\vartheta}) = 0 \quad \text{or} \quad \partial\psi^*(\boldsymbol{\vartheta})/\partial n = 0, \qquad \boldsymbol{\vartheta} \in \partial\mathbf{T}(\Omega), \tag{4.4}$$

which is different from $\psi^*(\boldsymbol{\vartheta}) \equiv 0$. Then, according to [B.S. 1, p. 562], or [B. 8, p. 124], there exists the kernel function

$$\begin{aligned} k_{\mathbf{T}(\Omega)}(\boldsymbol{\vartheta}, \mathbf{Z}) &= N(\boldsymbol{\vartheta}, \mathbf{Z}) - G(\boldsymbol{\vartheta}, \mathbf{Z}) \\ &= -\sum_{\nu=1}^{N} \psi_\nu(\boldsymbol{\vartheta})\psi_\nu(\mathbf{Z}) + \sum_{\nu=N+1}^{\infty} \psi_\nu(\boldsymbol{\vartheta})\psi_\nu(\mathbf{Z}), \quad \boldsymbol{\vartheta} \in \mathbf{T}(\Omega), \mathbf{Z} \in \mathbf{T}(\Omega), \end{aligned} \tag{4.5}$$

where $\psi_\nu(\boldsymbol{\vartheta})$ is a system of functions orthogonal in the sense of the metric (4.3), N and G are the respective Neumann and the Green's functions of (2.7b) in the domain $\mathbf{T}(\Omega)$. It holds

$$N < \infty. \tag{4.6}$$

The solution $\psi_0^*(\vartheta)$ of the boundary value problem can be written in the form

$$\begin{aligned} \psi_0^*(\boldsymbol{\vartheta}) &= \int_{\partial\mathbf{T}(\Omega)} b(\boldsymbol{\vartheta}_1)\, \frac{\partial k_{\mathbf{T}(\Omega)}(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta})}{\partial n_{\boldsymbol{\vartheta}_1}}\, ds_{\boldsymbol{\vartheta}_1} \\ &= \sum_{\nu=1}^{\infty} (\pm\psi_\nu^*(\boldsymbol{\vartheta}) \int_{\partial\mathbf{T}(\Omega)} b_1^*(\boldsymbol{\vartheta}_1)\, \frac{\partial\psi_\nu^*(\boldsymbol{\vartheta}_1)}{\partial n_{\boldsymbol{\vartheta}_1}}\, ds_{\boldsymbol{\vartheta}_1}, \end{aligned} \tag{4.7}$$

where $\psi_1^*(\vartheta_1)$ are the orthonormal functions of $\mathbf{T}(\Omega)$, and $b^*(\boldsymbol{\vartheta}_1)$ are the prescribed values (see [B.S. 1] or [B. 8, p. 122 ff.]).

REMARK. Instead of the metric (4.3) one can also use the metric

$$(u, v) = \int_{\partial\mathbf{T}(\Omega)} u(\boldsymbol{\vartheta})v(\boldsymbol{\vartheta})\, ds_{\boldsymbol{\vartheta}} \tag{4.8}$$

or

$$[u, v] = \int_{\partial\mathbf{T}(\Omega)} \frac{\partial u(\boldsymbol{\vartheta})}{\partial n_{\boldsymbol{\vartheta}}}\, \frac{\partial v(\boldsymbol{\vartheta})}{\partial n_{\boldsymbol{\vartheta}}}\, ds_{\boldsymbol{\vartheta}} \tag{4.9}$$

(see [S. 1] or [B.S. 2, p. 396 ff.]). It seems, however, that in most cases (4.3) is more convenient for numerical purposes. Concerning the numerical application of this method for the computation of ψ_0, see [B.H. 1].

Once the stream function $\psi(\boldsymbol{\vartheta}) = \psi_1(\boldsymbol{\vartheta}) + \psi_0(\boldsymbol{\vartheta})$ (or the potential) in the θ, λ-plane is determined, we obtain the stream lines $\psi(\boldsymbol{\vartheta}) =$ const. of the flow in the physical plane by using the formula

$$x(\theta, q) + iy(\theta, q) = \int \frac{e^{i\theta}}{\rho}\left[\left(\frac{\partial\psi}{\partial q} + i\,\frac{\partial\psi}{\partial\theta}\right)d\theta + \left(\frac{M^2 - 1}{q^2}\frac{\partial\psi}{\partial\theta} + \frac{i}{q}\frac{\partial\psi}{\partial q}\right)dq\right] \tag{4.9}$$

$$\psi(\theta, q) = \text{const.} \tag{4.9a}$$

in which, using (2.6) and (2.6a), the speed q can be replaced by λ. See [M. 1, p. 268].

$$\psi(\boldsymbol{\vartheta}) = \sum_{\nu=1}^{\infty} a_\nu g_\nu(\boldsymbol{\vartheta}) + \sum_{\mu=1}^{N}\sum_{\nu=1}^{2} A_\nu G_\nu(\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_\mu) \tag{4.10}$$

$$g_\nu(\boldsymbol{\vartheta}) = \sum_{\mu=1}^{\nu} a_{\nu\mu} p_\mu(\boldsymbol{\vartheta}), \tag{4.10a}$$

where $g_\nu(\boldsymbol{\vartheta})$ and $G_\nu(\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_\mu)$ are the stream functions corresponding to the potential functions $h_\nu(\boldsymbol{\vartheta})$ and $H_\nu(\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_\mu)$, respectively. $a_{\nu\mu}$ are orthogonalization constants. Therefore, *one can prepare for $g_\nu(\boldsymbol{\vartheta})$ and $G_\nu(\boldsymbol{\vartheta})$ once and for all, tables or standard punch cards.* Replacing g_ν by p_μ in (4.10), and ψ in (4.9) and (4.9a) by the right-hand side of (4.10), we obtain a *linear procedure for the determination of the flow pattern in the physical plane.*

Instead of representing $\psi_0(\boldsymbol{\vartheta})$ in terms of orthogonal functions of (2.7b) with respect to the domain $\mathbf{T}(\Omega)$, we can represent $\psi_0(\boldsymbol{\vartheta})$ in terms of the kernel function $k_{\mathbf{T}(\Omega)}(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2)$ for equation (2.7b).

Remark. Concerning the continuation of the subsonic flow to the transonic region, see [B. 3], [B. 6], [B. 7] and [S. 2].

5. Some relations between the boundary of Ω and of $\mathbf{T}(\Omega)$. If the domain of definition of the potential function Φ is known in the θ, λ-plane (the pseudo-logarithmic plane) or in the hodograph plane, then according to the considerations in Sec. 4 we can determine the potential function $\Phi(\theta, \lambda)$. Then, using the relations (4.9), see also [M. 1, p. 268], we determine the flow pattern in the physical plane. Suppose the flow $F(\Omega)$ in the x, y-plane is defined in the domain Ω, then we denote by $\mathbf{T}(F(\Omega))$ the image of the flow in the θ, λ-plane.

As a rule, the domain Ω of definition of Φ is given in the *physical plane* and the following questions arise:

(1) to study the conditions for the boundary curve $\partial\Omega$ and the pressure distribution along $\partial\Omega$ in order that the image $\mathbf{T}(\partial\Omega)$ of $\partial\Omega$ in the θ, λ-plane is of certain type;

(2) to draw conclusions from the properties of $\mathbf{T}(\partial\Omega)$ about the structure of $\partial\Omega$ and the pressure distribution in the neighborhood of $\partial\Omega$.

In the simplest case these relations are formulated in

Lemma 1. *If a segment s_ν of the boundary in the x, y-plane is a segment of the straight line or a segment of a "free boundary", then its image $\mathbf{T}(s_\nu)$ in the θ, λ-plane is a segment which is parallel to the θ or λ-axis, respectively.* See also [B. 7].

We shall generalize this result.

Remark. λ is a function of q given by (2.6), (2.6a). If x and y are coordinates of the point of a segment s_ν of $\partial\Omega$ and if t is the time, it holds

$$q = ((dx/dt)^2 + (dy/dt)^2)^{1/2} = ds/dt. \tag{5.1}$$

The images of Ω, s_ν, etc., in the θ, q-plane will be denoted by $\tilde{\mathbf{T}}(\Omega)$, $\tilde{\mathbf{T}}(s_\nu)$, etc. The transition from the θ, λ-plane to the θ, q-plane means a distortion of the λ-axis, see (2.6) and (2.6a). Instead of formulating relations between λ and θ along s, we shall formulate the corresponding relations between ds/dt and θ along $\tilde{\mathbf{T}}(\partial\Omega)$.

LEMMA 2. *Suppose that a flow is defined in a domain $\tilde{\mathbf{T}}(\Omega)$, and along a segment, say $\tilde{\mathbf{T}}(s_\nu)$, of the boundary of $\tilde{\mathbf{T}}(\Omega)$*

$$\tilde{R}_\nu(\theta, (ds/dt)^2) = 0 \tag{5.2}$$

holds. Here $\tilde{R}_\nu$ is a polynomial in θ and $q = ds/dt$. Then the segment s_ν in the physical plane can be represented in the form

$$x = \int_{t_0}^{t}\left[\rho_\nu\left(\frac{ds}{dt}\right)\right]dt, \qquad y = \int_{t_0}^{t}\left[\mathrm{P}_\nu\left(\frac{ds}{dt}\right)\right]dt, \tag{5.3}$$

where ρ_ν and P_ν are algebraic functions of ds/dt and of $\exp\,[iL(ds/dt)]$. *(L is an algebraic function.)*

Proof. One can write (5.2) in the form

$$\tilde{R}_\nu\left[\theta, \left(\frac{ds}{dt}\right)^2\right] \equiv \hat{R}_\nu\left[\text{arc cos}\left(\frac{dx}{dt}\Big/\frac{ds}{dt}\right), \left(\frac{ds}{dt}\right)^2\right] = 0,$$

$$\text{or} \quad \frac{dx}{dt} = \frac{1}{2}\frac{ds}{dt}(e^{iL} + e^{-iL}), \tag{5.4}$$

where $\hat{R}_\nu$ is a polynomial of its arguments. It follows that

$$dx/dt = \rho_\nu(ds/dt). \tag{5.5a}$$

Analogously, one gets

$$dy/dt = \mathrm{P}_\nu(ds/dt) \tag{5.5b}$$

from which (5.3) follows.

LEMMA 3. *Suppose that a flow is defined in the domain Ω of the physical plane. We assume that along a segment s_ν of $\partial\Omega$ the representation*

$$x = \sum_{\nu=0}^{M} a_\nu t_1^\nu, \quad M < \infty, \tag{5.6a}$$

$$y = \sum_{\nu=0}^{N} b_\nu t_1^\nu, \quad N < \infty, \tag{5.6b}$$

holds, where t_1 is a parameter. Then along s_ν the relation

$$\begin{vmatrix} -dx/dt_1 + a_1 & 2a_2 & 0 & \cdots \\ 0 & -dx/dt_1 + a_1 & 2a_2 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ -dy/dt_1 + b_1 & 2b_2 & 0 & \cdots \\ 0 & -dy/dt_1 + b_1 & 2b_2 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = 0 \tag{5.7}$$

holds.

If t denotes the time and $t = t(t_1)$ is a differentiable function, then $dx/dt_1 = (dx/dt)(dt/dt_1) = q_x dt/dt_1$. From (5.7) follows $dt_1/dt = S(q_x, q_y) \equiv S$ and from (5.6a), (5.6b) we obtain

$$\tan\theta = \frac{\sum \nu b_\nu t_1^{\nu-1}}{\sum \nu a_\nu t_1^{\nu-1}} \quad \text{and} \quad t_1 = \Gamma(e^{i\theta}) \equiv \Gamma,$$

where Γ is an algebraic function. Thus $(dx/dt_1)(dt_1/dt) = [\sum \nu a_\nu(\Gamma(e^{i\theta}))^{\nu-1}]S$ and

$$q^2 = S^2\{[\sum \nu a_\nu \Gamma]^2 + [\sum \nu b_\nu \Gamma]^2\} \tag{5.7a}$$

Since

$$q_x^2 = (dx/dt)^2 = (ds/dt)^2 \cos^2 \theta, \tag{5.8a}$$

$$q_y^2 = (dy/dt)^2 = (ds/dt)^2 \sin^2 \theta, \tag{5.8b}$$

(5.7a) represents a relation between q and θ. It is the equation of $\tilde{T}(s_\nu)$.

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