A NUMERICAL DETERMINATION OF ULTRA-SUBHARMONIC RESPONSE FOR THE DUFFING EQUATION $\ddot{x} + \alpha x + \beta x^3 = F \cos \omega t \quad (\alpha > 0)$*

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1. Introduction. It is the purpose of the present paper to present the results of the author's numerical digital computer investigation of some ultra-subharmonic solutions of the Duffing equation. These solutions are to be distinguished from the ordinary subharmonic solutions which were the subject of the author's previous investigation [1]. We repeat the definitions of these two types of periodic solutions. We first write the differential equation in the nondimensional form

$$\nu^2\xi'' + \xi + \delta\xi^3 = \cos\theta \tag{1}$$

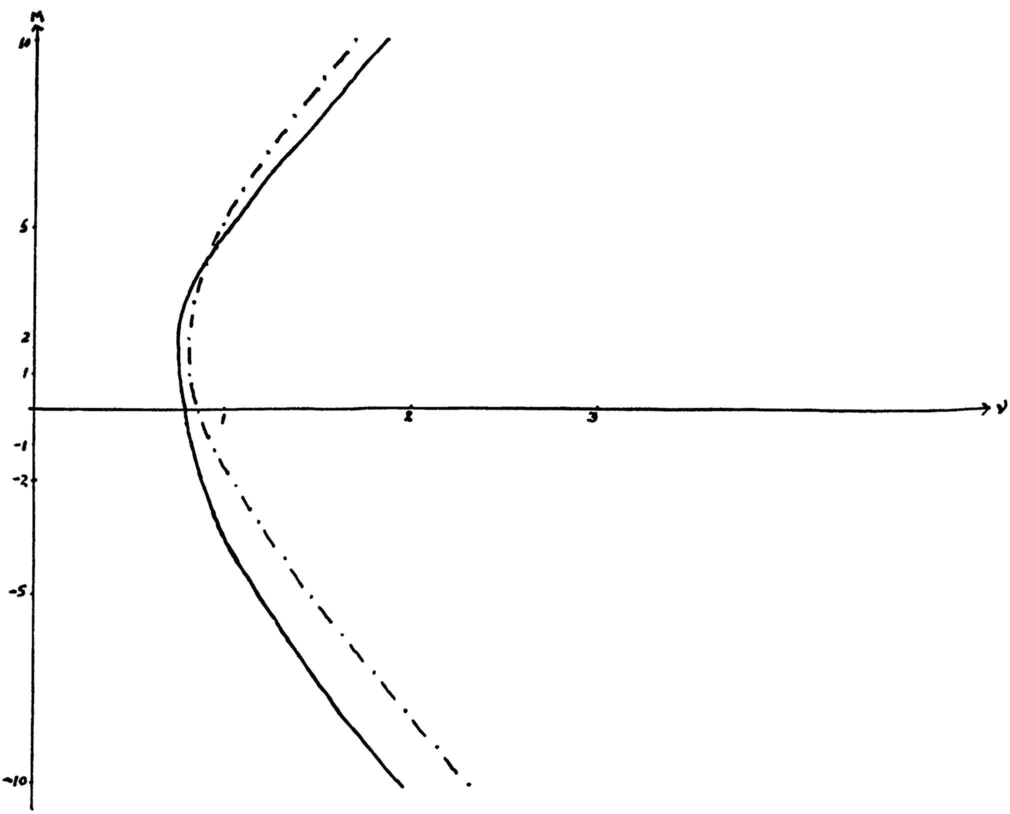

FIG. 1. Comparison between the perturbation and computer response curves for $\delta = .1$.
——— computer curve for periodic solutions of period 4π.
.—.— perturbation curve for the ultra-subharmonic solutions of period 4π for $p/q = 2/3$.

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under the initial conditions

$$\xi(0) = M; \qquad \xi'(0) = 0 \tag{2}$$

where the differentiation refers to the variable $\theta = \omega t$, and where

$$\nu^2 = \omega^2/\alpha, \quad \delta = \beta F^2/\alpha^3, \quad M = A\alpha/F, \quad \xi = \alpha x/F. \tag{3}$$

The periodic solutions with smallest period $2n\pi$, $(n = 2, 3, \cdots)$ of equation (1) under the initial conditions (2), which are known to exist for sufficiently small δ, and which, together with their nondimensional frequency ν, are analytic in δ for arbitrary M, are classified as follows:

The ordinary subharmonic solutions are those solutions with smallest period $2n\pi$, $(n = 2, 3, \cdots)$ for which ν reduces to n when $\delta = 0$. The ultra-subharmonic solutions are those solutions with smallest period $2n\pi$, $(n = 2, 3, \cdots)$ for which ν reduces to n/m where n and m are relatively prime and $\neq 1$ when $\delta = 0$.

The approximate response relation for both types of solutions, that is, the relation among the parameters yielding the periodic solutions, is given by the formula

$$\nu^2 = (p/q)^2\left\{1 + \tfrac{3}{4}\delta\left[M^2 + \frac{2M}{(p/q)^2 - 1} + \frac{3}{\{(p/q)^2 - 1\}^2}\right]\right\} \qquad (p/q \neq 3, 1/3, 1). \tag{4}$$

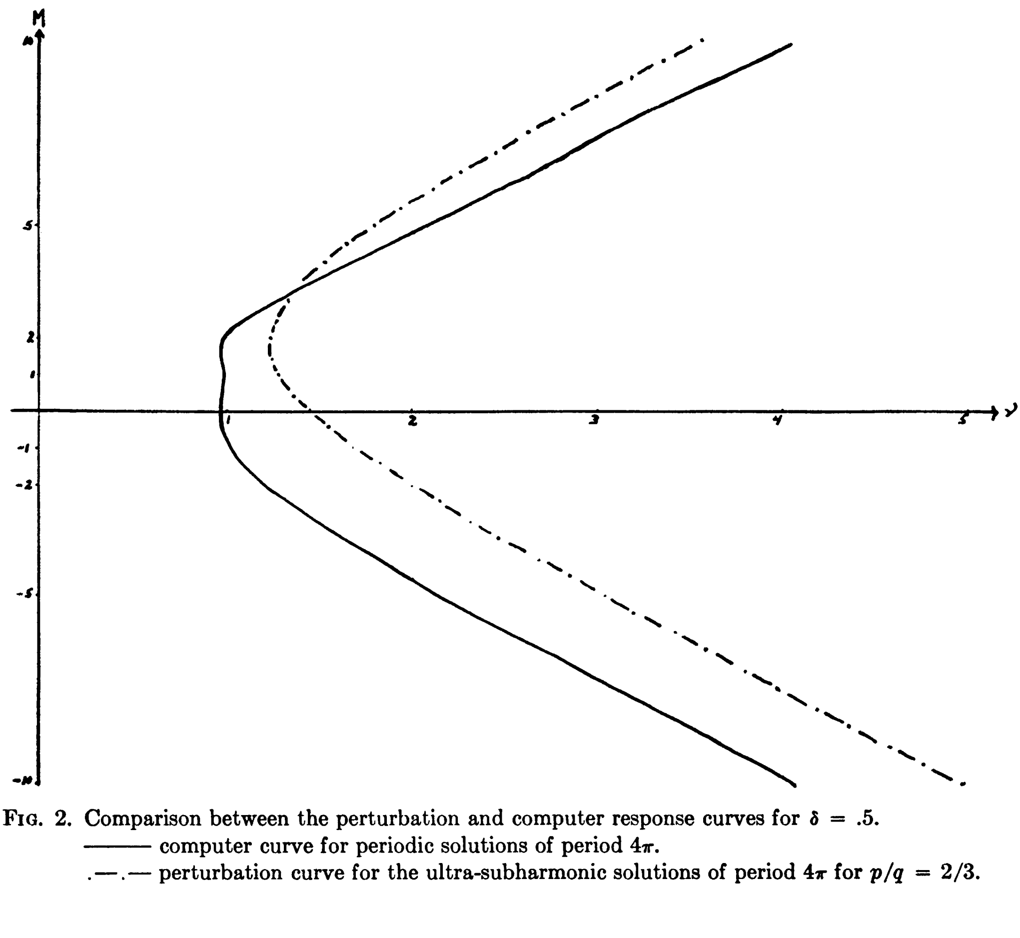

FIG. 2. Comparison between the perturbation and computer response curves for $\delta = .5$.
——— computer curve for periodic solutions of period 4π.
.—.— perturbation curve for the ultra-subharmonic solutions of period 4π for $p/q = 2/3$.

Since relation (4) is most easily obtained by a simple application of the perturbation procedure, it is referred to as the perturbation relation and the corresponding curves, the perturbation curves.

In the present paper we investigate the ultra-subharmonic solutions corresponding to $p/q = 2/3$ in relation (4). These are periodic with smallest period 4π. In our previous investigation we were concerned with the ordinary subharmonic solutions corresponding to $p/q = 2$ in relation (4) which also are periodic with smallest period 4π. The numerical procedure used is described in the author's previous paper [1]. The periodic solutions and their response curves are again obtained for values of M in the interval $-10 \leq M \leq 10$, and values of δ in the interval $0 < \delta \leq 10$ and values of ν suitably determined to insure the periodicity of the solutions. A check on the reliability of the computer was again made by comparing the values of a solution and its derivative at 4π with their corresponding initial values. We found good agreement but not quite as good as in the previous investigation. In particular for $\delta = 10$, as an example, we found that $|M - \xi(4\pi)|$ was less than 10^{-4} in all cases and less than 10^{-5} in one case, while $|\xi'(4\pi)|$ was less than 10^{-3} in all cases, less than 10^{-4} in two cases and less than 10^{-6} in one case.

2. Results. Fig. 1 through 5 show a comparison between the computer and perturbation response curves in the νM-plane for different values of δ. We have omitted

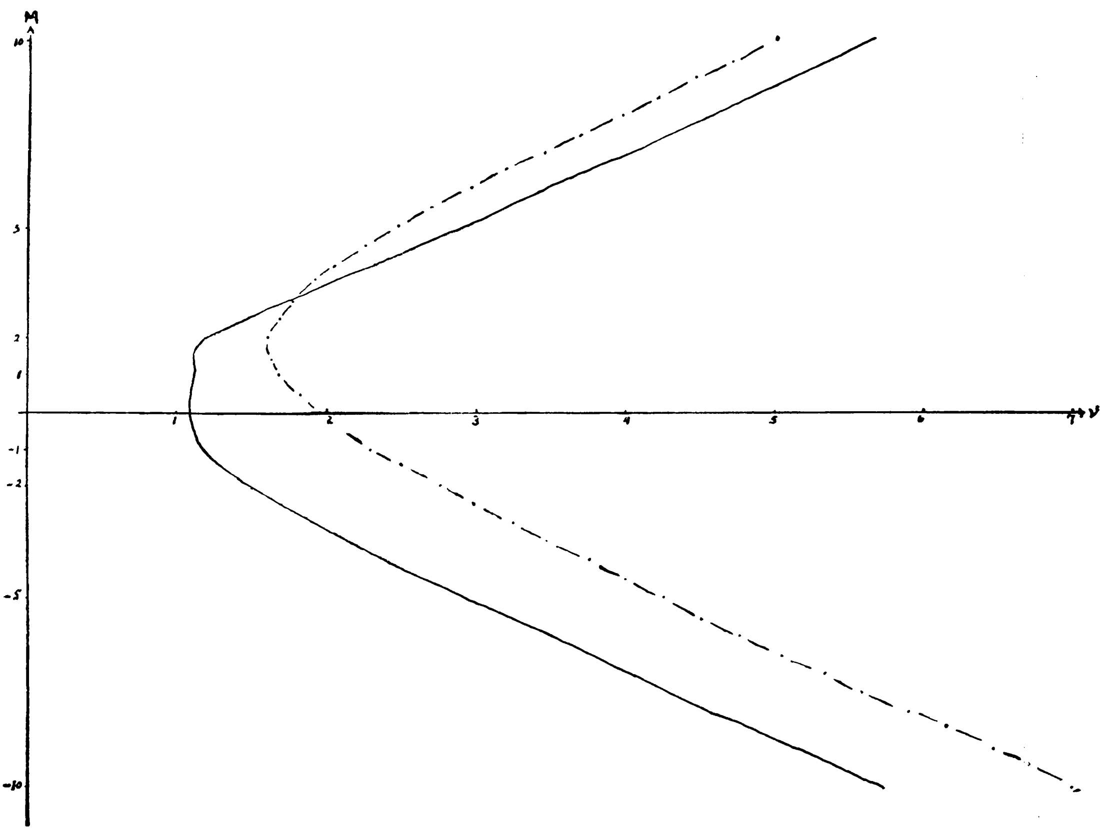

FIG. 3. Comparison between the perturbation and computer response curves for $\delta = 1$.
——— computer curve for periodic solutions of period 4π.
.—.— perturbation curve for the ultra-subharmonic solutions of period 4π for $p/q = 2/3$.

the figure for $\delta = .01$ in which the agreement between the two curves is excellent, as was to be expected. Except for those values of M which lie in the neighborhood of the intersection of the two curves in each of the figures, the agreement between the two curves is not very good. However, for $\delta = .1$ in Fig. 1, the agreement is not too bad. In particular, using the relative error $\lambda = |\nu_c - \nu_p|/\nu_c$ as a measure of agreement, we find that the maximum value of λ for $\delta = .1$ is approximately 21% occurring for $M = -10$, and for all positive M, the largest value of λ is approximately 10% occurring for $M = 10$. For the remaining values of δ, Figs. 2 through 5, the agreement is bad, especially for values of M in the interval $-10 \leq M \leq 2$ where λ changes from a maximum value of approximately 72% for $\delta = .5$, $M = -2$ to a maximum value of approximately 280% for $\delta = 10$, $M = 0$. For values of M in the interval $4 \leq M \leq 10$, the largest value of λ was approximately 16% for all the values of δ. These results should be compared with the results of our previous investigation in which we found excellent agreement for almost all the values of δ and M considered. The lack of agreement that we found in the present investigation indicates that for the ultra-subharmonic solutions the perturbation series is a slowly convergent series for the values of δ and M considered.

An interesting result is the following: If we set $\theta = \omega t$ in the original differential equation and apply the peturbation method taking F proportional to β, we obtain a

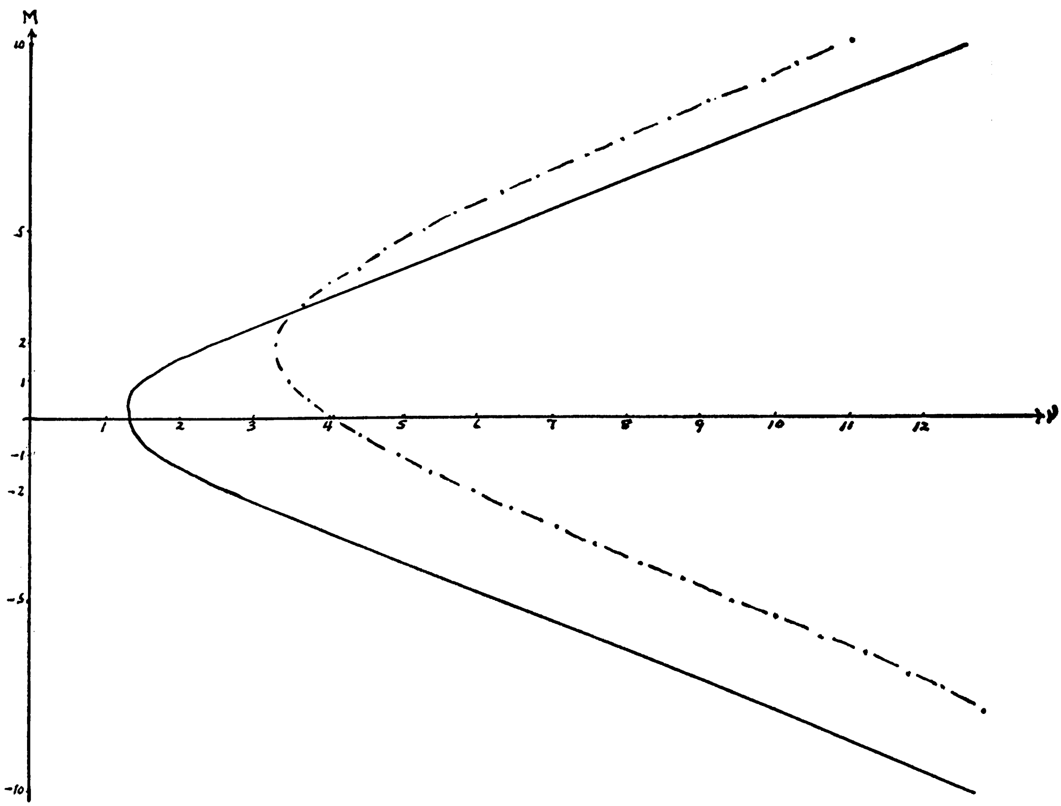

FIG. 4. Comparison between the perturbation and computer response curves for $\delta = 5$.
——— computer curve for periodic solutions of period 4π.
.—.— perturbation curve for the ultra-subharmonic solutions of period 4π for $p/q = 2/3$.

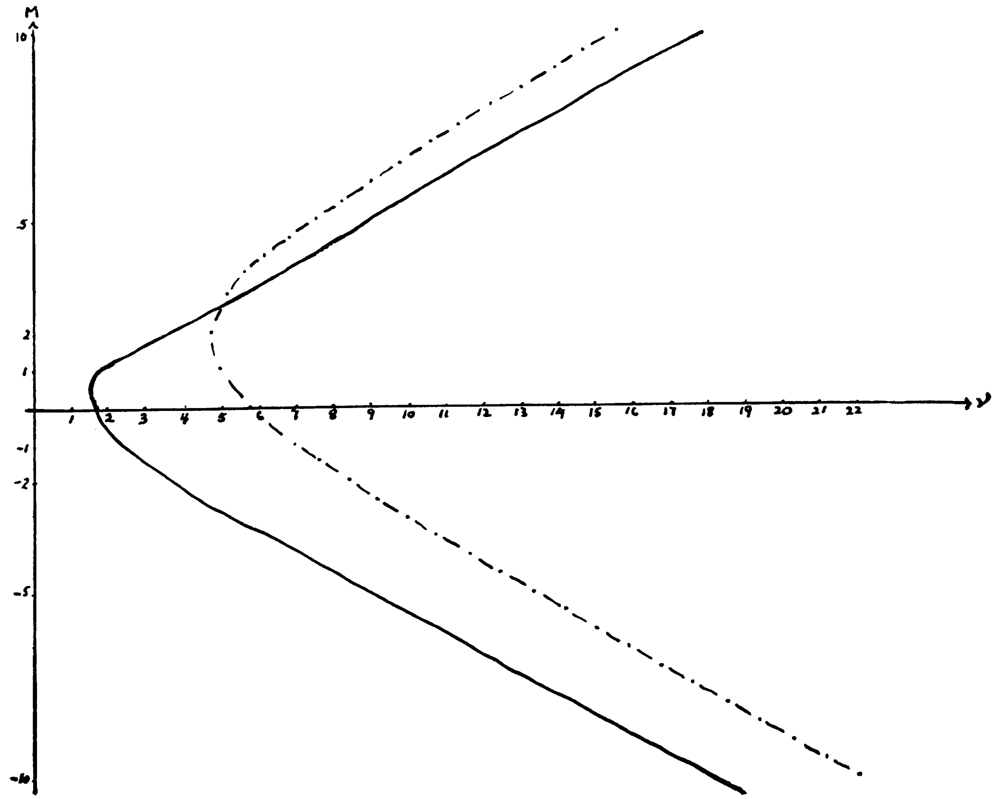

FIG. 5. Comparison between the perturbation and computer response curves for $\delta = 10$.
——— computer curve for periodic solutions of period 4π.
.—.— perturbation curve for the ultra-subharmonic solutions of period 4π for $p/q = 2/3$.

simple approximation for ω^2 that is independent of F and which, in terms of the non-dimensional parameters, reduces to

$$\nu^2 = (p/q)^2\{1 + \tfrac{3}{4}\,\delta M^2\} \tag{5}$$

instead of equation (4). There is very good agreement between the computer curves in Figs. 1 through 5 and the corresponding curves obtained from equation (5) for nearly all the values of δ and M considered for $p/q = 2/3$. The agreement is is also excellent between the computer curves in [1] and the corresponding curves of equation (5) for $p/q = 2$. This indicates that in the perturbation method the better procedure is to assume F proportional to β instead of keeping F fixed.

We conclude with two final remarks. First, Figs. 2 and 3 reveal an interesting feature which did not appear in our previous investigation, and that is the existence of a small interval of values of ν where four different values of M yield the same value of ν. This behaviour was also observed in the figures for $\delta = .3$ and $\delta = .4$ which we did not include in this paper. Secondly, the computer was programmed to print out solutions of period 4π. Some of these turned out to be of period 2π. The point $\nu = .78402617$, $M = 1.7$ on the computer response curve in Figure 1 corresponds to such a solution. This is a bifurcation point. The author is presently engaged in a numerical investigation of the question of bifurcation. I wish to thank my colleagues, Professor L. Heil, who is making the computer available to me, and Mr. H. Givner who did the programming.

Bibliography

1. M. E. Levenson, *A numerical determination of subharmonic response for the Duffing equation*, Quart. Appl. Math. **25**, 11–17 (1967)