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A NOTE ON THE POTENTIAL FLOW PAST A LEMNISCATE AND A GENERAL METHOD OF MILNE-THOMSON*

By J. W. MILES and G. E. BACKUS, *University of California, La Jolla*

Abstract. Basset's (1885) solution for potential flow past a lemniscate is (a) disproved and (b) reproduced by a naive application of "a general method" of Milne–Thomson. The correct solution is also given. The result is of interest for Rayleigh scattering, as well as for potential theory.

1. Lemniscate in uniform flow. We seek the complex potential associated with uniform, plane, potential flow past a lemniscate. We follow Basset's [1], [2] terminology; Moon and Spencer [3] designate the generalized lemniscate as a Cassini oval, and modern usage typically reserves *lemniscate* for the limiting figure ($\alpha = 0$ below) of Bernoulli.

Basset [1] introduces the coordinates ξ and η through the conformal mapping

$$z = c(1 + e^{2\zeta})^{1/2}, \qquad \zeta = \xi + i\eta \qquad (0 < \xi < \infty, -\pi < \eta < \pi) \tag{1}$$

and asserts that the perturbation stream function associated with the introduction of the lemniscate $\xi = \alpha$ in the otherwise uniform flow $\{U, V\}$ is given by

$$\psi = \tfrac{1}{2}i(U - iV)c[1 + e^{-2(\xi-2\alpha-i\eta)}]^{1/2} - \tfrac{1}{2}i(U + iV)c[1 + e^{-2(\xi-2\alpha+i\eta)}]^{1/2}. \tag{2}$$

[Basset actually gives separate stream functions, corresponding to the uniform flows $\{U, 0\}$ and $\{0, V\}$; the results §114(50), (51) in his 1888 treatise differ in sign, corresponding to translation of the lemniscate in an otherwise undisturbed flow, and contain a typographical error.] The corresponding complex potential is

$$w \equiv \phi + i\psi = (U + iV)c(1 + e^{4\alpha-2\zeta})^{1/2}, \tag{3}$$

which (since $e^{-2\zeta} \sim c^2/z^2$) has the asymptotic behaviour of a quadrupole. This cannot be correct, for the equivalent dipole strength of any moving body in a potential flow is proportional to the sum of the displaced and virtual masses [4]. In fact, w, as given by (3), has branch points at $\zeta = 2\alpha \pm \frac{1}{2}i\pi$, both of which lie in the flow field for $\alpha > 0$ or on the boundary for $\alpha = 0$.

The correct result for $w(z)$ must satisfy the boundary condition

$$\psi = -Uy + Vx \qquad (\xi = \alpha) \tag{4}$$

and be analytic in $\xi > \alpha$. We find that these conditions are satisfied by

$$w = (U + iV)c^2e^{2\alpha}(z^2 - c^2)^{-1/2} - (U - iV)[z - (z^2 - c^2)^{1/2}] \tag{5a}$$

$$= Qc[2e^{\alpha} \cosh (\zeta - \alpha - i\beta) - e^{-i\beta}(1 + e^{2\zeta})^{1/2}], \tag{5b}$$

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where

$$Qe^{i\beta} \equiv U + iV. \tag{6}$$

Letting $z \to \infty$ in (5a), we obtain the dipole behaviour

$$w \sim Qc^2(e^{2\alpha+i\beta} - \tfrac{1}{2}e^{-i\beta})z^{-1}. \tag{7}$$

We remark that the dipole moment given by (7) is of interest for Rayleigh scattering by a lemniscate by virtue of Rayleigh's general result that the scattering cross section of any small (compared with the wave length) cylinder depends essentially only on the dipole moment of the cross section [5].

2. Milne-Thomson's general method. Basset gives no details of his derivation of the incorrect result (2) in either his 1885 paper or his 1888 treatise, so that we can only conjecture as to the source of his error. We give here a derivation of (3) through a naive, and of (5b) through a correct, application of "a general method" of Milne–Thomson [6].

Let

$$z = f(\zeta) \equiv f_1(\zeta) + f_2(\zeta) \tag{8}$$

be a function that maps C on $\xi = \alpha$ and assume that f_1 and f_2 can be determined such that both $f_2(\zeta)$ and $f_1(2\alpha - \zeta)$ tend to constant values as $z \to \infty$ (Milne-Thomson requires these functions to vanish, but this difference is trivial. Then, the complex potential

$$Qe^{-i\beta}z + w = Q[e^{-i\beta}f_1(\zeta) + e^{i\beta}f_1^*(2\alpha - \zeta^*)], \tag{9}$$

where the asterisk implies complex conjugation, yields the uniform flow $\{U, V\}$ as $z \to \infty$ and is real on C. The corresponding perturbation potential is given by

$$w = Q[e^{i\beta}f_1^*(2\alpha - \zeta^*) - e^{-i\beta}f_2(\zeta)]. \tag{10}$$

The asterisks may be deleted if $f(\xi)$ is real. It is implicit, although Milne–Thomson does not explicitly state, that f_1 must be determined such that w be regular in $\xi > \alpha$.

We obtain the correct result (5b) by choosing $f_1 = ce^{\zeta}$. We obtain the incorrect result (3) by letting $f(\zeta)$ be given by (1) and choosing $f_1 \equiv f$ and $f_2 \equiv 0$.

References

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