A GENERAL CLASS OF UNSTEADY HEAT FLOW PROBLEMS IN A FINITE COMPOSITE HOLLOW CIRCULAR CYLINDER*

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1. Introduction. The solution of boundary-initial value problems in the conduction of heat in composite circular cylinders is of considerable technological importance, particularly to the aerospace, nuclear and ordnance industries where concentric hollow metallic cylinders of finite length are encountered under diverse heating and boundary- and initial-conditions. These problems have consequently attracted considerable attention and, in the course of time, a number of special solutions have been developed. Among these, for instance, Penner and Sherman [1] dealt with the problem of radial flow of heat in a cylindrical solid core surrounded by a cylindrical shell thermally insulated at the outer boundary, the initial temperature in each region being unequal but uniform. A similar problem, with a convective boundary condition periodic in time, is discussed by Lowell [2] and by Lowell and Patton [3]. A number of similar radial flow problems of slightly more general nature are treated by Jaeger [4]. Thiruvenkatachar and Ramakrishna [5] appear to be the first to consider a case of combined radial and axial flow of heat in a solid core cylinder of two materials under homogeneous boundary conditions and uniform initial conditions. The same problem where the outer radial surface instead of being thermally insulated is kept at a temperature varying sinusoidally with time is discussed by Kumar and Thiruvenkatachar [6] under homogeneous initial conditions. The problem of radial heat flow in a hollow core circular cylinder of two materials is studied by Jaeger [4] for the case of constant surface temperatures and homogeneous initial conditions, and also by Gerhard [7] for the case of constant convective heating on one face and adiabatic conditions on the other, the initial conditions being homogeneous. More recently, Kumar [8] treated the more general problem of combined radial and axial conduction of heat in a two material hollow circular cylinder, with homogeneous initial conditions and under convective type of time-independent boundary conditions prescribed at the inner and outer radial surfaces when the flat ends are kept at the zero initial temperature.

It is thus evident that, as far as heat conduction problems in composite hollow cylinders are concerned, Kumar's study [8] is the most general yet set forth, although it is restricted to (a) time-independent boundary conditions of a special nature, (b) cases of circular symmetry, (c) absence of internal heat sources, (d) assumption of homogeneous initial conditions, and (e) assumption of perfect thermal contact between the two concentric components of the hollow cylinder. The objective of the present paper is to solve the composite hollow cylinder problem in a very general form not subjected to these restrictions and limitations. To this end we first formulate the problem in its general form.

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2. Statement of the problem. Consider a hollow, right, circular cylinder composed of two concentric, radial layers, each of finite length $2l$. We assume that each rigid layer is homogeneous and isotropic, with thermal properties independent of temperature, and that the two layers are in imperfect thermal contact at the common interface, which is characterized by a finite interfacial conductance $h_1 > 0$. The internal heat sources and the boundary heat sources (i.e., source terms introduced by boundary conditions) in the two layers are prescribed in terms of arbitrary but integrable functions of space and time. These are also called volume (bulk) source functions and surface source functions, respectively. Initially, the temperature distribution in each layer is given by arbitrary but integrable functions of space. These latter are also known as initial excitations or initial source functions. Under these prescribed source conditions and considerably general boundary conditions, we wish to determine the resulting unsteady temperature field in each layer.

Using cylindrical polar co-ordinates r, φ, z, and choosing the z co-ordinate along the geometrical axis of the cylinders, we may express the governing differential equations for the temperature distributions T_1 and T_2 as

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\kappa_i}\frac{\partial}{\partial t}\right\} T_i(r, \varphi, z, t) + \frac{1}{k_i} Q_i(r, \varphi, z, t) = 0$$

$$(i = 1, 2;\, T_1 : a < r < b,\, 0 \le \varphi \le 2\pi,\, |z| < l;\, t > 0;$$
$$T_2 : b < r < c,\, 0 \le \varphi \le 2\pi,\, |z| < l;\, t > 0) \qquad (1)$$

where a, b, c denote, respectively, inner, interfacial and outer radii; $k_i > 0$ and $\kappa_i > 0$ denote, respectively, thermal conductivity and thermal diffusivity; $Q_i(r, \varphi, z, t)$ represents the prescribed, arbitrary volume source functions, per unit time and per unit volume, in the two layers, and t is the time. We specify the initial conditions for the Eqs. (1) as

$$T_i(r, \varphi, z, 0) = F_i(r, \varphi, z) \qquad (i = 1, 2;\, T_1 : a < r < b,\, 0 \le \varphi \le 2\pi,\, |z| < l;$$
$$T_2 : b < r < c,\, 0 \le \varphi \le 2\pi,\, |z| < l) \qquad (2)$$

where $F_i(r, \varphi, z)$ represents the prescribed, arbitrary initial sources in the layers. In addition to Eqs. (1) and (2), the temperature distributions $T_i(r, \varphi, z, t)$ are required to satisfy the conditions prescribed at the boundaries and at the interface. To unify the treatment of the most commonly encountered boundary conditions (prescribed surface temperature, prescribed surface heat flux, and linearized radiation or Newtonian type of convection), and thus to be readily able to obtain the corresponding results for the relevant special cases, we express the boundary- and interface-conditions in the following general form:

$$\left.\begin{aligned} &(-\partial/\partial r + h_0)T_1(r, \varphi, z, t) = \rho_1(\varphi, z, t) && (r = a) \\ &k_1\, \partial T_1(r, \varphi, z, t)/\partial r = k_2\, \partial T_2(r, \varphi, z, t)/\partial r && (r = b) \\ &\qquad = h_1\{T_2(r, \varphi, z, t) - T_1(r, \varphi, z, t)\} && \\ &(\partial/\partial r + h_2)T_2(r, \varphi, z, t) = \rho_2(\varphi, z, t) && (r = c) \end{aligned}\right\} (0 \le \varphi \le 2\pi;\, |z| < l;\, t > 0) \quad \begin{matrix}(3a)\\(3b,c)\\ \\(3d)\end{matrix}$$

$$\left.\begin{aligned} &(-\partial/\partial z + h_0')T_1(r, \varphi, z, t) = \zeta_1(r, \varphi, t) && (z = -l), \\ &(\partial/\partial z + h_2')T_1(r, \varphi, z, t) = \chi_1(r, \varphi, t) && (z = l), \end{aligned}\right\} (a < r < b;\, 0 \le \varphi \le 2\pi;\, t > 0) \quad \begin{matrix}(4a)\\(4b)\end{matrix}$$

$$\left.\begin{aligned}(-\partial/\partial z + h_0')T_2(r, \varphi, z, t) &= \zeta_2(r, \varphi, t) \quad (z = -l),\\ (\partial/\partial z + h_2')T_2(r, \varphi, z, t) &= \chi_2(r, \varphi, t) \quad (z = l),\end{aligned}\right\} (b < r < c;\ 0 \leq \varphi \leq 2\pi;\ t > 0). \quad \begin{matrix}(5a)\\(5b)\end{matrix}$$

In Eqs. (3a) and (3d), $h_0 \geq 0$ and $h_2 \geq 0$ are, respectively, given surface coefficients linearly related to the corresponding heat transfer coefficients at the internal and external radial surfaces $r = a$ and $r = c$, and the corresponding surface sources are represented by the arbitrary functions $\rho_1(\varphi, z, t)$ and $\rho_2(\varphi, z, t)$. Equations (3b, c) express the discontinuity of the temperature and the discontinuity of the radial gradient of temperature at the interface $r = b$. In the event of perfect interfacial thermal contact, we let the interfacial thermal conductance $h_1 \to \infty$, in which case Eqs. (3b, c) express the continuity of thermal flux and the continuity of temperature across the interface $r = b$. This simplifying assumption has been employed in all of the references [1]–[8] cited. Similarly, Eqs. (4) and (5) introduce arbitrary surface sources $\zeta_1(r, \varphi, t)$, $\zeta_2(r, \varphi, t)$ and $\chi_1(r, \varphi, t)$, $\chi_2(r, \varphi, t)$ at the flat ends $z = -l$ and $z = l$, respectively, where $h_0' \geq 0$ and $h_2' \geq 0$ are the corresponding prescribed surface coefficients proportional to the relevant surface heat transfer coefficients. By assigning appropriate values to the coefficients h_0 , h_2 , h_0' and h_2' , we can realize a wide variety of diverse combinations of Dirichlet, and/or Neumann, and/or Robin type of boundary conditions at the four boundaries $r = a$, $r = c$, $z = \pm l$. Equations (1), (2), (3), (4) and (5) constitute a general mathematical formulation of the problem.

3. Method of solution. We note that the technique of solution employed in [1], [2], [3] and [7] is the method of separation of variables, whereas the procedure followed in [4], [5], [6] and [8] employs the method of Laplace transform. It is well known that in the event of general and/or complex problems of heat flow, the application of the Laplace transform technique generally results in such complicated expressions for the inverse transform as to render its use prohibitive. This is particularly true in the case of the general boundary-initial value problem under consideration. Since the use of the separation of variables technique does not offer any particular advantage either, we shall utilize a different approach which will greatly facilitate the general solution of the problem. In order to deal effectively with the general problem at hand, it is, as will be seen, advantageous to resort to an inverse method of eigenfunction expansion, better known as the method of finite integral transformation. To this end, we define an auxiliary problem.

4. Auxiliary problem. Guided by a procedure outlined by Ölçer [9], we define the following auxiliary eigenvalue problem associated with the system of Eqs. (1), (3), (4) and (5).

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + \frac{\lambda}{\kappa_i}\right)\phi_i(r, \varphi, z) = 0$$

$$(i = 1, 2;\ \phi_1 : a < r < b,\ 0 \leq \varphi \leq 2\pi,\ |z| < l;\quad \phi_2 : b < r < c,\ 0 \leq \varphi \leq 2\pi,\ |z| < l) \quad (6)$$

$$\left.\begin{aligned}(-\partial/\partial r + h_0)\phi_1(r, \varphi, z) &= 0 \quad (r = a),\\ k_1\, \partial\phi_1(r, \varphi, z)/\partial r &= k_2\, \partial\phi_2(r, \varphi, z)/\partial r \quad (r = b)\\ &= h_1\{\phi_2(r, \varphi, z) - \phi_1(r, \varphi, z)\},\\ (\partial/\partial r + h_2)\phi_2(r, \varphi, z) &= 0 \quad (r = c)\end{aligned}\right\} (0 \leq \varphi \leq 2\pi;\ |z| < l) \quad (7)$$

$$\left.\begin{aligned}(-\partial/\partial z + h_0')\phi_1(r, \varphi, z) &= 0 \qquad (z = -l),\\ (\partial/\partial z + h_2')\phi_1(r, \varphi, z) &= 0 \qquad (z = l)\end{aligned}\right\} \quad (a < r < b;\, 0 \le \varphi \le 2\pi) \tag{8}$$

$$\left.\begin{aligned}(-\partial/\partial z + h_0')\phi_2(r, \varphi, z) &= 0 \qquad (z = -l),\\ (\partial/\partial z + h_2')\phi_2(r, \varphi, z) &= 0 \qquad (z = l)\end{aligned}\right\} \quad (b < r < c;\, 0 \le \varphi \le 2\pi) \tag{9}$$

where

$$\begin{aligned}\phi_i \equiv \phi_{i;\lambda} &= \phi_{1;\lambda}(r, \varphi, z) \qquad (a < r < b),\\ &= \phi_{2;\lambda}(r, \varphi, z) \qquad (b < r < c)\end{aligned}$$

denotes the real eigenfunctions in the two layers, and corresponds to the real eigenvalue λ. After lengthy manipulations, the solution to the eigenvalue problem is obtained as

$$\begin{aligned}\phi_1(r, \varphi, z) &= [k_2\mu_2 C_m'^{(2)}(\mu_2 b) C_m^{(1)}(\mu_1 r)]\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}[\nu_j \cos \nu_j(l + z) + h_0' \sin \nu_j(l + z)],\\ \phi_2(r, \varphi, z) &= [k_1\mu_1 C_m'^{(1)}(\mu_1 b) C_m^{(2)}(\mu_2 r)]\begin{Bmatrix}\cos(m\varphi)\\ \sin(m\varphi)\end{Bmatrix}[\nu_j \cos \nu_j(l + z) + h_0' \sin \nu_j(l + z)]\end{aligned} \tag{10}$$

where

$$\begin{aligned}C_m^{(1)}(\mu_1 r) &= \mu_1 C_{1,0}^{(m)}(\mu_1 a, \mu_1 r) - h_0 C_{0,0}^{(m)}(\mu_1 a, \mu_1 r),\\ C_m^{(2)}(\mu_2 r) &= \mu_2 C_{0,1}^{(m)}(\mu_2 r, \mu_2 c) + h_2 C_{0,0}^{(m)}(\mu_2 r, \mu_2 c)\end{aligned} \tag{11}$$

and the primes over $C_m^{(1)}$ and $C_m^{(2)}$ indicate differentiation with respect to their arguments. The cylinder functions (of two arguments and of order m) appearing in (11) are defined in terms of the cross products of Bessel functions of order m:

$$\begin{aligned}C_{0,0}^{(m)}(x, y) &= C^{(m)}(x, y) = J_m(x) Y_m(y) - Y_m(x) J_m(y),\\ C_{p,q}^{(m)}(x, y) &= \partial^{p+q} C^{(m)}(x, y)/\partial x^p\, \partial y^q \qquad (p, q = 0, 1, 2, \cdots)\end{aligned} \tag{12}$$

where $J_m(x)$ is the Bessel function of the first kind of order m, and $Y_m(x)$ is the Bessel function of the second kind of order m. In equation (10), $m = 0, 1, 2, \cdots$; ν_j is the jth nonnegative root of the axial frequency equation

$$\nu_j(h_0' + h_2') \cos(2\nu_j l) + (h_0' h_2' - \nu_j^2) \sin(2\nu_j l) = 0, \tag{13}$$

μ_1 and μ_2 are related by the coupling equation

$$\mu_2^2 = (\kappa_1/\kappa_2)\mu_1^2 + (\kappa_1/\kappa_2 - 1)\nu_j^2 \tag{14}$$

and are determined, for a given value of m and a given value of ν_j, as the kth nonnegative root of the radial frequency equation

$$k_1 h_1 \mu_1 C_m^{(2)}(\mu_2 b) C_m'^{(1)}(\mu_1 b) = k_2 \mu_2 C_m'^{(2)}(\mu_2 b)\{k_1 \mu_1 C_m'^{(1)}(\mu_1 b) + h_1 C_m^{(1)}(\mu_1 b)\}. \tag{15}$$

The resultant eigenvalues λ are obtained as

$$\lambda = \kappa_1(\mu_1^2 + \nu_j^2) = \kappa_2(\mu_2^2 + \nu_j^2). \tag{16}$$

The eigenvalues λ and the eigenfunctions $\phi_i(r, \varphi, z)$, $(i = 1, 2)$, each comprise triple index sets. Thus, in a more explicit form,

$$\left.\begin{aligned} \lambda &\equiv \lambda_{j,k,m}\,, \\ \phi_i &\equiv \phi_{i;\lambda} = \phi_{i;j,k,m}(r,\varphi,z) \end{aligned}\right\} \qquad (j,k,m = 0,1,2,\cdots). \tag{17}$$

An important property of the eigenfunctions $\phi_{i;\lambda}$ is the orthogonality relation expressed by

$$A(\lambda)\left\{\frac{k_1}{\kappa_1}\int_a^b\int_0^{2\pi}\int_{-l}^{l}\phi_{1;\lambda}\phi_{1;\lambda'}r\,dr\,d\varphi\,dz + \frac{k_2}{\kappa_2}\int_b^c\int_0^{2\pi}\int_{-l}^{l}\phi_{2;\lambda}\phi_{2;\lambda'}r\,dr\,d\varphi\,dz\right\} = \delta_{\lambda\lambda'}$$

where $\delta_{\lambda\lambda'}$ is the Kronecker delta. This expression is obtained from the system of Eqs. (6), (7), (8) and (9). We thus have

$$\frac{1}{A_{j,k,m}(\lambda)} = \frac{k_1}{\kappa_1}\int_a^b\int_0^{2\pi}\int_{-l}^{l}\{\phi_{1;j,k,m}(r,\varphi,z)\}^2 r\,dr\,d\varphi\,dz + \frac{k_2}{\kappa_2}\int_b^c\int_0^{2\pi}\int_{-l}^{l}\{\phi_{2;j,k,m}(r,\varphi,z)\}^2 r\,dr\,d\varphi\,dz \tag{18}$$

where

$$A_{j,k,m}(\lambda) \equiv A(\lambda_{j,k,m}) \equiv A(\lambda).$$

Another important relation between the eigenfunctions ϕ_i and the eigenvalues λ, which is obtained from the system of (6), (7), (8) and (9), is the following:

$$\begin{aligned} \lambda = A(\lambda)\Bigg\{ & k_1\int_a^b\int_0^{2\pi}\int_{-l}^{l}\left[\left(\frac{\partial\phi_1}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi_1}{\partial\varphi}\right)^2 + \left(\frac{\partial\phi_1}{\partial z}\right)^2\right] r\,dr\,d\varphi\,dz \\ & + k_2\int_b^c\int_0^{2\pi}\int_{-l}^{l}\left[\left(\frac{\partial\phi_2}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi_2}{\partial\varphi}\right)^2 + \left(\frac{\partial\phi_2}{\partial z}\right)^2\right] r\,dr\,d\varphi\,dz \\ & + k_1\left[ah_0\int_0^{2\pi}\int_{-l}^{l}\phi_1^2(a,\varphi,z)\,d\varphi\,dz + h_0'\int_a^b\int_0^{2\pi}\phi_1^2(r,\varphi,-l)r\,dr\,d\varphi\right. \\ & \left. + h_2'\int_a^b\int_0^{2\pi}\phi_1^2(r,\varphi,l)r\,dr\,d\varphi\right] + k_2\left[ch_2\int_0^{2\pi}\int_{-l}^{l}\phi_2^2(c,\varphi,z)\,d\varphi\,dz\right. \\ & \left. + h_0'\int_b^c\int_0^{2\pi}\phi_2^2(r,\varphi,-l)r\,dr\,d\varphi + h_2'\int_b^c\int_0^{2\pi}\phi_2^2(r,\varphi,l)r\,dr\,d\varphi\right] \\ & + bh_1\int_0^{2\pi}\int_{-l}^{l}[\phi_2(b,\varphi,z) - \phi_1(b,\varphi,z)]^2\,d\varphi\,dz\Bigg\}. \end{aligned}$$

Since, from (18), $A(\lambda) > 0$, and since all the parameters appearing in the right-hand side of the above expression are nonnegative, it follows from this relation that all the eigenvalues λ are nonnegative. This expression shows that $\lambda = 0$ is possible only when $h_0 = h_0' = h_2 = h_2' = 0$, and $\phi_1 = \phi_2 = \text{constant} \neq 0$. Thus, except for the case of prescribed heat flux conditions over the entire boundary of the composite hollow cylinder, the eigenvalues λ are all positive.

We now use the expressions (10) to carry out the integrations indicated in (18), and obtain, after lengthy algebra,

$$\frac{1}{A_{j,k,m}(\lambda)} = \frac{\pi}{4}(1 + \delta_{0m})k_1k_2\left[\frac{2l(\nu_j^2 + h_0'^2)(\nu_j^2 + h_2'^2) + (h_0' + h_2')(\nu_j^2 + h_0'h_2')}{(\nu_j^2 + h_2'^2)}\right]$$
$$\cdot[(k_2/\kappa_1)\mu_2^2[C_m^{(2)}(\mu_2 b)]^2\{(b^2 - m^2/\mu_1^2)[C_m^{(1)}(\mu_1 b)]^2$$
$$+ b^2[C_m'^{(1)}(\mu_1 b)]^2 - (4/\pi^2)[1 + (1/\mu_1^2)(h_0^2 - m^2/a^2)]\}$$
$$- (k_1/\kappa_2)\mu_1^2[C_m^{(1)}(\mu_1 b)]^2\{(b^2 - m^2/\mu_2^2)[C_m^{(2)}(\mu_2 b)]^2$$
$$+ b^2[C_m'^{(2)}(\mu_2 b)]^2 - (4/\pi^2)[1 + (1/\mu_2^2)(h_2^2 - m^2/c^2)]\}] \qquad (19)$$

where we have used the frequency equations (13) and (15), and the Wronskian relations for the relevant cylinder functions. Finally, we let

$$R_1(\mu_1 r) = k_2\mu_2 C_m'^{(2)}(\mu_2 b)C_m^{(1)}(\mu_1 r),$$
$$R_2(\mu_2 r) = k_1\mu_1 C_m'^{(1)}(\mu_1 b)C_m^{(2)}(\mu_2 r), \qquad (20)$$

$$Z_j(z) = \nu_j \cos \nu_j(l + z) + h_0' \sin \nu_j(l + z) \qquad (21)$$

from which it follows that

$$R_1(\mu_1 a) = -(2/\pi)(k_2\mu_2/a)C_m'^{(2)}(\mu_2 b),$$
$$R_2(\mu_2 c) = (2/\pi)(k_1\mu_1/c)C_m'^{(1)}(\mu_1 b), \qquad (22)$$

$$Z_j(l) = \beta_j\nu_j \ ,$$
$$Z_j(-l) = \nu_j \ , \qquad (23)$$

$$\beta_j = ((\nu_j^2 + h_0'^2)/(h_0' + h_2'))((\sin 2\nu_j l)/\nu_j)) = \pm((\nu_j^2 + h_0'^2)/(\nu_j^2 + h_2'^2))^{1/2}. \qquad (24)$$

5. Resolution of the problem. We now define a pair of transformations by

$$T_1^*(\lambda, t) = \frac{k_1}{\kappa_1}\int_a^b \int_0^{2\pi} \int_{-l}^{l} \phi_{1;\lambda}(r, \varphi, z)T_1(r, \varphi, z, t)r\,dr\,d\varphi\,dz,$$
$$T_2^*(\lambda, t) = \frac{k_2}{\kappa_2}\int_b^c \int_0^{2\pi} \int_{-l}^{l} \phi_{2;\lambda}(r, \varphi, z)T_2(r, \varphi, z, t)r\,dr\,d\varphi\,dz. \qquad (25)$$

The inverse transform of (25) is readily obtained if we note that the eigenfunctions $\phi_{i;\lambda}(r, \varphi, z)$ form a complete set. In view of this and the orthogonality property expressed by the equation preceding (18), we have the inversion formula for the transformations (25):

$$T_i(r, \varphi, z, t) = \sum_\lambda A(\lambda)\phi_{i;\lambda}(r, \varphi, z) \sum_{i=1}^{2} T_i^*(\lambda, t) \qquad (i = 1, 2) \qquad (26)$$

where the coefficient $A(\lambda)$ is given by (18) or, more explicitly, by (19).

We next apply the transformations (25) to the differential equations (1). To this end we assume that the two cylindrical layers are independent of time and, bearing in mind that the eigenfunctions $\phi_{i;\lambda}$ satisfy the differential equations (6), apply the Gauss divergence theorem in each cylindrical layer. In view of the boundary- and interface-conditions (3), (4), (5) and (7), (8), (9), this transformation finally results in

$$(d/dt + \lambda) \sum_{i=1}^{2} T_i^*(\lambda, t) = G(\lambda, t) \qquad (27)$$

where

$$G(\lambda, t)$$

$$= \int_a^b \int_0^{2\pi} \int_{-l}^{l} \phi_{1;\lambda}(r, \varphi, z) Q_1(r, \varphi, z, t) r\, dr\, d\varphi\, dz + ak_1 \int_0^{2\pi} \int_{-l}^{l} \phi_{1;\lambda}(a, \varphi, z) \rho_1(\varphi, z, t)\, d\varphi\, dz$$

$$+ k_1 \int_a^b \int_0^{2\pi} \phi_{1;\lambda}(r, \varphi, -l) \zeta_1(r, \varphi, t) r\, dr\, d\varphi + k_1 \int_a^b \int_0^{2\pi} \phi_{1;\lambda}(r, \varphi, l) \chi_1(r, \varphi, t) r\, dr\, d\varphi$$

$$+ \int_b^c \int_0^{2\pi} \int_{-l}^{l} \phi_{2;\lambda}(r, \varphi, z) Q_2(r, \varphi, z, t) r\, dr\, d\varphi\, dz + ck_2 \int_0^{2\pi} \int_{-l}^{l} \phi_{2;\lambda}(c, \varphi, z) \rho_2(\varphi, z, t)\, d\varphi\, dz$$

$$+ k_2 \int_b^c \int_0^{2\pi} \phi_{2;\lambda}(r, \varphi, -l) \zeta_2(r, \varphi, t) r\, dr\, d\varphi + k_2 \int_b^c \int_0^{2\pi} \phi_{2;\lambda}(r, \varphi, l) \chi_2(r, \varphi, t) r\, dr\, d\varphi. \tag{28}$$

Integration of Eq. (27), subject to the initial conditions (2) transformed by (25), leads to

$$\sum_{i=1}^{2} T_i^*(\lambda, t) = \exp(-\lambda t) \sum_{i=1}^{2} F_i^*(\lambda) + \int_0^t \exp(-\lambda(t - \tau)) G(\lambda, \tau)\, d\tau \tag{29}$$

where

$$\begin{aligned} F_1^*(\lambda) &= \frac{k_1}{\kappa_1} \int_a^b \int_0^{2\pi} \int_{-l}^{l} \phi_{1;\lambda}(r, \varphi, z) F_1(r, \varphi, z) r\, dr\, d\varphi\, dz, \\ F_2^*(\lambda) &= \frac{k_2}{\kappa_2} \int_b^c \int_0^{2\pi} \int_{-l}^{l} \phi_{2;\lambda}(r, \varphi, z) F_2(r, \varphi, z) r\, dr\, d\varphi\, dz. \end{aligned} \tag{30}$$

Introducing (29) into (26) we have the formal solution for the temperature distributions $T_i(r, \varphi, z, t)$:

$$T_i(r, \varphi, z, t) = \sum_\lambda A(\lambda) \phi_{i;\lambda}(r, \varphi, z) \exp(-\lambda t) \left\{ \sum_{i=1}^{2} F_i^*(\lambda) + \int_0^t \exp(\lambda\tau) G(\lambda, \tau)\, d\tau \right\}$$

$$(i = 1, 2). \tag{31}$$

Substituting (28) and (30) into (31), and recalling the expressions (10) for the eigenfunctions, we may express the general solution in a more explicit form as

$$T_i(r, \varphi, z, t) = \sum_j \sum_k \sum_m A_{j,k,m}(\lambda) R_i(\mu_i r) Z_j(z) \exp(-\lambda_{j,k,m} t)$$

$$\cdot \int_0^{2\pi} \left\{ \int_{-l}^{l} \left\{ \frac{k_1}{\kappa_1} \int_a^b R_1(\mu_1 r) F_1(r, \varphi', z) r\, dr + \frac{k_2}{\kappa_2} \int_b^c R_2(\mu_2 r) F_2(r, \varphi', z) r\, dr \right\} Z_j(z)\, dz \right.$$

$$+ \int_0^t \exp(\lambda_{j,k,m}\tau) \left\{ \int_{-l}^{l} \left[\left\{ \int_a^b R_1(\mu_1 r) Q_1(r, \varphi', z, \tau) r\, dr + \int_b^c R_2(\mu_2 r) Q_2(r, \varphi', z, \tau) r\, dr \right\} \right. \right.$$

$$\left. + (2/\pi) k_1 k_2 \{ \mu_1 C_m'^{(1)}(\mu_1 b) \rho_2(\varphi', z, \tau) - \mu_2 C_m'^{(2)}(\mu_2 b) \rho_1(\varphi', z, \tau) \} \right] Z_j(z)\, dz$$

$$+ \nu_j \left[k_1 \int_a^b R_1(\mu_1 r) \{ \zeta_1(r, \varphi', \tau) + \beta_j \chi_1(r, \varphi', \tau) \} r\, dr \right.$$

$$\left. \left. \left. + k_2 \int_b^c R_2(\mu_2 r) \{ \zeta_2(r, \varphi', \tau) + \beta_j \chi_2(r, \varphi', \tau) \} r\, dr \right] \right\} d\tau \right\} \cos m(\varphi - \varphi')\, d\varphi'$$

$$(i = 1, 2) \tag{32}$$

where $A_{j,k,m}$, R_i , Z_j , β_j are, respectively, given by (19), (20), (21), (24). The j- and k-summations extend, respectively, over the nonnegative roots of the frequency equations (13) and (15) coupled by (14), the resultant eigenvalues $\lambda_{j,k,m}$ are given by (16), and $m = 0, 1, 2, \cdots$.

The general solution (32) shows how the two temperature distributions depend on the thermal properties of and the source functions in the two cylindrical layers. The initial sources characterized by $F_i(r, \varphi, z)$ give rise to transient terms only, unless $h_0 = h_0' = h_2 = h_2' = 0$, in which case an additional term, steady-state in time, enters the solution. The volume sources characterized by $Q_i(r, \varphi, z, t)$ and the surface sources characterized by $\rho_i(\varphi, z, t)$, $\zeta_i(r, \varphi, t)$, $\chi_i(r, \varphi, t)$ $(i = 1, 2)$, even when independent of time, give rise to both transient terms (which die out exponentially with time) and steady-state terms. Only in the event that all the time-dependent source functions are Dirac delta functions in time (i.e., instantaneous pulses at zero time) does the general expression (32) give rise to entirely transient terms provided, again, that the surface coefficients h_0 , h_0' , h_2 and h_2' are not simultaneously zero.

From the general solution (32) numerous particular solutions corresponding to the very many specialized versions of the general problem defined by the system of equations (1), (2), (3), (4) and (5) can be readily recovered. For example, if the problem is one of axial symmetry, that is, if the source functions F_i , Q_i , ρ_i , ζ_i , χ_i , $(i = 1, 2)$ are independent of the polar angle φ, then in the summation over m only the term corresponding to $m = 0$ contributes in (32); integration with respect to φ' gives a factor of 2π; (32) reduces to a double infinite series over j, k, and the temperature distributions become independent of φ. Similarly, in the event of no axial conduction, i.e., in the case where the source functions F_i , Q_i , ρ_i are independent of z and, furthermore, the cylinder ends $z = \pm l$ are insulated $(h_0' = h_2' = \zeta_i = \chi_i = 0)$, the situation reduces to that of the two-dimensional, unsteady flow of heat in a composite, hollow, circular, thin disk. In this case the frequency equation (13) reduces to $\sin(2\nu_j l) = 0$, and only the term corresponding to $\nu_0 = 0$ contributes to the j-summation appearing in (32) which thus becomes independent of z, and reduces to a double summation over k and m.

Lastly, it is of interest to note that the general solution (32) can also be specialized with respect to geometry. Thus, for example, the simpler case of a solid core composite cylinder problem can be readily obtained from (32) as a limiting case by letting $a = h_0 = \rho_1 = 0$. This simpler problem does not appear to have been previously solved in complete generality. Another special case which is contained in the solution (32) is the limiting case of a single component hollow circular cylinder. In this case it is sufficient to let $h_1 = \infty$, $k_2 = k_1$, $\kappa_2 = \kappa_1$, and $c = b$ in (32) which then becomes the general solution to the corresponding hollow cylinder problem recently treated in detail by Ölçer [10]. The axially symmetric case of a somewhat simpler version of this problem has been treated previously by Marchi and Zgrablich [11], and the special two-dimensional case of the same problem in a circular ring has been studied by Kleiner [12].

6. Alternative forms of the solution. The general solution as given in the form (32) does not converge uniformly unless the surface sources ρ_i , ζ_i and χ_i $(i = 1, 2)$ are all identically zero. It is therefore desirable to convert the solution (32) into uniformly convergent alternative forms more applicable to detailed computation. In this section we shall derive two such alternative forms. For this purpose we shall make the additional assumptions that the time-dependent forcing functions all possess first order partial derivatives with respect to time, and that the surface coefficients do not all vanish simultaneously. In the event that $h_0 = h_0' = h_2 = h_2' = 0$, the procedure outlined in this section requires a fundamental modification which will be discussed in a future study.

We now introduce an associated steady temperature field by $T_i^{(0)}(r, \varphi, z, t)$ $(i = 1, 2)$ in which t is regarded as a parameter. We choose $T_i^{(0)}$ to satisfy the following system of equations:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right)T_i^{(0)}(r, \varphi, z, t) + \frac{1}{k_i} Q_i(r, \varphi, z, t) = 0$$

$$(i = 1, 2;\ T_1^{(0)}: a < r < b,\ 0 \leq \varphi \leq 2\pi,\ |z| < l;\quad T_2^{(0)}: b < r < c,\ 0 \leq \varphi \leq 2\pi,\ |z| < l) \tag{33}$$

$$\left.\begin{aligned} &(-\partial/\partial r + h_0)T_1^{(0)}(r, \varphi, z, t) = \rho_1(\varphi, z, t) && (r = a),\\ &k_1\, \partial T_1^{(0)}(r, \varphi, z, t)/\partial r = k_2\, \partial T_2^{(0)}(r, \varphi, z, t)/\partial r && (r = b),\\ &\qquad = h_1\{T_2^{(0)}(r, \varphi, z, t) - T_1^{(0)}(r, \varphi, z, t)\} &&\\ &(\partial/\partial r + h_2)T_2^{(0)}(r, \varphi, z, t) = \rho_2(\varphi, z, t) && (r = c) \end{aligned}\right\} (0 \leq \varphi \leq 2\pi;\ |z| < l) \tag{34}$$

$$\left.\begin{aligned} &(-\partial/\partial z + h_0')T_1^{(0)}(r, \varphi, z, t) = \zeta_1(r, \varphi, t) && (z = -l),\\ &(\partial/\partial z + h_2')T_1^{(0)}(r, \varphi, z, t) = \chi_1(r, \varphi, t) && (z = l) \end{aligned}\right\} (a < r < b; 0 \leq \varphi \leq 2\pi) \tag{35}$$

$$\left.\begin{aligned} &(-\partial/\partial z + h_0')T_2^{(0)}(r, \varphi, z, t) = \zeta_2(r, \varphi, t) && (z = -l),\\ &(\partial/\partial z + h_2')T_2^{(0)}(r, \varphi, z, t) = \chi_2(r, \varphi, t) && (z = l) \end{aligned}\right\} (b < r < c; 0 \leq \varphi \leq 2\pi). \tag{36}$$

The result of transforming this system by the transformation (25) is

$$\sum_{i=1}^{2} T_i^{(0)*}(\lambda, t) = \frac{1}{\lambda} G(\lambda, t) \tag{37}$$

where $G(\lambda, t)$ is given by expression (28). In view of (37), equation (27) may be rewritten as

$$\left(\frac{d}{dt} + \lambda\right) \sum_{i=1}^{2} T_i^*(\lambda, t) = \lambda \sum_{i=1}^{2} T_i^{(0)*}(\lambda, t) \tag{38}$$

the result of integration of which, subject to the initial conditions (2) transformed by (25), can be expressed in the form of

$$\sum_{i=1}^{2} \{T_i^*(\lambda, t) - T_i^{(0)*}(\lambda, t)\} = \exp(-\lambda t) \sum_{i=1}^{2} \{F_i^*(\lambda) - T_i^{(0)*}(\lambda, 0)\}$$
$$- \int_0^t \exp(-\lambda(t - \tau)) \frac{\partial}{\partial \tau} \sum_{i=1}^{2} T_i^{(0)*}(\lambda, \tau)\, d\tau. \tag{39}$$

We now rewrite the inversion formula (26) in the following form:

$$T_i(r, \varphi, z, t)$$
$$= T_i^{(0)}(r, \varphi, z, t) + \sum_{\lambda} A(\lambda)\phi_{i;\lambda}(r, \varphi, z) \sum_{i=1}^{2} \{T_i^*(\lambda, t) - T_i^{(0)*}(\lambda, t)\} \qquad (i = 1, 2)$$

which, upon combination with equation (39), becomes

$$T_i(r, \varphi, z, t) = T_i^{(0)}(r, \varphi, z, t) + \sum_{\lambda} A(\lambda)\phi_{i;\lambda}(r, \varphi, z)$$
$$\cdot \exp(-\lambda t) \sum_{i=1}^{2} \left\{ F_i^*(\lambda) - T_i^{(0)*}(\lambda, 0) - \int_0^t \exp(\lambda\tau) \frac{\partial T_i^{(0)*}(\lambda, \tau)}{\partial \tau} d\tau \right\} \quad (i = 1, 2). \quad (40a)$$

In view of relation (37) we can rewrite the expression (40a) in the form of

$$T_i(r, \varphi, z, t) = T_i^{(0)}(r, \varphi, z, t) + \sum_{\lambda} A(\lambda)\phi_{i;\lambda}(r, \varphi, z)$$
$$\cdot \exp(-\lambda t) \left\{ \sum_{i=1}^{2} F_i^*(\lambda) - \frac{1}{\lambda} G(\lambda, 0) - \frac{1}{\lambda} \int_0^t \exp(\lambda\tau) \frac{\partial G(\lambda, \tau)}{\partial \tau} d\tau \right\} \quad (i = 1, 2) \quad (40b)$$

where now the $T_i^{(0)}$ function does not appear in the summation over λ. Expressions (40a) and (40b) are the two alternative forms of (31). In more explicit forms they are, respectively,

$$T_i(r, \varphi, z, t) = T_i^{(0)}(r, \varphi, z, t) + \sum_{j} \sum_{k} \sum_{m} {}_{(j=k=m\neq 0)} A_{j,k,m}(\lambda) R_i(\mu_i r) Z_j(z) \exp(-\lambda_{j,k,m} t)$$
$$\cdot \int_0^{2\pi} \left[\int_{-l}^{l} \left\{ \frac{k_1}{\kappa_1} \int_a^b R_1(\mu_1 r)[F_1(r, \varphi', z) - T_1^{(0)}(r, \varphi', z, 0)] r\, dr \right.\right.$$
$$\left. + \frac{k_2}{\kappa_2} \int_b^c R_2(\mu_2 r)[F_2(r, \varphi', z) - T_2^{(0)}(r, \varphi', z, 0)] r\, dr \right\} Z_j(z)\, dz$$
$$- \int_0^t \exp(\lambda_{j,k,m}\tau) \int_{-l}^{l} \left\{ \frac{k_1}{\kappa_1} \int_a^b R_1(\mu_1 r) \frac{\partial T_1^{(0)}(r, \varphi', z, \tau)}{\partial \tau} r\, dr \right.$$
$$\left.\left. + \frac{k_2}{\kappa_2} \int_b^c R_2(\mu_2 r) \frac{\partial T_2^{(0)}(r, \varphi', z, \tau)}{\partial \tau} r\, dr \right\} Z_j(z)\, dz\, d\tau \right] \cos m(\varphi - \varphi')\, d\varphi' \quad (i = 1, 2) \quad (41a)$$

$$T_i(r, \varphi, z, t) = T_i^{(0)}(r, \varphi, z, t) + \sum_{j} \sum_{k} \sum_{m} {}_{(j=k=m\neq 0)} A_{j,k,m}(\lambda) R_i(\mu_i r) Z_j(z) \exp(-\lambda_{j,k,m} t)$$
$$\cdot \int_0^{2\pi} \left\{ \int_{-l}^{l} \left\{ \frac{k_1}{\kappa_1} \int_a^b R_1(\mu_1 r) F_1(r, \varphi', z) r\, dr + \frac{k_2}{\kappa_2} \int_b^c R_2(\mu_2 r) F_2(r, \varphi', z) r\, dr \right\} Z_j(z)\, dz \right.$$
$$- \frac{1}{\lambda_{j,k,m}} \left\{ \int_{-l}^{l} \left[\left\{ \int_a^b R_1(\mu_1 r) Q_1(r, \varphi', z, 0) r\, dr + \int_b^c R_2(\mu_2 r) Q_2(r, \varphi', z, 0) r\, dr \right\} \right.\right.$$
$$\left. + \frac{2}{\pi} k_1 k_2 \left\{ \mu_1 C_m'^{(1)}(\mu_1 b) \rho_2(\varphi', z, 0) - \mu_2 C_m'^{(2)}(\mu_2 b) \rho_1(\varphi', z, 0) \right\} \right] Z_j(z)\, dz$$
$$+ \nu_j \left[k_1 \int_a^b R_1(\mu_1 r) \{ \zeta_1(r, \varphi', 0) + \beta_j \chi_1(r, \varphi', 0) \} r\, dr \right.$$
$$\left.\left. + k_2 \int_b^c R_2(\mu_2 r) \{ \zeta_2(r, \varphi', 0) + \beta_j \chi_2(r, \varphi', 0) \} r\, dr \right] \right\}$$
$$- \int_0^t \frac{\exp(\lambda_{j,k,m}\tau)}{\lambda_{j,k,m}} \left\{ \int_{-l}^{l} \left[\left\{ \int_a^b R_1(\mu_1 r) \frac{\partial Q_1(r, \varphi', z, \tau)}{\partial \tau} r\, dr + \int_b^c R_2(\mu_2 r) \frac{\partial Q_2(r, \varphi', z, \tau)}{\partial \tau} r\, dr \right\} \right.\right.$$

$$+ \frac{2}{\pi} k_1 k_2 \left\{ \mu_1 C_m'^{(1)}(\mu_1 b) \frac{\partial \rho_2(\varphi', z, \tau)}{\partial \tau} - \mu_2 C_m'^{(2)}(\mu_2 b) \frac{\partial \rho_1(\varphi', z, \tau)}{\partial \tau} \right\} \Bigg] Z_j(z) \, dz$$

$$+ \nu_j \Bigg[k_1 \int_a^b R_1(\mu_1 r) \left\{ \frac{\partial \zeta_1(r, \varphi', \tau)}{\partial \tau} + \beta_j \frac{\partial \chi_1(r, \varphi', \tau)}{\partial \tau} \right\} r \, dr$$

$$+ k_2 \int_b^c R_2(\mu_2 r) \left\{ \frac{\partial \zeta_2(r, \varphi', \tau)}{\partial \tau} + \beta_j \frac{\partial \chi_2(r, \varphi', \tau)}{\partial \tau} \right\} r \, dr \Bigg] \Bigg\} d\tau \Bigg\} \cos m(\varphi - \varphi') \, d\varphi'$$

$$(i = 1, 2). \qquad (41b)$$

Expressions (41a) and (41b) are the two alternative forms of (32). In the event that the source functions $Q_i(r, \varphi, z, t)$, $\rho_i(\varphi, z, t)$, $\zeta_i(r, \varphi, t)$, $\chi_i(r, \varphi, t)$ $(i = 1, 2)$ are all independent of t, the functions $T_i^{(0)}(r, \varphi, z, t)$ become time-independent and represent the steady-state temperature field. In this case all the terms in the time-integrals appearing in the general solutions (40) and (41) vanish, and the resulting terms in the infinite series of (40) and (41) become exponentially decreasing. The solutions (40) and (41) then take the form of transient terms (which converge uniformly) superimposed upon steady-state temperatures. Expressions (40) and (41) are, therefore, particularly well suited for the treatment of cases in which the volume and the surface sources are in the form of continuous pulses at zero time. On the other hand, expressions (31) and (32) are especially suitable for treating cases with instantaneous pulses at zero time.

7. Determination of $T_i^{(0)}(r, \varphi, z, t)$. There remains now the problem of determining the solution of $T_i^{(0)}(r, \varphi, z, t)$, the so-called pseudo-steady temperature distribution of order zero. In view of (26) and (37), it is possible to write the following eigenfunction expansion for $T_i^{(0)}(r, \varphi, z, t)$:

$$T_i^{(0)}(r, \varphi, z, t) = \sum_\lambda \frac{A(\lambda)}{\lambda} \phi_{i;\lambda}(r, \varphi, z) G(\lambda, t) \qquad (i = 1, 2) \qquad (42)$$

where $G(\lambda, t)$ is given by (28). Since the eigenvalues λ comprise triple index sets, representation (42) for the $T_i^{(0)}$ functions is in the form of triply infinite series. We can, however, express the $T_i^{(0)}$ functions in the form of doubly infinite series. A comparison of the two equivalent forms will then result in certain summation or expansion formulae.

In order to determine the $T_i^{(0)}(r, \varphi, z, t)$ functions from the system of equations (33), (34), (35) and (36), it is advantageous to employ a repeated application of one-dimensional finite integral transforms, either with respect to φ and r, or with respect to φ and z. For the problem at hand these finite transforms are defined as follows.

(a) Finite cosine transform with respect to φ:

$$T_i^{(0)*}(r, m, z, t; \varphi') = \int_0^{2\pi} T_i^{(0)}(r, \varphi, z, t) \cos m(\varphi - \varphi') \, d\varphi \qquad (43)$$

the inverse transform being

$$T_i^{(0)}(r, \varphi, z, t) = \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{T_i^{(0)}(r, m, z, t; \varphi)}{(1 + \delta_{0m})} \qquad (44)$$

where δ_{0m} is the Kronecker delta, and $m = 0, 1, 2, \cdots$.

(b) Finite Hankel transform with respect to r:

$$T_1^{(0)*}(k, m, z, t; \varphi') = \int_a^b T_1^{(0)*}(r, m, z, t; \varphi') k_1 R_1(\mu_k r) r \, dr,$$
$$T_2^{(0)*}(k, m, z, t; \varphi') = \int_b^c T_2^{(0)*}(r, m, z, t; \varphi') k_2 R_2(\mu_k r) r \, dr \tag{45}$$

where

$$R_1(\mu r) = k_2 \mu C_m'^{(2)}(\mu b) C_m^{(1)}(\mu r),$$
$$R_2(\mu r) = k_1 \mu C_m'^{(1)}(\mu b) C_m^{(2)}(\mu r) \tag{46}$$

and $\mu = \mu_k$ is the kth nonnegative root of

$$k_1 h_1 C_m^{(2)}(\mu_k b) C_m'^{(1)}(\mu_k b) = k_2 \, C_m'^{(2)}(\mu_k b) \{ k_1 \mu_k C_m'^{(1)}(\mu_k b) + h_1 C_m^{(1)}(\mu_k b) \} \tag{47}$$

in which the relevant cylinder functions have been defined in (11) and (12). From the orthogonality relation

$$p(\mu_k) \left\{ k_1 \int_a^b R_1(\mu_k r) R_1(\mu_{k'} r) r \, dr + k_2 \int_b^c R_2(\mu_k r) R_2(\mu_{k'} r) r \, dr \right\} = \delta_{kk'} \tag{48}$$

follow, for (45), the inversion formulae

$$T_i^{(0)*}(r, m, z, t; \varphi') = \sum_k p_k R_i(\mu_k r) \sum_{i=1}^{2} T_i^{(0)*}(k, m, z, t; \varphi') \qquad (i = 1, 2) \tag{49}$$

where

$$\frac{1}{p_k} = \frac{1}{p(\mu_k)} = k_1 \int_a^b R_1^2(\mu_k r) r \, dr + k_2 \int_b^c R_2^2(\mu_k r) r \, dr \tag{50a}$$

or, upon evaluation of the integrals,

$$\begin{aligned} 1/p_k = (b^2/2) k_1 k_2 (\mu_k^2 - m^2/b^2) \{ & k_2 [C_m'^{(2)}(\mu_k b)]^2 [\{C_m^{(1)}(\mu_k b)\}^2 + \{C_m'^{(1)}(\mu_k b)\}^2] \\ & - k_1 [C_m'^{(1)}(\mu_k b)]^2 [\{C_m^{(2)}(\mu_k b)\}^2 + \{C_m'^{(2)}(\mu_k b)\}^2] \} \\ & + (2/\pi^2) k_1 k_2 [k_1 \{C_m'^{(1)}(\mu_k b)\}^2 \{\mu_k^2 + (h_2^2 - m^2/c^2)\} \\ & - k_2 \{C_m'^{(2)}(\mu_k b)\}^2 \{\mu_k^2 + (h_0^2 - m^2/a^2)\}]. \end{aligned} \tag{50b}$$

We note that the kernels $R_1(\mu r)$ and $R_2(\mu r)$ as defined by (46) satisfy the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{m^2}{r^2} + \mu^2 \right) R_i(\mu r) = 0 \qquad (i = 1, 2; R_1 : a < r < b, R_2 : b < r < c)$$

and the boundary- and interface-conditions (7) satisfied by ϕ_1 and ϕ_2, respectively. The transform pair of (45) and (49) may be regarded as defining a new and extended finite Hankel transform suitable for the composite hollow cylinder geometry.

(c) Finite trigonometric transform with respect to z:

$$T_i^{(0)*}(r, m, \nu_i, t; \varphi') = \int_{-l}^{l} T_i^{(0)*}(r, m, z, t; \varphi') Z_i(z) \, dz \qquad (i = 1, 2) \tag{51}$$

where $Z_i(z)$ is given by (21). The inversion formula is

$$T_i^{(0)*}(r, m, z, t; \varphi') = \sum_i L_i Z_i(z) T_i^{(0)*}(r, m, \nu_i, t; \varphi') \qquad (i = 1, 2) \tag{52}$$

where L_i is given by

$$\frac{1}{L_i} = \int_{-l}^{l} Z_i^2(z)\,dz = \frac{2l(\nu_i^2 + h_0'^2)(\nu_i^2 + h_2'^2) + (h_0' + h_2')(\nu_i^2 + h_0'h_2')}{2(\nu_i^2 + h_2'^2)} \tag{53}$$

and the summation extends over the nonnegative roots of (13). Again, we note that the kernel $Z_i(z)$ as defined by (21) satisfies the differential equation $(d^2/dz^2 + \nu_i^2)Z_i(z) = 0$ $(|z| < l)$ and the boundary conditions (8) and (9).

In the definition of these transforms we have indicated the distinction between single and double transforms only in the arguments of the transformed functions, and have, in the interest of not complicating the notation any further, used a single asterisk (as superscript) to denote both kinds of transforms.

These three sets of transform pairs permit, in a particularly concise manner, the determination of the $T_i^{(0)}(r, \varphi, z, t)$ $(i = 1, 2)$ functions from the system of (33), (34), (35) and (36). To facilitate the solution of $T_i^{(0)}$ it is convenient to split it up into three simpler parts so that

$$T_i^{(0)}(r, \varphi, z, t) = U_i(r, \varphi, z, t) + V_i(r, \varphi, z, t) + W_i(r, \varphi, z, t) \qquad (i = 1, 2) \tag{54}$$

where U_i satisfies the system of (33), (34), (35) and (36) in which $Q_i = \zeta_i = \chi_i = 0$; V_i satisfies the same system with $Q_i = \rho_i = 0$; and W_i again satisfies the same system where $\rho_i = \zeta_i = \chi_i = 0$, $(i = 1, 2)$.

In order to determine $U_i(r, \varphi, z, t)$, we transform the system defining U_i, first by (43) and then by (51), to get

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\nu_i^2 + \frac{m^2}{r^2}\right)\right\}U_i^*(r, m, \nu_i, t; \varphi') = 0$$

$$(i = 1, 2;\ U_1^* : a < r < b,\ U_2^* : b < r < c) \tag{55}$$

subject to the conditions (34), as applied to $U_i(r, \varphi, z, t)$, and transformed similarly by (43) and (51) in succession. Under the restriction that h_0, h_0', h_2 and h_2' are not to be simultaneously zero, the solution of this system is given by

$$\begin{aligned} U_1^*(r, m, \nu_i, t; \varphi') = {} & \frac{\rho_1^*(m, \nu_i, t; \varphi')}{\Delta_u}\,[h_1\{k_2 D_m'^{(2)}(\nu_i b) D^{(m)}(\nu_i r, \nu_i b) \\ & - k_1 D_m^{(2)}(\nu_i b) D_{0,1}^{(m)}(\nu_i r, \nu_i b)\} + k_1 k_2 \nu_i D_m'^{(2)}(\nu_i b) D_{0,1}^{(m)}(\nu_i r, \nu_i b)] \\ & - \frac{\rho_2^*(m, \nu_i, t; \varphi')}{\Delta_u}\left[\left(\frac{h_1 k_2}{\nu_i b}\right) D_m^{(1)}(\nu_i r)\right] \end{aligned} \tag{56a}$$

$$\begin{aligned} U_2^*(r, m, \nu_i, t; \varphi') = {} & \frac{-\rho_2^*(m, \nu_i, t; \varphi')}{\Delta_u}\,[h_1\{k_1 D_m'^{(1)}(\nu_i b) D^{(m)}(\nu_i r, \nu_i b) \\ & - k_2 D_m^{(1)}(\nu_i b) D_{0,1}^{(m)}(\nu_i r, \nu_i b)\} - k_1 k_2 \nu_i D_m'^{(1)}(\nu_i b) D_{0,1}^{(m)}(\nu_i r, \nu_i b)] \\ & + \frac{\rho_1^*(m, \nu_i, t; \varphi')}{\Delta_u}\left[\left(\frac{h_1 k_1}{\nu_i b}\right) D_m^{(2)}(\nu_i r)\right] \end{aligned} \tag{56b}$$

where

$$\Delta_u = -[h_1\{k_2 D_m'^{(2)}(\nu_i b) D_m^{(1)}(\nu_i b) - k_1 D_m^{(2)}(\nu_i b) D_m'^{(1)}(\nu_i b)\} + k_1 k_2 \nu_i D_m'^{(1)}(\nu_i b) D_m'^{(2)}(\nu_i b)], \tag{57}$$

$$
\begin{aligned}
D_m^{(1)}(\nu_j r) &= \nu_j D_{1,0}^{(m)}(\nu_j a, \nu_j r) - h_0 D_{0,0}^{(m)}(\nu_j a, \nu_j r), \\
D_m^{(2)}(\nu_j r) &= \nu_j D_{0,1}^{(m)}(\nu_j r, \nu_j c) + h_2 D_{0,0}^{(m)}(\nu_j r, \nu_j c).
\end{aligned} \tag{58}
$$

The primes over $D_m^{(1)}$ and $D_m^{(2)}$ indicate differentiation with respect to their arguments. The modified cylinder functions (of two arguments and of order m) appearing in (56) are defined in terms of cross products of modified Bessel functions of order m:

$$
\begin{aligned}
D_{0,0}^{(m)}(x, y) &= D^{(m)}(x, y) = I_m(x)K_m(y) - K_m(x)I_m(y), \\
D_{p,q}^{(m)}(x, y) &= \partial^{p+q} D^{(m)}(x, y)/\partial x^p \, \partial y^q \qquad (p, q = 0, 1, 2, \cdots)
\end{aligned} \tag{59}
$$

where $I_m(x)$ is the modified Bessel function of the first kind of order m, and $K_m(x)$ is the modified Bessel function of the second kind of order m.

The combination of (56), (52) and (44) yields the solutions $U_i(r, \varphi, z, t)$:

$$
\begin{aligned}
&U_i(r, \varphi, z, t) \\
&= \frac{1}{\pi} \sum_j \sum_m \frac{L_j Z_j(z)}{(1 + \delta_{0m})} \left\{ f_i^{(1)}(m, \nu_j, r) \int_0^{2\pi} \int_{-l}^{l} \rho_1(\varphi', z, t) \cos m(\varphi - \varphi') Z_j(z) \, d\varphi' \, dz \right. \\
&\quad \left. + f_i^{(2)}(m, \nu_j, r) \int_0^{2\pi} \int_{-l}^{l} \rho_2(\varphi', z, t) \cos m(\varphi - \varphi') Z_j(z) \, d\varphi' \, dz \right\} \qquad (i = 1, 2)
\end{aligned} \tag{60}
$$

where $f_i^{(1)}(m, \nu_j, r)$ and $f_i^{(2)}(m, \nu, r)$ are the coefficients of $\rho_1^*(m, \nu_j, t; \varphi')$ and $\rho_2^*(m, \nu_i, t; \varphi')$ in (56).

Similarly, in order to determine $V_i(r, \varphi, z, t)$, we transform the system defining V_i, this time first by (43) and then by (45), to get

$$
\left(\frac{\partial^2}{\partial z^2} - \mu_k^2 \right) \sum_{i=1}^{2} V_i^*(k, m, z, t; \varphi') = 0 \qquad (|z| < l) \tag{61}
$$

which is subject to (35) and (36) as applied to $V_i(r, \varphi, z, t)$, and transformed similarly by (43) and (45) in succession. Under the restriction that h_0, h_0', h_2 and h_2' are not to vanish simultaneously, the solution of this system is obtained as

$$
\begin{aligned}
\sum_{i=1}^{2} V_i^*(k, m, z, t; \varphi') &= \frac{1}{\Delta_v} \left\{ [\mu_k \cosh \mu_k(l - z) + h_2' \sinh \mu_k(l - z)] \sum_{i=1}^{2} \zeta_i^*(k, m, t; \varphi') \right. \\
&\quad \left. + [\mu_k \cosh \mu_k(l + z) + h_0' \sinh \mu_k(l + z)] \sum_{i=1}^{2} \chi_i^*(k, m, t; \varphi') \right\}
\end{aligned} \tag{62}
$$

where

$$
\Delta_v = (\mu_k^2 + h_0' h_2') \sinh (2\mu_k l) + \mu_k (h_0' + h_2') \cosh (2\mu_k l). \tag{63}
$$

Inverting (62) by (49) and (44) in succession, we have

$$
\begin{aligned}
V_i(r, \varphi, z, t) &= \frac{1}{\pi} \sum_k \sum_m \frac{p_k R_i(\mu_k r)}{(1 + \delta_{0m}) \, \Delta_v} \int_0^{2\pi} \left\{ [\mu_k \cosh \mu_k(l - z) + h_2' \sinh \mu_k(l - z)] \right. \\
&\quad \cdot \left[\int_a^b \zeta_1(r, \varphi', t) k_1 R_1(\mu_k r) r \, dr + \int_b^c \zeta_2(r, \varphi', t) k_2 R_2(\mu_k r) r \, dr \right] \\
&\quad + [\mu_k \cosh \mu_k(l + z) + h_0' \sinh \mu_k(l + z)] \left[\int_a^b \chi_1(r, \varphi', t) k_1 R_1(\mu_k r) r \, dr \right. \\
&\quad \left. \left. + \int_b^c \chi_2(r, \varphi', t) k_2 R_2(\mu_k r) r \, dr \right] \right\} \cos m(\varphi - \varphi') \, d\varphi' \qquad (i = 1, 2).
\end{aligned} \tag{64}
$$

We note that we could have obtained $V_i(r, \varphi, z, t)$ in another form by using the combination of transformations (43) and (51), instead of applying (43) and (45). Similarly, another expression may be obtained for $U_i(r, \varphi, z, t)$ in a different form, through an application of (43) and (45) rather than through the combined use of (43) and (51). We shall now illustrate this point by determining two equivalent expressions for $W_i(r, \varphi, z, t)$. To this end, we first apply, in succession, the transformations (43) and (45) to the system defining W_i, and obtain

$$\left(\frac{\partial^2}{\partial z^2} - \mu_k^2\right) \sum_{i=1}^{2} W_i^*(k, m, z, t; \varphi') + \sum_{i=1}^{2} \frac{1}{k_i} Q_i^*(k, m, z, t; \varphi') = 0 \qquad (|z| < l) \tag{65}$$

$$\begin{aligned} \left(-\frac{\partial}{\partial z} + h_0'\right) \sum_{i=1}^{2} W_i^*(k, m, z, t; \varphi') &= 0 \qquad (z = -l), \\ \left(\frac{\partial}{\partial z} + h_2'\right) \sum_{i=1}^{2} W_i^*(k, m, z, t; \varphi') &= 0 \qquad (z = l). \end{aligned} \tag{66}$$

Subject to the restriction that h_0, h_0', h_2 and h_2' are not simultaneously zero, the solution to the system of (65) and (66) is obtained as

$$\begin{aligned} \sum_{i=1}^{2} W_i^*(k, m, z, t; \varphi') &= \frac{1}{\mu_k} \int_0^z \sinh \mu_k(z' - z) \sum_{i=1}^{2} \frac{1}{k_i} Q_i^*(k, m, z', t; \varphi')\, dz' \\ &+ \frac{1}{\mu_k \Delta_v} \int_{-l}^{l} \left\{\sum_{i=1}^{2} \frac{1}{k_i} Q_i^*(k, m, z', t; \varphi')\right\}\{H(z')\Omega_k(z', z) + H(-z')\Omega_k(z, z')\}\, dz' \end{aligned} \tag{67}$$

where

$$\begin{aligned} \Omega_k(z, z') = \{&\mu_k \cosh \mu_k(l - z) + h_2' \sinh \mu_k(l - z)\} \\ &\cdot \{\mu_k \cosh \mu_k(l + z') + h_0' \sinh \mu_k(l + z')\}, \end{aligned} \tag{68}$$

$H(z)$ being the Heaviside unit function, and Δ_v is given by (63). Inverting (67) by means of (49) and (44), we have

$$\begin{aligned} &W_i(r, \varphi, z, t) \\ &= \frac{1}{\pi} \sum_k \sum_m \frac{p_k R_i(\mu_k r)}{(1 + \delta_{0m})\mu_k} \int_0^{2\pi} \left\{\int_0^z \sinh \mu_k(z' - z)\left[\int_a^b Q_1(r, \varphi', z', t) R_1(\mu_k r) r\, dr \right.\right. \\ &\left. + \int_b^c Q_2(r, \varphi', z', t) R_2(\mu_k r) r\, dr\right] dz' \\ &+ \frac{1}{\Delta_v} \int_{-l}^{l} [H(z')\Omega_k(z', z) + H(-z')\Omega_k(z, z')]\left[\int_a^b Q_1(r, \varphi', z', t) R_1(\mu_k r) r\, dr\right. \\ &\left.\left. + \int_b^c Q_2(r, \varphi', z', t) R_2(\mu_k r) r\, dr\right] dz'\right\} \cos m(\varphi - \varphi')\, d\varphi' \qquad (i = 1, 2) \end{aligned} \tag{69}$$

which is the first form of solution for W_i. To obtain a second form of solution, we apply this time the transformations (43) and (51) to the system defining W_i. As a result, we have

$$\left\{\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) - \left(\nu_i^2 + \frac{m^2}{r^2}\right)\right\} W_i^*(r, m, \nu_i, t; \varphi') + \frac{1}{k_i} Q_i^*(r, m, \nu_i, t; \varphi') = 0$$

$$(i = 1, 2;\ W_1^* : a < r < b,\ W_2^* : b < r < c) \tag{70}$$

and the conditions (7) as applied to $W_i^*(r, m, \nu_i, t; \varphi')$. Excepting again the special case where $h_0 = h_0' = h_2 = h_2' = 0$, the solution of W_i^* may be expressed in the form of

$$W_i^*(r, m, \nu_i, t; \varphi') = f_i^{(1)}(m, \nu_i, r) \int_a^b \frac{1}{k_1} Q_1^*(r, m, \nu_i, t; \varphi') D_m^{(1)}(\nu_i r) r\, dr$$

$$- f_i^{(2)}(m, \nu_i, r) \int_b^c \frac{1}{k_2} Q_2^*(r, m, \nu_i, t; \varphi') D_m^{(2)}(\nu_i r) r\, dr$$

$$+ \int_b^r \frac{1}{k_i} Q_i^*(r', m, \nu_i, t; \varphi') D^{(m)}(\nu_i r', \nu_i r) r'\, dr' \qquad (i = 1, 2) \qquad (71)$$

which, upon inversion successively by (52) and (44), yields

$$W_i(r, \varphi, z, t) = \frac{1}{\pi} \sum_j \sum_m \frac{L_j Z_j(z)}{(1 + \delta_{0m})} \int_0^{2\pi} \int_{-l}^{l} \left\{ f_i^{(1)}(m, \nu_i, r) \int_a^b \frac{1}{k_1} Q_1(r, \varphi', z, t) D_m^{(1)}(\nu_i r) r\, dr \right.$$

$$- f_i^{(2)}(m, \nu_i, r) \int_b^c \frac{1}{k_2} Q_2(r, \varphi', z, t) D_m^{(2)}(\nu_i r) r\, dr$$

$$\left. + \int_b^r \frac{1}{k_i} Q_i(r', \varphi', z, t) D^{(m)}(\nu_i r', \nu_i r) r'\, dr' \right\} Z_j(z) \cos m(\varphi - \varphi')\, d\varphi'\, dz \quad (i = 1, 2). \quad (72)$$

Equation (72) is a second form of the solution for $W_i(r, \varphi, z, t)$. The first form is given by (69). Both forms are in terms of doubly infinite summations. For purposes of numerical evaluation we prefer to work with (72) rather than (69), since the computation of the roots μ_k from (47) is much harder than that of the roots ν_j from (13). In addition, the roots μ_k are coupled with m, that is, for each value of m, a new set of μ_k has to be determined. On the other hand, ν_j is determined independently of m. This concludes the determination of $T_i^{(0)}(r, \varphi, z, t)$ expressed by (54).

It should be noted that, if (70) is further transformed by (45), and the conditions (7) as applied to $W_i^*(r, m, \nu_i, t; \varphi')$ utilized, the following equation is obtained:

$$(\nu_i^2 + \mu_k^2) \sum_{i=1}^{2} W_i^*(k, m, \nu_i, t; \varphi') = \sum_{i=1}^{2} \frac{1}{k_i} Q_i^*(k, m, \nu_i, t; \varphi') \qquad (73)$$

where

$$\sum_{i=1}^{2} \frac{1}{k_i} Q_i^*(k, m, \nu_i, t; \varphi') = \int_{-l}^{l} \int_0^{2\pi} \left\{ \int_a^b Q_1(r, \varphi', z, t) R_1(\mu_k r) r\, dr \right.$$

$$\left. + \int_b^c Q_2(r, \varphi', z, t) R_2(\mu_k r) r\, dr \right\} \cos m(\varphi - \varphi') Z_j(z)\, d\varphi'\, dz. \qquad (74)$$

Inversion of (73) with respect to r is, by (49),

$$W_i^*(r, m, \nu_i, t; \varphi') = \sum_k \frac{p_k R_i(\mu_k r)}{(\nu_i^2 + \mu_k^2)} \sum_{i=1}^{2} \frac{1}{k_i} Q_i^*(k, m, \nu_i, t; \varphi') \qquad (i = 1, 2) \qquad (75)$$

where $W_i^*(r, m, \nu_i, t; \varphi')$ is given by (71). Similarly, inversion of (73) with respect to z is, by (52),

$$\sum_{i=1}^{2} W_i^*(k, m, z, t; \varphi') = \sum_j \frac{L_j Z_j(z)}{(\nu_j^2 + \mu_k^2)} \sum_{i=1}^{2} \frac{1}{k_i} Q_i^*(k, m, \nu_i, t; \varphi') \qquad (76)$$

where $\sum_{i=1}^{2} W_i^*(k, m, z, t; \varphi')$ is given by (67). Equations (75) and (76) serve as summation formulae. Similar formulae may be obtained for U_i^* and V_i^* in the same way.

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