ON THE CONCEPT OF RATE-INDEPENDENCE*

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I. Introduction. A great many materials of interest in continuum physics are described by constitutive equations which are rate-independent, i.e., which are invariant under a change of time scale. The theories of classical plasticity and the theory of hypoelasticity are important examples of rate-independent theories.

Mechanical and thermodynamical theories of material behavior based on a general concept of rate-independence recently have been developed by Pipkin and Rivlin [1] and by Owen [2]. In these theories the physical notion of rate-independence, stated in terms of the mechanical theory, is taken to be the following: *the stress at any time depends upon all past values of strain but is independent of the rate at which these values are assumed.* (For the thermodynamical theory read "stress, heat-flux and entropy" for "stress" and "strain, temperature and temperature gradient" for "strain".)

Two mathematical statements of this concept have been proposed; both appear to be reasonable interpretations of the physical principle. Pipkin and Rivlin introduce the concept of rate-independence by requiring that the present value of stress depend upon the strain history through the arc-length parameterization of the strain path. Thus if E is some strain measure for the motion and $E^t(\cdot)$ the history of strain up to time t.

$$E^t(s) = E(t - s), \qquad s \in [0, \infty),$$

they assume the value of stress at time t, $T(t)$, is given by

$$T(t) = \underset{\sigma=0}{\overset{\infty}{\mathfrak{I}}} (\tilde{E}^t(\sigma)), \tag{1}$$

where $\mathfrak{I}$ is some functional and $\tilde{E}^t(\cdot)$ represents the arc-length description of $E^t(\cdot)$.

Truesdell and Noll [3] introduce a second definition of rate-independence in which they consider the usual time description of the response to a given history of strain

$$T(t) = \underset{s=0}{\overset{\infty}{\mathfrak{I}^*}} (E^t(s))$$

but in which they require that the functional $\mathfrak{I}^*$ assign the same value to all histories of strain which assume the given values $\{E^t(s) \mid s \in [0, \infty)\}$ in the same order as does the given history. In mathematical terms this requirement becomes

$$\underset{s=0}{\overset{\infty}{\mathfrak{I}^*}} (E^t(\phi(s))) = \underset{s=0}{\overset{\infty}{\mathfrak{I}^*}} (E^t(s)) \tag{2}$$

for any monotone function ϕ mapping $[0, \infty)$ onto $[0, \infty)$.[1]

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[1]The idea embodied in (2) first appears in a theorem on hypoelastic materials due to Noll [4].

Intuitively, the two definitions appear to be equivalent; in this paper we show that they are indeed equivalent if the definitions are slightly modified. We consider the following generalized form of the problem: we take $\mathfrak{F}$ to be a set of functions mapping $[0, \infty)$ into R^m and π some function on the set $\mathfrak{F}$ (we assume that the range of π is in some normed vector space, but this assumption is not of relevance for our proof). We then show that π is invariant under a set of mappings as in (2) if and only if it may be written in terms of the arc-length parameterizations of functions in $\mathfrak{F}$, as in (1).

Essentially the problem is the same as that of the demonstration in calculus that the arc-length parameterization of a curve is equivalent to the description of the curve as an equivalence class of parameterizations. In the case in question complications arise from two requirements imposed by the physical context in which the problem appears: first, the range of the parameterization variable, time (measured into the past), may be infinite; second, zero derivatives of the time parameterization must be included, that is, we must include the case of a deformation constant over some interval of time. The first complication is met by enlarging slightly the class of reparameterization functions (the functions ϕ of (2)); the second merely necessitates a certain degree of smoothness be required of the functions in $\mathfrak{F}$.

In sections II and III we develop preliminary results which are necessary in order that the definitions of rate-independence can be stated in meaningful terms. In section IV we present the two definitions of rate-independence and the proof that they are equivalent. In the final section we show that the concept of fading memory is only trivially compatible with the notion of rate-independence and suggest a sense in which it can be made compatible.

II. **The domain of the functional; the invariance mappings.** Let π denote a functional with domain $\mathfrak{F}$, $\mathfrak{F}$ being a set of functions each of whose elements f maps the half-line $H = [0, \infty)$ into m-dimensional vector space R^m, and range included in some normed vector space $\mathfrak{X}$. We investigate the invariance of the functional π under transformations of the form $f \to f \circ \phi$ where $\phi \in \Phi$, a class of real-valued monotone functions on H; $f \circ \phi$ denotes the usual composition of two functions. As an example, in the context of a mechanical theory of material behavior f can be interpreted as the history function of the strain tensor[2] ($m = 6$ or 9), and $\mathfrak{X}$ is the space of symmetric tensors on R^3.

Clearly, if the arc-length of a function $f \in \mathfrak{F}$ is to be defined, f necessarily must be of locally bounded variation. However, in order to eliminate the physically undesirable class of "singular functions" (nonconstant functions which are continuous, of locally bounded variation and which have zero derivative almost everywhere), we require further that each $f \in \mathfrak{F}$ be absolutely continuous. Recall that f is said to be absolutely continuous on the interval $[a, b]$ if $\sum |f(\beta_i) - f(\alpha_i)|$ tends to zero with $\sum(\beta_i - \alpha_i)$ where $\{(\alpha_i, \beta_i)\}$ is any countable family of nonoverlapping intervals contained in $[a, b]$. Here $|\cdot|$ denotes the Euclidean norm in R^m. If f is absolutely continuous then f has a derivative $\dot{f}(s)$ for almost every s and the function $\dot{f}$ is integrable with f as primitive. (These results are proved, for example, in [5] for the case $m = 1$; the generalization

[2]For the reparameterizations below to be meaningful we necessarily assume that the quantities $f(s)$ and their arguments s (time) are dimensionless, and hence some dimensional transformation may be necessary to pose the problem in these terms. (The dimensionless variable corresponding to time will arise naturally in a rate-independent material since such a material cannot admit a modulus with a time dimension: cf. Truesdell and Noll [3, p. 402]).

to $1 < m < \infty$ is trivial.) A simple argument then shows that $|\dot{f}|$ is also integrable. In the next section we use this fact in order to define the arc-length function for f.

Now let us define

$$\mathfrak{F} = \{f : H \to R^m \mid f \quad \text{is absolutely continuous on any finite subinterval of } H\}.$$

Of course $f \in \mathfrak{F}$ implies f is of locally bounded variation on H.

For any $f \in \mathfrak{F}$ we define

$$s_f = \inf \{s \geq 0 \mid \dot{f}(s') = 0 \quad \text{for almost every } s' \geq s\}$$

where we allow s_f to be infinite. Since f is absolutely continuous this implies f is constant in $[s_f, \infty)$, a fact which is of importance in our definition of the invariance transformations ϕ. (We use the term invariance transformation anticipating Definition 2, section IV.)

In proceeding to the specification of the class of invariance transformations, we point out that it is necessary to define many such classes, each of which depends weakly upon a particular function $f \in \mathfrak{F}$. This weak dependence is admitted in order to include the arc-length function corresponding to a given f in at least one invariance class. We could develop the results which follow without requiring this weak dependence, but in relaxing this requirement we would have to exclude from the class $\mathfrak{F}$ all functions f which have finite arc-length but for which $s_f = +\infty$. In no way can we justify this exclusion. Thus we shall define

$$\Phi_f = \{\phi : H \to H \mid \phi \text{ is monotone nondecreasing, absolutely continuous on any finite subinterval of } H, \text{ and such that } \phi(H) \text{ includes } [0, s_f)\}.$$

The set Φ_f includes all monotone nondecreasing finitely absolutely continuous functions which map H onto H. It is important to note that every $\phi \in \Phi_f$ is such that $f \circ \phi$ takes on all of the values, in the same order, as does f.

It is a standard result that an absolutely continuous function composed with an absolutely continuous monotone function is absolutely continuous: thus *for any* $f \in \mathfrak{F}$ *and* $\phi \in \Phi_f$ *we have also* $f \circ \phi \in \mathfrak{F}$.

III. **The arc-length parameterization.** Let f be any function in the class $\mathfrak{F}$. The *arc-length function for* f, $\ell_f : H \to H$, is given by

$$\ell_f(s) = \int_0^s |\dot{f}(\sigma)| \, d\sigma.$$

If f is taken to represent the past history of a quantity (such as a strain measure) relative to the present time t then $\ell_f(s)$ is the arc-length of the path traversed in the time interval $[t - s, t]$.

Clearly ℓ_f is an absolutely continuous monotone nondecreasing function; $\dot{\ell}_f(s)$ represents the speed of traversal of the path at s, i.e., $\dot{\ell}_f(s) = |\dot{f}(s)|$ almost everywhere in H. Since ℓ_f is monotone the limit

$$L_f = \lim_{s \to \infty} \ell_f(s)$$

exists (it may be infinite). Whether finite or infinite, L_f will be called the total path length corresponding to the function f or, more concisely, the path length of f. We note that $\ell_f \in \Phi_{\tilde{f}}$ whenever $\tilde{f} \in \mathfrak{F}$ is such that $s_{\tilde{f}} \leq L_f$.

To obtain a parameterization of f in terms of arc-length we introduce the rightinverse function for ℓ_f . Thus we define for every $s \in [0, L_f)$

$$\ell_f^i(s) = \min \{\sigma \mid \ell_f(\sigma) = s\}.$$

If ℓ_f is a strictly increasing function then the function ℓ_f^i is the inverse of ℓ_f . We observe that whatever the nature of ℓ_f , ℓ_f^i itself is a strictly increasing function on $[0, L_f)$: if $s_1 < s_2$ in $[0, L_f)$ and we assume $\ell_f^i(s_1) \geq \ell_f^i(s_2)$ then we may use the obvious relation $\ell_f(\ell_f^i(s)) = s$ to conclude, since ℓ_f is monotone, $s_1 \geq s_2$, a contradiction. Being monotone, ℓ_f^i has at most a countable number of jump discontinuities; it is easy to see that each point of discontinuity corresponds to exactly one finite interval on which ℓ_f is constant.

The following relation will be needed in discussing properties of the arc-length parameterization of f:

$$\lim_{s \nearrow L_f} \ell_f^i(s) = s_f .$$

Since ℓ_f^i is strictly increasing the limit on the left exists. To establish that the limit is s_f we first show that

$$s^* = \lim_{s \nearrow L_f} \ell_f^i(s) \leq s_f .$$

This is trivially true if $s_f = +\infty$. Now if s_f is finite then so is L_f . Let $0 \leq s < L_f$. Then $\sigma = \ell_f^i(s)$ must be less than s_f ; if it were greater than s_f then $\ell_f(\sigma) = s = L_f$, for ℓ_f must be constant on $[s_f , \infty)$. It follows then that $s^* \leq s_f$. Suppose now that $s^* < s_f$; we will show that this supposition leads to a contradiction. Indeed, if $s^* = +\infty$ the contradiction is immediate. If $s^* < \infty$ we choose $\bar{s}$ finite such that $s^* < \bar{s}$. Using the relation

$$\ell_f(s^*) = \ell_f(\lim_{s \nearrow L_f} \ell_f^i(s)) = \lim_{s \nearrow L_f} \ell_f(\ell_f^i(s)) = \lim_{s \nearrow L_f} s = L_f$$

and the monotonicity of ℓ_f we obtain the inequality

$$L_f \leq \ell_f(\bar{s}).$$

The definition of L_f implies this must be an equality. Since $\bar{s}$ is any element of H greater than s^* it follows that ℓ_f is constant on $[s^*, \infty)$ and, therefore, that f is constant on $[s^*, \infty)$. But s_f is the least number for which this is true and hence $s_f \leq s^*$, which contradicts the supposition $s^* < s_f$. This establishes the desired result, i.e., $s^* = s_f$.

We now define $\hat{f} : [0, L_f) \to R^m$ by

$$\hat{f} = f \circ \ell_f^i .$$

The function $\hat{f}$ is called the *arc-length parameterization corresponding to* f. The function $\hat{f}$ must be continuous[3] on $[0, L_f)$ although ℓ_f^i is not, in general. To show that $\hat{f}$ is continuous, we recall that ℓ_f^i has only a countable number of jump discontinuities. Suppose $c \in [0, L_f)$ is a point of discontinuity of ℓ_f^i . Then there exist two elements of H, $a < b$, such that ℓ_f takes on the value c throughout the interval $[a, b]$ and at no other point of H. Then $\dot{\ell}_f = |\dot{f}| = 0$ on $[a, b]$ and hence f is constant on $[a, b]$. Now, the limits $\ell_f^i(c^-)$ and $\ell_f^i(c^+)$ exist and have the values a and b respectively (these relations can be demonstrated by means of arguments quite similar to the one used in showing $\lim_{s \nearrow L_f} \ell_f^i(s) =$

[3]As is stated by Pipkin and Rivlin [1, p. 315].

s_f). Since $f(a) = f(b)$ the conclusion $\hat{f}(c^-) = \hat{f}(c^+)$ is then immediate.

Now we shall extend (if $L_f < \infty$) the function $\hat{f}$ to the domain H and then show that this extended function is in $\mathfrak{F}$. In order to do this we need the following result: if f is such that $L_f < \infty$, f is not only of locally bounded variation but of bounded variation on H. That this is so is immediate when one recognizes that since f is absolutely continuous the total variation of f on any interval $[0, a]$ is just

$$\ell_f(a) = \int_0^a |\dot{f}(\sigma)| \, d\sigma.$$

Since in this case f is of bounded variation on H we may conclude, by a standard argument, that $f_\infty = \lim_{s \to \infty} f(s)$ exists and has finite norm. Moreover, since f is continuous

$$\lim_{s \nearrow L_f} \hat{f}(s) = \lim_{s \nearrow L_f} f(\ell_f^i(s)) = f(\lim_{s \nearrow L_f} \ell_f^i(s)) = f(s_f) = f_\infty .$$

We define the *extended arc-length parameterization corresponding to* f by

$$\hat{f}^e(s) = \begin{cases} \hat{f}(s) & 0 \leq s < L_f , \\ f_\infty & L_f \leq s < \infty \end{cases}$$

when $L_f < \infty$; otherwise we make the identification $\hat{f}^e = \hat{f}$. In either case the function is continuous.

To show that $\hat{f}^e \in \mathfrak{F}$ we must prove it to be absolutely continuous on finite subintervals of H. Central to the proof of this result is the identity

$$\hat{f}^e \circ \ell_f = f. \tag{3}$$

To prove this identity we first suppose $s_f < \infty$. If $s \geq s_f$ the definitions of s_f and of f_∞ imply that $f(s) = f_\infty$. To show that $(\hat{f}^e \circ \ell_f)(s) = f_\infty$ as well, we note that since s_f is finite, L_f will be finite; hence, $\ell_f(s) = L_f$ whenever $s \geq s_f$ and

$$(\hat{f}^e \circ \ell_f)(s) = \hat{f}^e(\ell_f(s)) = \hat{f}^e(L_f) = f_\infty .$$

Now we consider, regardless of the value of s_f, the case $s < s_f$. Since ℓ_f cannot be constant in $[s, s_f)$ it follows that $\ell_f(s) < L_f$. Then

$$\hat{f}^e(\ell_f(s)) = f(\ell_f^i(\ell_f(s))) = f(\min \{\sigma \mid \ell_f(\sigma) = \ell_f(s)\})$$

and it is clear that $\ell_f(\sigma) = \ell_f(s)$ implies $f(\sigma) = f(s)$. These arguments suffice to establish the identity (3) for all s.

To show that $\hat{f}^e$ is absolutely continuous on finite subintervals of H, we consider first the case $L_f = \infty$. Let $[a, b]$ be any finite interval in H. Then since ℓ_f^i is a monotone function, since $\ell_f^i(b) < \infty$ and since f is of locally bounded variation, it is easy to show $\hat{f}^e = \hat{f} = f \circ \ell_f^i$ is of bounded variation on $[a, b]$. Hence we can write $\hat{f}^e = g + h$ where g is absolutely continuous on $[a, b]$ and h is of bounded variation with $\dot{h} = 0$ almost everywhere on $[a, b]$ and $h(a) = 0$. We consider the interval $[\alpha, \beta]$ where $\alpha = \ell_f^i(a_f)$ and $\beta = \ell_f^i(b)$. Then $\alpha < \beta < \infty$ and thus ℓ_f and f are absolutely continuous on $[\alpha, \beta]$. The relation $\hat{f}^e \circ \ell_f = f$ then implies $h \circ \ell_f = f - g \circ \ell_f$, so that $h \circ \ell_f$ is absolutely continuous on $[\alpha, \beta]$. We observe that

$$\frac{d}{ds}(h \circ \ell_f)(s) = \dot{h}(\ell_f(s))\dot{\ell}_f(s)$$

for almost every $s \in [\alpha, \beta]$, and thus for all $s \in [\alpha, \beta]$

$$(h \circ \ell_f)(s) = \int_\alpha^s \dot{h}(\ell_f(\sigma))\dot{\ell}_f(\sigma)\, d\sigma + (h \circ \ell_f)(\alpha)$$

$$= \int_a^{\ell_f(s)} \dot{h}(\xi)\, d\xi + (h \circ \ell_f)(\alpha)$$

since $h \circ \ell_f$ is absolutely continuous, $\dot{h}$ is integrable, and ℓ_f is absolutely continuous. However, $\dot{h}(\xi) = 0$ for almost every $\xi \in [a, b]$, so

$$(h \circ \ell_f)(s) = (h \circ \ell_f)(\alpha) = h(a) = 0.$$

This means h is identically zero on the set $\ell_f([\alpha, \beta])$; since $\ell_f([\alpha, \beta]) = [a, b]$, h is identically zero on $[a, b]$ and therefore $\hat{f}^e$ is absolutely continuous on $[a, b]$.

Now we turn to the case $L_f < \infty$. If $[a, b] \subset [0, L_f)$ we may use the above argument to show that $\hat{f}^e$ is absolutely continuous on $[a, b]$. Since $\hat{f}^e$ is constant on $[L_f, \infty)$, we need only show that $\hat{f}^e$ is absolutely continuous on $[0, L_f]$ in order to complete the proof that $\hat{f}^e$ is absolutely continuous on finite subintervals of H. We establish the result on $[0, L_f]$ indirectly. This approach is based on the fact that $\hat{f}^e$ is absolutely continuous on closed subintervals of $[0, L_f)$ and continuous on $[0, L_f]$; if we show it is of bounded variation on $[0, L_f]$ it follows that it is absolutely continuous on $[0, L_f]$ (see, for example, [6, p. 334]). We have shown that f is of bounded variation on H when $L_f < \infty$; since ℓ_f^i maps $[0, L_f]$ into $[0, s_f] \subset H$ and is monotone, and since $\hat{f}^e$ is continuous on $[0, L_f]$, it follows that $\hat{f}^e$ is of bounded variation on $[0, L_f]$. This completes the proof: $\hat{f}^e \in \mathfrak{F}$.

IV. **Rate independence; equivalence of definitions.** Now we may formally define the problem. We let π be a mapping from $\mathfrak{F}$ into some normed vector space $\mathfrak{X}$. The work in the previous sections ensures that the following definitions are meaningful.

Definition 1 (After Pipkin and Rivlin). *The functional π is* rate-independent *if for every $f \in \mathfrak{F}$*

$$\pi(f) = \pi(\hat{f}^e).$$

Definition 2 (After Truesdell and Noll). *The functional π is* rate-independent *if for every $f \in \mathfrak{F}$*

$$\pi(f \circ \phi) = \pi(f) \qquad \text{for all } \phi \in \Phi_f\,.$$

These definitions contain the spirit if not the exact detail of the originals. Thus, as remarked above, Φ_f contains all of the monotone transformations considered by Truesdell and Noll; Φ_f will contain additional transformations only when $s_f < \infty$.

Our main result is the following

Theorem. *Definition 1 and Definition 2 are equivalent.*

That Definition 2 implies Definition 1 follows immediately from our earlier results. Thus in Sec. III we showed that $f \in \mathfrak{F}$ implies that $\hat{f}^e \in \mathfrak{F}$, that $\hat{f}^e \circ \ell_f = f$ and that $\ell_f \in \Phi_{\tilde{f}}$ for any $\tilde{f}$ such that $s_{\tilde{f}} \leq L_f$. But in fact $s_{\hat{f}^e} = L_f$, so

$$\pi(\hat{f}^e) = \pi(\hat{f}^e \circ \ell_f) = \pi(f).$$

The proof of the converse is somewhat more involved. We shall show that for any f and any $\phi \in \Phi_f$

$$\widehat{f \circ \phi}^e = \hat{f}^e$$

which then implies $\pi(f \circ \phi) = \pi(f)$. Here $\widehat{f \circ \phi}^e$ denotes the extended arc-length function corresponding to $f \circ \phi$. We note first

$$\ell_{f \circ \phi}(s) = \int_0^s |\dot{\overline{f(\phi(\sigma))}}|\, d\sigma = \int_0^s |\dot{f}(\phi(\sigma))|\, \dot{\phi}(\sigma)\, d\sigma$$

since the derivatives exist almost everywhere and $\dot{\phi}(\sigma) \geq 0$. But ϕ is absolutely continuous so

$$\ell_{f \circ \phi}(s) = \int_0^{\phi(s)} |\dot{f}(\sigma)|\, d\sigma = \ell_f(\phi(s))$$

and in particular, since $\phi([0, \infty))$ includes $[0, s_f)$,

$$L_{f \circ \phi} = \lim_{s \to \infty} \ell_{f \circ \phi}(s) = \lim_{s \to \infty} \ell_f(\phi(s)) = L_f \, .$$

Moreover, in the case where $L_f < \infty$ it is clear that

$$\lim_{\sigma \to s_{f \circ \phi}} f(\phi(\sigma)) = f_\infty$$

so that $\widehat{f \circ \phi}^e$ and $\hat{f}^e$ agree by definition on the interval $[L_f, \infty)$. Then, noting that

$$\widehat{(f \circ \phi)}(s) = f(\phi(\ell^i_{f \circ \phi}(s))) \qquad \text{and} \qquad \hat{f}(s) = f(\ell^i_f(s)),$$

it suffices to show $\phi(\ell^i_{f \circ \phi}(s)) = \ell^i_f(s)$ for $s < L_f$. To this end we note that

$$\begin{aligned} \phi(\ell^i_{f \circ \phi}(s)) &= \phi(\min \{\sigma \mid \ell_{f \circ \phi}(\sigma) = s\}) \\ &= \min \{\phi(\sigma) \mid \ell_{f \circ \phi}(\sigma) = s\} \end{aligned}$$

since ϕ is nondecreasing. Because $\ell_{f \circ \phi}(\sigma) = \ell_f(\phi(\sigma))$ the last relation becomes

$$\phi(\ell^i_{f \circ \phi}(s)) = \min \{\sigma \mid \ell_f(\sigma) = s\} = \ell^i_f(s).$$

This completes the proof.

It is of course possible to prove this theorem for a more restricted class of functions than the set $\mathfrak{F}$. For example, instead of $\mathfrak{F}$ one could have chosen the class

$$\tilde{\mathfrak{F}} = \{f : H \to R^m \mid f \text{ is piecewise continuously differentiable on finite subintervals of } H \text{ and } f \text{ has a finite number of intervals of constancy}\},$$

with $\tilde{\Phi}_f$ defined in a similar manner. In this case the proofs that $\hat{f}^e \in \tilde{\mathfrak{F}}$ and that $f \circ \phi \in \tilde{\mathfrak{F}}$ whenever $\phi \in \tilde{\Phi}_f$ become much more involved. Finally, we remark that the class $\mathfrak{F}$ above seems to be the largest class of functions for which the identity $\hat{f}^e \circ \ell_f = f$ necessarily holds.

V. **Rate-independence and fading memory.** For the purposes of this section let us add an additional assumption on the set of functions $\mathfrak{F}$. We require that $f \in \mathfrak{F}$ *only if f is bounded at* ∞ (which implies f is uniformly bounded on H). It is clear that this in no way interferes with the analysis of the previous sections. This condition is sufficient to guarantee $\mathfrak{F}$ is included in the space of functions on H into R^m with finite norm

$$||f|| = k(0)\, |f(0)| + \int_0^\infty |f(\sigma)|\, k(\sigma)\, d\sigma \tag{5}$$

where $k(s)$ is a positive real-valued function, integrable over $[0, \infty)$. A norm of this sort is used in the theory of fading memory introduced by Coleman and Noll [7], [8] and generalized by Coleman and Mizel [9]. In this theory, a functional is said to obey the postulate of fading memory if it is continuous with respect to the topology generated by such a norm. As is remarked by Truesdell and Noll [3, p. 402] the assumption of rate-independence is in general incompatible with this theory of fading memory We make this precise below: let us say that π obeys the *CMN principle of fading memory* if π is continuous in the (relative) topology on $\mathfrak{F}$ defined by (5).

PROPOSITION. *A rate-independent functional obeys the CMN principle of fading memory if and only if it is elastic, i.e., if and only if there exists a continuous function* $\Pi : R^m \to \mathfrak{X}$ *such that for each* $f \in \mathfrak{F}$

$$\pi(f) = \Pi(f(0)). \tag{6}$$

If π satisfies (6) then, trivially, π is rate-independent and satisfies the CMN principle of fading memory. The proof of the converse follows directly from the definition of rate-independence. For any $a \in R^m$ we let $a^\dagger$ denote the function in $\mathfrak{F}$ with constant value a and define $\Pi(a) = \pi(a^\dagger)$. Clearly Π is continuous; we must show that $\pi(f) = \pi(f(0)^\dagger)$ for any $f \in \mathfrak{F}$.

For any $\sigma \in H$ we define the function $\phi^{(\sigma)} : H \to H$ by

$$\phi^{(\sigma)}(t) = \begin{cases} 0 & 0 \le t < \sigma, \\ t - \sigma & \sigma \le t < \infty. \end{cases}$$

Then if f is any function in $\mathfrak{F}$, $\phi^{(\sigma)} \in \Phi_f$ for each $\sigma \in H$. The function $f \circ \phi^{(\sigma)} \in \mathfrak{F}$ is the *static continuation* of f of amount σ considered by Coleman and Noll ([8]; the centrality of static continuations in the theory of fading memory is shown by Coleman and Mizel [9]).

Now let us consider the difference $f \circ \phi^{(\sigma)} - f(0)^\dagger$:

$$\begin{aligned} \|f \circ \phi^{(\sigma)} - f(0)^\dagger\| &= \int_0^\infty |f(\phi^{(\sigma)}(s)) - f(0)|\, k(s)\, ds \\ &= \int_\sigma^\infty |f(s - \sigma) - f(0)|\, k(s)\, ds. \end{aligned}$$

Our modifications of the set $\mathfrak{F}$ imply that if $f \in \mathfrak{F}$, $|f(s)| \le K_f$, a constant, for all s. Hence

$$\|f \circ \phi^{(\sigma)} - f(0)^\dagger\| \le 2K_f \int_\sigma^\infty k(s)\, ds;$$

by choosing σ sufficiently large we can make this quantity arbitrarily small. Thus since π is presumed continuous with respect to the topology generated by this norm it follows that, given $\epsilon > 0$, we can choose σ sufficiently large that

$$\|\pi(f \circ \phi^{(\sigma)}) - \pi(f(0)^\dagger)\| < \epsilon$$

(here $\|\cdot\|$ denotes the norm in $\mathfrak{X}$). But since π is rate-independent, $\pi(f \circ \phi^{(\sigma)}) = \pi(f)$ for any σ. Thus $\|\pi(f) - \pi(f(0)^\dagger)\|$ is less than ϵ; since ϵ is arbitrary this implies $\pi(f) = \pi(f(0)^\dagger) = \Pi(f(0))$.

Thus the concept of fading memory as considered by Coleman, Mizel, and Noll is only trivially compatible with the concept of rate-independence. The same argument may be applied to Wang's first treatment of fading memory [10] since, as is mentioned by Coleman and Mizel [9], his norm is equivalent to one of the form (5). The same type of result can also be proved for Wang's second treatment of fading memory [11]. For Wang's materials of order zero $\mathfrak{F}$ admits $f \circ \phi^{(\sigma)}$ whenever it admits f and it is easy to show that in this topology also $\lim_{\sigma \to \infty} f \circ \phi^{(\sigma)} = f(0)$. For Wang's materials of order $p \geq 1$ one can consider the *α-retardation*, f_α, of f,

$$f_\alpha(s) = f(\alpha s), \qquad \alpha \in H$$

and easily show that $f_\alpha \to f(0)$ as $\alpha \to 0$, which yields the same result since $\pi(f_\alpha) = \pi(f)$. Hence the above proposition remains valid if one substitutes "Wang" for "CMN". Of course the result is valid for any topology on $\mathfrak{F}$ in which a constant function can be approximated arbitrarily closely by means of static continuations or retardations (as long as the static continuations or retardations, themselves, are in $\mathfrak{F}$).

Nevertheless, the concept of fading memory is not completely empty for rate-independent functionals; we may choose to suppose the memory fades with arc-length rather than time (by applying the norm (5) to appropriate modifications of the arc-length parameterizations and supposing the functional continuous in the generated topology).

Of course there are many other topologies which are compatible with rate-independence. One example is that given by Pipkin and Rivlin [1]; the topology they consider allows integral approximations of continuous rate-independent functionals.

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