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UNIQUENESS THEOREM FOR A MULTI-MODE SURFACE WAVE DIFFRACTION PROBLEM*

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Abstract. Uniqueness is demonstrated for the solution to the reduced wave equation subject to a mixed boundary value condition that excites two surface wave modes. The configuration is taken as a right-angled wedge and the edge condition assumed has the form

$$\sum_{i=0}^{2} \left|\frac{\partial^i u}{\partial x^i}\right| = O\left(\frac{1}{r^{1+h}}\right), \qquad 0 \le h < \tfrac{2}{3}, \qquad \text{for} \quad r \to 0.$$

It is conjectured that the same procedure may be used to prove uniqueness for the corresponding N-mode problem under the edge condition

$$\sum_{i=0}^{N} \left|\frac{\partial^i u}{\partial x^i}\right| = O(r^{-[(2N-1)/3+h]}), \qquad 0 \le h < \tfrac{2}{3}, \qquad \text{as} \quad r \to 0.$$

In this paper, we prove a uniqueness theorem for a mixed boundary value problem that occurs in the phenomenological theory of multi-mode surface wave diffraction (see Morgan, Karp, and Karal [1]). The method is essentially an extension of that employed to the single-mode case in Morgan and Karp [2] and the formulation is a modification of Stoker and Peter's work [3] on plane incidence for the Sommerfeld problem. We prove the following:

THEOREM I. *Let $u(x, y)$ have continuous second order derivatives in the wedge-shaped region D defined by the inequalities $0 < r \equiv (x^2 + y^2)^{1/2}$, $0 \le \theta \equiv \arctan y/x \le 3\pi/2$ (see Fig. 1). Let u be a solution of the following boundary value problem:*

I.1 *$u(x, y)$ satisfies the reduced wave equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + K^2 u = 0$, in D. Here K is real.*

I.2. *$\partial u/\partial y = 0$ for $y = 0, x > 0$ while $(\partial/\partial x - \lambda_1)(\partial/\partial x - \lambda_2)u = 0$ for $y < 0, x = 0$, where λ_1 and λ_2 are positive and distinct.*

I.3. *u and its derivatives satisfy the following conditions in D:*

(a) $$\sum_{j=0}^{3} \sum_{i=0}^{j} \left|\frac{\partial^j u}{\partial x^{j-i}\, \partial y^i}\right| < M \qquad \text{for} \quad r > R_0,$$

where M is independent of r and θ and R_0 is some positive constant.

(b) $$\sum_{i=0}^{2} \left|\frac{\partial^i u}{\partial x^i}\right| = O\left(\frac{1}{r^{1+h}}\right) \qquad \text{as} \quad r \to 0 \quad \text{with} \quad 0 \le h < \tfrac{2}{3}.$$

* Received December 20, 1966. Based in part on the author's doctoral dissertation. The research reported in this paper was sponsored by the Air Force Cambridge Research Laboratories, Office of Aerospace Research under Contract No. AF 19(628)3868. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

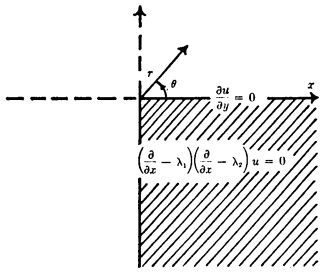

FIG. 1

I.4 *u can be written as $u(x, y) = u_{incident} + u_{reflected} + u_{radiated}$, where*

$$u_{inc.} = \begin{cases} \sum_{m=1}^{2} A_m \exp\left[+\lambda_m x + i(K^2 + \lambda_m^2)^{1/2} y\right], & \begin{matrix} x < 0, \\ y < 0, \end{matrix} \\ 0, & y > 0, \end{cases}$$

$$u_{refl.} = \begin{cases} \sum_{m=1}^{2} B_m \exp\left[+\lambda_m x - i(K^2 + \lambda_m^2)^{1/2} y\right], & \begin{matrix} x < 0, \\ y < 0, \end{matrix} \\ 0, & y > 0, \end{cases}$$

and B_1, B_2 are constants representing reflection coefficients.

I.5 *$u_{rad.} \equiv u - u_{inc.} - u_{refl.}$ obeys the radiation condition*

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u_{rad.}}{\partial r} - iKu_{rad.}\right) = 0$$

uniformly in θ, $0 \leq \theta \leq 3\pi/2$ and vanishes at infinity.

Then, $u(x, y)$ is unique.

Proof. As mentioned before, we extend the method of [2] to this case. Hence, we allow two solutions:

$$u^{(n)} = u_{inc.} + u_{refl.}^{(n)} + u_{rad.}^{(n)}, \qquad n = 1, 2 \tag{1.1}$$

where

$$u_{refl.}^{(n)} = \begin{cases} \sum_{m=1}^{2} B_m^{(n)} \exp\left[+\lambda_m x - i(K^2 + \lambda_m^2)^{1/2} y\right], & \begin{matrix} x < 0, \\ y < 0, \end{matrix} \\ 0, & y > 0. \end{cases} \tag{1.2}$$

These functions are taken to have possibly different reflection coefficients and possibly different but radiating diffracted fields. Next we form the difference function

$$\psi = u^{(1)} - u^{(2)} \tag{1.3}$$

which is essentially a solution of the previously posed boundary value problem without an incident field. Lastly, we introduce the auxiliary function

$$v(x, y) = (\partial/\partial x - \lambda_1)(\partial/\partial x - \lambda_2)\psi(x, y) \tag{1.4}$$

which is then a solution of the reduced wave equation satisfying homogeneous boundary conditions and obeys pseudo-radiation conditions.[1] Furthermore, by the postulated edge behavior of $u(x, y)$, it is easy to see that

$$v = O(1/r^{1+h}) \quad \text{as} \quad r \to 0 \quad \text{with} \quad 0 \le h < \tfrac{2}{3}. \tag{1.5}$$

Thus on expanding v in the form

$$v(r, \theta) = \sum_{n=0}^{\infty} C_n(r) \cos \frac{2n+1}{3} \theta \tag{1.6}$$

where

$$C_n(r) = C_n \int_0^{3\pi/2} v(r, \theta) \cos \frac{2n+1}{3} \theta \, d\theta, \qquad n = 0, 1, 2, \cdots, \tag{1.7}$$

it follows that

$$v(r, \theta) = D_0 H_{1/3}^{(1)}(Kr) \cos \theta/3 + D_1 H_1^{(1)}(Kr) \cos \theta. \tag{1.8}$$

The remainder of the proof consists in inverting v to obtain $\psi(x, y)$ in the form

$$\begin{aligned} \psi(x, y) = {} & \begin{cases} 0, & y > 0, \\ \sum_{m=1}^{2} (B_m^{(1)} - B_m^{(2)}) \exp[+\lambda_m x - i(K^2 + \lambda_m^2)^{1/2} y], & \begin{matrix} x < 0, \\ y < 0, \end{matrix} \end{cases} \\ & + \sum_{m=1}^{2} a_m e^{+\lambda_m x} \int_x^{\infty} e^{-\lambda_m \xi} \Big\{ D_0 H_{1/3}^{(1)}[K(\xi^2 + y^2)^{1/2}] \cos \frac{\theta}{3} \\ & + D_1 H_1^{(1)}[K(\xi^2 + y^2)^{1/2}] \cos \theta \Big\} \, d\xi \end{aligned} \tag{1.9}$$

where $a_m = (-1)^m/(\lambda_2 - \lambda_1)$. Then on applying the continuity conditions

$$\begin{aligned} [u] &= u(x, 0^+) - u(x, 0^-) = 0, \\ \left[\frac{\partial u}{\partial y}\right] &= \frac{\partial u}{\partial y}(x, 0^+) - \frac{\partial u}{\partial y}(x, 0^-) = 0, \qquad x < 0, \end{aligned} \tag{1.10}$$

we obtain a homogeneous system of equations for the four unknowns D_0, D_1, $B_1^{(1)} - B_1^{(2)}$,

[1]This effectively means that

$$\lim_{r \to \infty} \int_0^{3\pi/2} r^{1/2}(\partial v/\partial r - iKv) \cos[(2n+1)/3]\theta \, d\theta = 0$$

$n = 0, 1, 2, \cdots$ or equivalently this implies that (1.7) following obeys the Sommerfeld radiation condition required of u_{rad}.

The demonstration of this radiation condition is accomplished by first dividing the range of integration into three parts $[0, \pi - 1/r]$, $[\pi - 1/r, \pi + 1/r]$, and $[\pi + 1/r, (3/2)\pi]$. Then we estimate each of the resulting integrals separately. On the first and third intervals, the conditions set forth in Morgan [4] are satisfied, thus

$$\lim_{r \to \infty} \int_\alpha^\beta r^{1/2}(\partial v/\partial r - iKv) \cos[(2n+1)/3]\theta \, d\theta = 0$$

where $[\alpha, \beta] = [0, \pi - 1/r]$ or $[\pi + 1/r, (3/2)\pi]$. Lastly, the integral over $[\pi - 1/r, \pi + 1/r]$ is small by virtue of the smallness of the range and condition I.3a.

and $B_2^{(1)} - B_2^{(2)}$. However, the only solution to this system is the trivial one. Hence ψ is identically zero and $u(x, y)$ is unique.

Final comment. Uniqueness for the analogous problem having a source incident field($u_{inc} = \pi i H_0^{(1)}[K(x - x_0)^2 + (y - y_0)^2]^{1/2}$) may be formulated and proven in the same manner as above. In fact, this was done for a plane structure under the Nth order boundary condition, $\prod_{m=1}^{N} (\partial/\partial y + \lambda_m)u = 0$ on $y = 0$, see Morgan [5]. Furthermore, it is conjectured that the solution to the right-angled wedge under an Nth order condition will be unique if we require

$$\sum_{i=0}^{N} \left| \frac{\partial^i u(r, \theta)}{\partial x^i} \right| = O(r^{-[(2N-1)/3+h]}), \qquad 0 \leq h < \tfrac{2}{3}, \text{ as } r \to 0.$$

Acknowledgement. The author wishes to express his appreciation to Professor S. N. Karp for his continued encouragement and guidance.

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