THEORY OF IRREDUCIBLE OPERATORS OF LINEAR SYSTEMS*

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Abstract. The complementary differential operator is devised to decouple two coupled partial differential equations with variable coefficients. This complementary operator transforms two coupled partial differential equations into a pair of uncoupled equations either of (i) fourth order, (ii) third order, or (iii) second order. The conditions for the existence of linearly dependent solutions for two coupled equations with constant coefficient is also discussed. The method is applied to the propagation of two sound waves in liquid helium at finite temperatures.

1. Introduction. A physical system that can be described in terms of a certain number of variables, say u_1, u_2, $\cdots$, u_n, is frequently characterized by a system of partial differential equations that relate these variables. We shall consider a system characterized by following two partial differential equations:

$$Lu = \sum_{i,j} f_{ij} A_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_i g_i B_i \frac{\partial u}{\partial x_i} + hCu = 0, \tag{1}$$

where f_{ij}, g_i, and h are continuous scalar functions of the space and time variables unless stated otherwise, A_{ij}, B_i, C are 2×2 matrices with constant elements, and u is a column vector function with two components u_1 and u_2, i.e.

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The two variables u_1 and u_2 will appear in both equations if at least one of the matrices has a nonvanishing off-diagonal element.

In the theory of the system of equations, the first problem is, often, to deduce an equivalent partial differential equation for u_1, or u_2, alone. The well-adapted technique is the method of elimination [1].

Despite the straightforward calculation involved in the method of elimination, it has the following two basic deficiencies. First, the method has only limited applicability. The success, there, depends primarily on the coefficient f_{ij}, g_i, and h. Second, the order of the highest derivative appearing in the coupled equation is, in general, higher than that of the original equation.

The raising of the order of the derivative is an inevitable operation imbedded in the elimination process. Consequently, the initial and boundary value problems associated with the equivalent equation deduced by the method of elimination are expectedly complicated.

The coupled partial differential equation occurs in various fields of applied science such as fluid mechanics, theory of elasticity, wave propagation, nonequilibrium thermo-

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dynamics, etc. Yet many interesting initial and boundary value problems are still largely unexplored. Among many practical reasons, basically this may be attributed to the complexity in dealing with the higher order system.

It appears, therefore, worthwhile to develop a method of decoupling whereby the system of equations may be decomposed into two equations of the lowest possible order, and each equation contains only one dependent variable. From the methodologic point of view, this problem resembles the inverse problem of decomposing the Klein–Gordon equation into the Dirac equation.

The present investigation is motivated by the pioneering work of Clauser [2], who analyzed the behavior of the magnetohydrodynamic field in terms of independent modes.

The present method developed herein is called, for the sake of convenience, the method of irreducible operators. It consists of transforming system (1) into two equations, each of which is of the lowest possible order and contains only one dependent variable. The two operators governing two dependent variables are irreducible operators. In contrast to the method of elimination in which the uncoupled, or equivalent, equations govern u_1 or u_2, the method of irreducible operators, in general, deals with the two variables, say v_1 or v_2, which are linear combinations of the original dependent variables u_1 and u_2. At the outset, the form of the linear combinations is left arbitrary. This arbitrariness introduced by the transformation of two variables together with the complementary operator, devised in the present investigation, are subsequently utilized in decoupling system (1). When the proper complementary operators required for the decoupling contain no derivatives, the two decoupled equations obtained will be of second order. If the proper complementary operator with constant coefficients does not exist, we proceed to search for the complementary operator containing first derivatives. If this decoupling is successful, the resulting decoupled equation will be of the third order, and so on.

Besides the advantage of decoupling system (1) into a system of equations of order lower than the equivalent equation obtained by the elimination, the present method appears to have greater applicability than the method of elimination.

In Sec. 2, some basic properties of the similarity transformation of 2×2 matrices are recapitulated under the spin matrix formalism. The reduction to irreducible operators, the complementary differential operators, and four basic theorems concerning irreducible operators of fourth, third, and second orders are presented in Sec. 3. Finally, in Sec. 4, an application of the method of irreducible operators is demonstrated in the study of the propagation of two sound waves in liquid helium at low temperatures. It is found that the density and thermal waves each have two different propagation speeds. The solution yields the well-known fact that at zero temperature the density wave propagates at the first speed of sound, and the thermal wave propagates at the second speed of sound.

2. Similarity transformation and parallel matrices. It is well known that four Pauli spin matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

constitute a basis for the full matrix algebra of dimension 2 and order 4. The matrix algebra in the present paper will be carried out in the space spanned by spin matrices. Any 2×2 matrix A is, thus, expressed as

$$A = a_\rho \sigma_\rho \, . \tag{2}$$

Here, and in what follows, we use Greek suffixes for the range of values 0, 1, 2, 3, and Latin suffixes for the range 1, 2, 3, with the usual summation convention. The quantity with a bar, for example $\bar{A}$, denotes $a_k \sigma_k$ for $k = 1, 2, 3$, and is regarded as a three-dimensional vector. $\bar{A} \cdot \bar{B} = a_k b_k$, $\bar{A} \times \bar{B} = \sigma_k \epsilon_{klm} a_l b_m$ represent the usual dot product and cross product of two vectors.

Within the present formulation, the multiplication of two matrices is

$$AB = a_\rho b_\rho \sigma_0 + (a_0 \bar{B} + b_0 \bar{A} + i\bar{A} \times \bar{B}). \tag{3}$$

Define the complementary matrix[1] A^* as follows:

$$A^* = a_\rho^* \sigma_\rho \tag{4}$$

where

$$a_0^* = a_0 \, , \qquad a_k^* = -a_k \, . \tag{5}$$

The inner product of A and B, denoted by (A, B), or (B, A), is defined as

$$(A, B) = (B, A) = \tfrac{1}{2}(AB^* + BA^*) = a_\rho b_\rho \sigma_0 \, . \tag{6}$$

The norm of A is

$$(A, A) = \tfrac{1}{2}(AA^* + AA^*) = AA^* = a_\rho a_\rho \sigma_0 \, . \tag{7}$$

Consider the matrix $A = a_\rho \sigma_\rho$, and let $\alpha = \alpha_\rho \sigma_\rho$ be a nonsingular matrix; then the similarity transformation of A by α can be formally expressed as a rotation of the four-dimensional vector $A = a_\rho \sigma_\rho$ by a 4×4 orthogonal matrix S.

$$A \to A' = \alpha^{-1} A \alpha = (\det \alpha)^{-1} a_\rho \tilde{S}_{\rho\tau} \sigma_\tau = S_{\rho\tau} a_\rho \sigma_\tau = a'_\tau \sigma_\tau \, , \tag{8}$$

or

$$A' = SA, \tag{9}$$

where $\tilde{S}_{\rho\tau}$ are the elements of the matrix S multiplied by det α and are given as follows:

$$\begin{array}{lll}
\tilde{S}_{00} = 1, & \tilde{S}_{0k} = \tilde{S}_{k0} = 0, & \\
\tilde{S}_{11} = \alpha_0^2 - \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \, , & \tilde{S}_{12} = -2(i\alpha_0\alpha_3 + \alpha_1\alpha_2), & \tilde{S}_{13} = 2(i\alpha_0\alpha_2 - \alpha_1\alpha_3), \\
\tilde{S}_{21} = 2(i\alpha_0\alpha_3 - \alpha_1\alpha_2), & \tilde{S}_{22} = \alpha_0^2 + \alpha_1^2 - \alpha_2^2 + \alpha_3^2 \, , & \tilde{S}_{23} = -2(i\alpha_0\alpha_1 + \alpha_2\alpha_3), \\
\tilde{S}_{31} = -2(i\alpha_0\alpha_2 + \alpha_1\alpha_3), & \tilde{S}_{32} = 2(i\alpha_0\alpha_1 - \alpha_2\alpha_3), & \tilde{S}_{33} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 - \alpha_3^2 \, .
\end{array} \tag{10}$$

The following important properties of the S matrix can be shown directly from (10):

(a) Orthogonality relation:

$$S_{\mu\nu} S_{\rho\nu} = \delta_{\mu\rho} \, . \tag{11}$$

(b) Proper rotation:

$$\det S = 1. \tag{12}$$

[1]The complementary matrix A^* is related to the inverse of A by $A^* = (\det A)\, A^{-1}$.

(c) Four-valued representation:

$$S = S(\pm\alpha) = S(\pm i\alpha). \tag{13}$$

A basic operation occurring in the theory of irreducible operators is the simultaneous diagonalization or triangularization of two or more matrices.

Whereas a condition for the simultaneous diagonalization of two matrices A, B is the commutation of A and B, this condition possesses a simple geometrical interpretation under present formalism. This and other related subjects are discussed in the remaining part of this section.

Definition 1. Two matrices A, B, are said to be parallel, written $A \parallel B$, if they satisfy the following condition:

$$\pm(\bar{A}\cdot\bar{B})[(\bar{A}\cdot\bar{A})(\bar{B}\cdot\bar{B})]^{-1/2} = \pm(a_k b_k)[(a_l a_l)(b_m b_m)]^{-1/2} = 1, \tag{14a}$$

or equivalently,

$$(\bar{A} \times \bar{B})\cdot(\bar{A} \times \bar{B}) = 0. \tag{14b}$$

Note that the zeroth component does not enter. Also notice that if the elements of $\bar{A}$ and $\bar{B}$ are all real, then the only solution satisfying Eq. (14a) or Eq. (14b) is $\bar{A} = K\bar{B}$, where K is an arbitrary real number.

Lemma 1. *The necessary and sufficient condition that a matrix A can be diagonalized and a matrix B can be triangularized simultaneously by a similarity transformation is that $A \parallel B$.*

Proof. To prove necessity, we use the fact A is diagonalized by a similarity transformation S, i.e. $A \rightarrow A' = SA$ with

$$a_0' = a_0 \,, \tag{15a}$$

$$a_1' = S_{k1}a_k = 0, \tag{15b}$$

$$a_2' = S_{k2}ak = 0, \tag{15c}$$

$$a_3' = S_{k3}a_k \,. \tag{15d}$$

By eliminating successively a_1, a_2, and a_3 between Eq. (15b) and Eq. (15c), we obtain

$$\frac{S_{21}S_{12} - S_{11}S_{22}}{S_{31}S_{12} - S_{11}S_{32}} = -\frac{S_{33}}{S_{23}} = -\frac{a_3}{a_2}\,,$$

$$\frac{S_{11}S_{22} - S_{21}S_{12}}{S_{31}S_{32} - S_{21}S_{32}} = -\frac{S_{33}}{S_{13}} = -\frac{a_3}{a_1}\,,$$

$$\frac{S_{31}S_{12} - S_{11}S_{32}}{S_{31}S_{22} - S_{21}S_{32}} = -\frac{S_{23}}{S_{13}} = -\frac{a_2}{a_1}.$$

Hence

$$\frac{a_1}{S_{13}} = \frac{a_2}{S_{23}} = \frac{a_3}{S_{33}} = N. \tag{16}$$

Substituting Eq. (16) into Eq. (15), we have

$$a_3' = NS_{k3}S_{k3} = N, \tag{17}$$

by virtue of the orthogonality relation. Since $a_k a_k$ is an invariant under similarity transformation,

$$a_k a_k = a_l' a_l' = a_3'^2 = N^2,$$

this gives

$$N = \pm(a_k a_k)^{1/2}. \tag{18}$$

Substituting Eq. (18) into Eq. (16) yields

$$S_{k3} = \pm a_k (a_l a_l)^{-1/2}. \tag{19}$$

When B is triangularized by the corresponding similarity transformation, then $b_2' = \pm i b_1'$ (the upper sign corresponding to the vanishing of the 21 element of B', and the lower sign corresponding to the vanishing of the 12 element of B').

From Eq. (19), we have

$$b_3' = S_{k3} b_3 = \pm a_k b_k (a_l a_l)^{-1/2}, \tag{20a}$$

and also

$$b_i b_i = b_i' b_i' = b_3'^2. \tag{20b}$$

Substituting Eq. (20b) into Eq. (20a) yields the condition

$$\pm a_k b_k [(a_l a_l)(b_m b_m)]^{-1/2} = 1. \tag{21}$$

Since the sufficiency can be proved without recourse to the particular formalism adopted in this paper, together with the fact that the proof requires additional lemmas and theorems in linear algebra (see, for example, Gel'fand [3]), it is not reproduced here.

From Lemma 1, the following corollary is readily obtained.

Corollary. *The necessary and sufficient condition that two matrices A and B can be diagonalized simultaneously by a similarity transformation is that $A \parallel B$.*

The following remarks, stated without proof, are useful in the later development.

Remarks. (1) Any two nonparallel matrices A and B cannot be made parallel by repeated similarity transformations.

(2) The transitivity law of parallel matrices does not necessarily hold: i.e., if $A \parallel B$, and $B \parallel C$, then A may or may not be parallel to C.

(3) If $A \parallel B \parallel C$, then $AB \parallel AC$.

The following lemma is derived from Remark 3.

Lemma 2. *If $A \parallel B$, and A is a nonsingular matrix, then $A \parallel A^n B$, where n is a positive or negative integer.*

Proof. For $n = 0$, the assertion obviously holds. For $n = 1$, the proof of $A \parallel AB$ consists of showing the following equality:

$$a_0^2(\bar{A} \times \bar{B})\cdot(\bar{A} \times \bar{B}) - [\bar{A} \times (\bar{A} \times \bar{B})]\cdot[\bar{A} \times (\bar{A} \times \bar{B})] = 0. \tag{22}$$

The first term is zero by virtue of $A \parallel B$; the second term possesses the following vector identity

$$(\bar{A}\cdot\bar{A})(\bar{A}\cdot\bar{B})^2 - (\bar{A}\cdot\bar{A})^2(\bar{B}\cdot\bar{B}). \tag{23}$$

The above expression vanishes on account of Lemma 1. This proves $A \parallel AB$.

The proof is completed for the case $n > 1$ by mathematical induction as follows. Suppose $A \parallel A^n B$ holds. Let $A^n B = B^{(n)}$, then $A \parallel B^{(n)}$ implies $A \parallel AB^{(n)} \overset{\parallel}{=} A^{n+1}B$. The proof for the case of negative integers is essentially the same as the case of positive integers. Note that since A is nonsingular, it is sufficient to prove $A \parallel A^*B$, instead of $A \parallel A^{-1}B$. This is done by replacing $\bar{A}$ by $\bar{A}^*$ in Eq. (22). The result is $A \parallel A^*B$, and therefore $A \parallel A^{-1}B$. Mathematical induction is then used to complete the proof.

Corollary. *If $A \parallel B$, and A is nonsingular, then $A \parallel BA^n$, where n is a positive or negative integer.*

Proof. From Lemma 2 we have $A \parallel A$, $B \parallel A^*B$. Since $A^*B = 2a_p b_p^* \sigma_0 - B^*A$, the following relations hold:

$$A \parallel A^*B \overset{\parallel}{=} 2a_p b_p^* \sigma_0 - B^*A \parallel, B^*A \overset{\parallel}{=} (-B + 2b_0\sigma_0)A \parallel BA.$$

This proves the case of $n = 1$. The remaining proof is completed by mathematical induction.

Remark (4). If A, B, C, $\cdots$ are not parallel to each other, then the cosets A^*A, A^*B, A^*C, $\cdots$ are parallel to each other if and only if A is a singular matrix. The proof consists of showing that the condition of parallelism between A^* and A^*B can be written as

$$(a_0^2 - \bar{A}^* \cdot \bar{A}^*)\{(\bar{A}^* \cdot \bar{A}^*)(\bar{B} \cdot \bar{B}) - (\bar{A}^* \cdot \bar{B})^2\} = 0.$$

The quantity inside the braces does not vanish because A^* and B are nonparallel. Hence, in order that the above equality hold, $a_0^2 - \bar{A}^* \cdot \bar{A}^* = 0$, implying that A is a singular matrix. Since the proof is independent of matrix B, the result holds for any matrices. It is, however, to be pointed out that all the left cosets with respect to A^* are also singular.

3. Reduction to irreducible operators from eq. (1).

Definition 2. u_1 and u_2 are called modes of nth-order if they are solutions of the equations $L_1 u_1 = 0$ and $L_2 u_2 = 0$, when L_1 and L_2 are nth-order linear operators. If u_1 is governed by the nth-order operator with a known inhomogeneous function $l_1(u_2)$ (i.e., $L_1 u_1 = l_1(u_2)$, $L_2 u_2 = 0$) where l_1 is some linear operator, then u_1 is called a mode of quasi-nth order. The modes which are governed by the irreducible operator are called fundamental modes.

The system (1) may possess modes of fourth, third and second order, or quasi-fourth -third and -second order, depending on the nature of the functions f_{ij}, g_i, and h_i, and the matrices A_{ij}, B_i, C. In the following, the class of Eq. (1), which may be decomposed into two fourth order operators or quasi-fourth order, will be studied first. Subsequently, we shall study lower-than-fourth order operators.

Let us define the following complementary operator to the operator L.

$$L^* = \sum_{i,j}^{n} A_{ij}^* \left(\alpha_{ij} f_{ij} \frac{\partial^2}{\partial x_i\, \partial \chi_j} + \beta_{ij} \frac{\partial}{\partial x_i} f_{ij} \frac{\partial}{\partial x_j} + \gamma_{ij} \frac{\partial}{\partial x_j} f_{ij} \frac{\partial}{\partial x_i} + \epsilon_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} f_{ij}\right) + \sum_{k}^{n} B_k^* \left(\alpha_k g_k \frac{\partial}{\partial x_k} + \beta_k \frac{\partial}{\partial x_k} g_k\right) + C^* h, \qquad (24)$$

where α_{ij}, β_{ij}, γ_{ij}, ϵ_{ij}, α_i and β_i are, in general, arbitrary complex numbers with the following restrictions:

$$\begin{aligned} \alpha_{ij} + \beta_{ij} + \gamma_{ij} + \epsilon_{ij} &= 1 \quad \text{for all } i,\, j, \\ \alpha_i + \beta_i &= 1 \quad \text{for all } i. \end{aligned} \tag{25}$$

Operating on the left of Eq. (1) by L^* yields

$$\begin{aligned} L^*Lu = {} & \sum_{i,j}^{n} A_{ij}^* A_{ij} \left\{ f_{ij} \frac{\partial^4 u}{\partial x_i^2\, \partial x_j^2} + (1 + \beta_{ij} + \epsilon_{ij}) f_{ij} \frac{\partial f_{ij}}{\partial x_i} \frac{\partial^3 u}{\partial x_i\, \partial x_j^2} \right. \\ & + (1 + \gamma_{ij} + \epsilon_{ij}) f_{ij} \frac{\partial f_{ij}}{\partial x_j} \frac{\partial^3 u}{\partial x_i^2\, \partial x_j} + \left[(1 + \epsilon_{ij}) f_{ij} \frac{\partial^2 f_{ij}}{\partial x_i\, \partial x_j} + (\beta_{ij} + \gamma_{ij} + 2\epsilon_{ij}) \right. \\ & \left.\left. \cdot \frac{\partial f_{ij}}{\partial x_i} \frac{\partial f_{ij}}{\partial x_j} \right] \frac{\partial^2 u}{\partial x_i\, \partial x_j} \right\} + \sum_{\substack{i,j,k,l \\ ij \neq kl}}^{n} (A_{ij}^* A_{kl} + A_{kl}^* A_{ij}) f_{ij} f_{kl} \frac{\partial^4 u}{\partial x_i\, \partial x_j\, \partial x_k\, \partial x_l} \\ & + \sum_{i,j,k}^{n} (A_{ij}^* B_k + B_k^* A_{ij}) f_{ij} g_k \frac{\partial^3 u}{\partial x_i\, \partial x_j\, \partial x_k} + \sum_i B_i^* B_i \left\{ g_i^2 \frac{\partial^2 u}{\partial x_i^2} + (1 + \beta_i) \right. \\ & \left. \cdot g_i \frac{\partial g_i}{\partial x_i} \frac{\partial u}{\partial x_i} \right\} + \sum_{\substack{i,j \\ i \neq j}}^{n} (B_i^* B_j + B_j^* B_i) g_i g_j \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i,j}^{n} f_{ij} h (A_{ij}^* C \\ & + C^* A_{ij}) \frac{\partial^2 u}{\partial x_i\, \partial x_j} + \sum_{i}^{n} (B_i^* C + C^* B_i) g_i h \frac{\partial u}{\partial x_i} + h^2 C^* C u \\ & + \sum_{\substack{i,j,k,l \\ ij \neq kl}}^{n} (A_{ij}^* A_{kl} + A_{kl}^* A_{ij}) \mathfrak{L}_{ij,kl} u + \sum_{\substack{i,j,k,l \\ ij \neq kl}}^{n} A_{kl}^* A_{ij} (\mathfrak{L}_{kl,ij} - \mathfrak{L}_{ij,kl}) u \\ & \cdot \sum_{ij,k}^{n} (A_{ij}^* B_k + B_k^* A_{ij}) \mathfrak{L}_{ij,k} u + \sum_{ij,k}^{n} B_k^* A_{ij} (\mathfrak{L}_{k,ij} - \mathfrak{L}_{ij,k}) u \\ & \cdot \sum_{i,j}^{n} (A_{ij}^* C + C^* A_{ij}) \mathfrak{L}_{ij,0} u + \sum_{i,j}^{n} C^* A_{ij} (\mathfrak{L}_{0,ij} - \mathfrak{L}_{ij,0}) u \\ & + \sum_{\substack{i,j \\ i \neq j}}^{n} (B_i^* B_j + B_j^* B_i) \mathfrak{L}_{i,j} u + \sum_{\substack{i,j \\ i \neq j}}^{n} B_j^* B_i (\mathfrak{L}_{j,i} - \mathfrak{L}_{i,j}) u \\ & + \sum_{i}^{n} (B_i^* C + C^* B_i) \mathfrak{L}_{i,0} u + \sum_{i}^{n} C^* B_i (\mathfrak{L}_{0,i} - \mathfrak{L}_{i,0}) u = 0, \end{aligned} \tag{26}$$

where $\mathfrak{L}_{ij,kl}$, $\mathfrak{L}_{ij,k}$, $\mathfrak{L}_{ij,0}$, $\mathfrak{L}_{i,j}$ and $\mathfrak{L}_{i,0}$ shall be called residual operators, $\mathfrak{L}_{kl,ij}$, $\mathfrak{L}_{k,ij}$, $\mathfrak{L}_{0,ij}$, $\mathfrak{L}_{j,i}$ and $\mathfrak{L}_{0,i}$, will be called the complementary residual operators of $\mathfrak{L}_{ij,kl}$, $\mathfrak{L}_{ij,k}$, etc. The residual and complementary residual operators are given as follows:

$$\begin{aligned} \mathfrak{L}_{ij,kl} = {} & \left\{ f_{ij} \frac{\partial f_{kl}}{\partial x_i} + (\beta_{ij} + \epsilon_{ij}) f_{kl} \frac{\partial f_{ij}}{\partial x_i} \right\} \frac{\partial^3}{\partial x_j\, \partial x_k\, \partial x_l} + \left\{ f_{ij} \frac{\partial f_{kl}}{\partial x_j} \right. \\ & \left. + (\gamma_{ij} + \epsilon_{ij}) f_{kl} \frac{\partial f_{ij}}{\partial x_j} \right\} \frac{\partial^3}{\partial x_i\, \partial x_k\, \partial x_l} + \left\{ f_{ij} \frac{\partial^2 f_{kl}}{\partial x_i\, \partial x_j} + (\beta_{ij} + \epsilon_{ij}) \frac{\partial f_{ij}}{\partial x_i} \frac{\partial f_{kl}}{\partial x_j} \right. \\ & \left. + (\gamma_{ij} + \epsilon_{ij}) \frac{\partial f_{ij}}{\partial x_j} \frac{\partial f_{kl}}{\partial x_i} + \epsilon_{ij} \frac{\partial^2 f_{ij}}{\partial x_i\, \partial x_j} f_{kl} \right\} \frac{\partial^2}{\partial x_k\, \partial x_l}. \end{aligned} \tag{27a}$$

$$\mathfrak{L}_{ij,k} = \left\{f_{ij}\frac{\partial g_k}{\partial x_i} + (\beta_{ij} + \epsilon_{ij})\frac{\partial f_{ij}}{\partial x_i} g_k\right\}\frac{\partial^2}{\partial x_j\,\partial x_k} + \left\{f_{ij}\frac{\partial g_k}{\partial x_j} + (\epsilon_{ij} + \gamma_{ij})\frac{\partial f_{ij}}{\partial x_j} g_k\right\}\frac{\partial^2}{\partial x_i\,\partial x_k} + \left\{f_{ij}\frac{\partial^2 g_k}{\partial x_i\,\partial x_j} + (\beta_{ij} + \epsilon_{ij})\frac{\partial f_{ij}}{\partial x_i}\frac{\partial g_k}{\partial x_j} + (\gamma_{ij} + \epsilon_{ij})\frac{\partial f_{ij}}{\partial x_j}\frac{\partial g_k}{\partial x_i} + \epsilon_{ij}\frac{\partial^2 f_{ij}}{\partial x_i\,\partial x_j} g_k\right\}. \tag{27b}$$

$$\mathfrak{L}_{k,ij} = \left(g_k\frac{\partial f_{ij}}{\partial x_k} + \beta_k f_{ij}\frac{\partial g_k}{\partial x_k}\right)\frac{\partial^2}{\partial x_i\,\partial x_j}. \tag{27c}$$

$$\mathfrak{L}_{ij,0} = \left\{f_{ij}\frac{\partial h}{\partial x_j} + (\gamma_{ij} + \epsilon_{ij})\frac{\partial f_{ij}}{\partial x_j} h\right\}\frac{\partial}{\partial x_i} + \left\{f_{ij}\frac{\partial h}{\partial x_i} + (\beta_{ij} + \epsilon_{ij})\frac{\partial f_{ij}}{\partial x_i} h\right\}\cdot\frac{\partial}{\partial x_j} + \left\{f_{ij}\frac{\partial^2 h}{\partial x_i\,\partial x_j} + (\beta_{ij} + \epsilon_{ij})\frac{\partial f_{ij}}{\partial x_i}\frac{\partial h}{\partial x_i} + (\gamma_{ij} + \epsilon_{ij})\frac{\partial f_{ij}}{\partial x_j}\frac{\partial h}{\partial x_i} + \epsilon_{ij}\frac{\partial^2 f_{ij}}{\partial x_i\,\partial x_j} h\right\}. \tag{27d}$$

$$\mathfrak{L}_{0,ij} = 0. \tag{27e}$$

$$\mathfrak{L}_{i,j} = \left(g_i\frac{\partial g_j}{\partial x_i} + \beta_i g_j\frac{\partial g_i}{\partial x_i}\right)\frac{\partial}{\partial x_i}. \tag{27f}$$

$$\mathfrak{L}_{i,0} = \left(g_i\frac{\partial h}{\partial x_i} + \beta_i\frac{\partial g_i}{\partial x_i} h\right). \tag{27g}$$

$$\mathfrak{L}_{0,i} = 0. \tag{27h}$$

Note that $\mathfrak{L}_{kl,ij}$ is obtained by interchanging ij by kl in $\mathfrak{L}_{ij,kl}$ and $\mathfrak{L}_{j,i}$ is obtained by interchanging i by j in $\mathfrak{L}_{i,j}$.

There are two types of cosets appearing in Eq. (26). One type consists of inner products of two matrices, such as $A_{ij}^* A_{ij}$ or $A_{ij}^* B_k + B_k^* A_{ij}$. These cosets have only a σ_0 component as is shown in Eqs. (6) and (7).

Another type of coset consists of the multiplication of two different matrices, like $A_{kl}^* A_{ij}$, $B_k^* A_{ij}$ etc., which are not, in general, diagonal or triangular. These cosets appear only in terms containing the difference of the residual and complementary residual operators, for example $\mathfrak{L}_{kl,ij} - \mathfrak{L}_{ij,kl}$, $\mathfrak{L}_{k,ij} - \mathfrak{L}_{ij,k}$, etc. Thus, Eq. (26) may be written symbolically as

$$L^*Lu = D_0^{(4)}\sigma_0 u + (R_0^{(3)}\sigma_0 + R_1^{(3)}\sigma_1 + R_2^{(3)}\sigma_2 + R_3^{(3)}\sigma_3)u, \tag{28a}$$

where $D_0^{(4)}\sigma_0$ is the operator containing the diagonal cosets, and $R_p^{(3)}\sigma_p$ is the operator containing the difference of the residual and the complementary residual operators. Superscripts represent the highest order of derivative in the operator.

In general, Eq. (26), or Eq. (28a), is a coupled partial differential equation. However, Eq. (26) becomes either a pair of uncoupled fourth order equations, or a quasi-fourth order equation, when one of the following three conditions is satisfied: (i) When the cosets $A_{kl}^* A_{ij}$, $B_k^* A_{ij}$, are diagonal or triangular with zeros at the same entry, Eq. (26) becomes two fourth order equations

$$L_1^{(4)} u_1 = l_1^{(3)}(u_2), \qquad L_2^{(4)} u_2 = 0, \tag{28b}$$

where $l_1^{(3)}(u_2)$ is at most of third order, and is zero if all the cosets $A_{kl}^* A_{ij}$, $B_k^* A_{ij}$, are diagonal. (ii) (a) When the coefficients associated with each derivative appearing in $R_1^{(3)}$ and $R_2^{(3)}$ are zero, Eq. (28a) becomes

$$L^*Lu = D_0^{(4)}\sigma_0 u + R_0^{(3)}\sigma_0 u + R_3^{(3)}\sigma_3 u$$

representing two fourth order partial differential equations. For the coefficients to be zero, 18 independent conditions listed in Table 1 must be satisfied. These conditions hold for Eq. (1) with two independent variables x and y. (b) When the coefficients associated with the derivatives are such that $R_1^{(3)} - iR_2^{(3)} = 0$ or $R_1^{(3)} + iR_2^{(3)} = 0$, Eq. (28a) becomes a quasi-fourth order equation similar to Eq. (28b), with the proper inhomogeneous term $l_1^{(3)}(u_2)$ or $l_2^{(3)}(u_1)$. For $R_1^{(3)}\sigma_1 \mp iR_2^{(3)}\sigma_2 = 0$, 10 independent conditions, which can be easily deduced from Table 1, are to be satisfied for each case.

TABLE 1.
Conditions for the Vanishings of the Coefficients Associated with the Derivatives in $R_\rho^{(3)}\sigma_\rho$.

Derivatives	Coefficients
$\dfrac{\partial^3}{\partial x^3}$	$f_{12}\dfrac{\partial f_{11}}{\partial y} + (\gamma_{12} + \epsilon_{12})f_{11}\dfrac{\partial f_{12}}{\partial y} = 0.$
$\dfrac{\partial^3}{\partial y^3}$	$f_{12}\dfrac{\partial f_{22}}{\partial x} + (\beta_{12} + \epsilon_{12})\dfrac{\partial f_{12}}{\partial x}f_{22} = 0.$
$\dfrac{\partial^3}{\partial x^2\,\partial y}$	$(A_{12}^*A_{11})^{\dagger}_{[12,21]}\left\{(2 - \beta_{11} - \epsilon_{12})f_{11}\dfrac{\partial f_{12}}{\partial x} - (1 - \beta_{11} - \gamma_{11} - 2\epsilon_{11})\dfrac{\partial f_{11}}{\partial x}f_{12}\right\} - (A_{22}^*A_{11})_{[12,21]}\left\{2\dfrac{\partial f_{11}}{\partial y}f_{22} + (\beta_{22} + \gamma_{22} + 2\epsilon_{22})f_{11}\dfrac{\partial f_{22}}{\partial y}\right\} = 0$
$\dfrac{\partial^3}{\partial x\,\partial y^2}$	$(A_{22}^*A_{12})_{[12,21]}\left\{(2 - \gamma_{12} - \epsilon_{12})\dfrac{\partial f_{12}}{\partial y}f_{22} - (1 - \beta_{22} - \gamma_{22} - 2\epsilon_{22})f_{12}\dfrac{\partial f_{22}}{\partial y}\right\} - (A_{22}^*A_{11})_{[12,21]}\left\{2f_{11}\dfrac{\partial f_{22}}{\partial x} + (\beta_{11} + \gamma_{11} + 2\epsilon_{11})\dfrac{\partial f_{11}}{\partial x}f_{22}\right\} = 0.$
$\dfrac{\partial^2}{\partial x^2}$	$(A_{12}^*A_{11})_{[12,21]}\left\{f_{12}\dfrac{\partial^2 f_{11}}{\partial x\,\partial y} + (\beta_{12} + \epsilon_{12})\dfrac{\partial f_{11}}{\partial y}\dfrac{\partial f_{12}}{\partial x} + (\gamma_{12} + \epsilon_{12})\dfrac{\partial f_{11}}{\partial x}\dfrac{\partial f_{12}}{\partial y}\right\} + (A_{22}^*A_{11})_{[12,21]}\left\{\dfrac{\partial^2 f_{11}}{\partial y^2}f_{22} + (\beta_{22} + \gamma_{22} + 2\epsilon_{22})\dfrac{\partial f_{11}}{\partial y}\dfrac{\partial f_{22}}{\partial y} + \epsilon_{22}f_{11}\dfrac{\partial^2 f_{22}}{\partial y^2}\right. + (B_1^*A_{11})_{[12,21]}\Big\}(1 - \beta_{11} - \gamma_{11} - 2\epsilon_{11})\dfrac{\partial f_{11}}{\partial x}g_1 - (2 - \beta_1)f_{11}\dfrac{\partial g_1}{\partial x}$

†$(A_{ij}{}^*A_{kl,})_{[12],[21]}$, means either the 12 or 21 element of the coset $A_{ij}{}^*A_{kl}$.

Note 1; For $R_\rho^{(3)}\sigma_\rho$ to be diagonal, all the coefficients associated with the 12 and 21 element must be zero. This requires a total of 18 conditions to be satisfied.

Note 2; For $R_\rho^{(3)}\sigma_\rho$ to be triangular, only the coefficients associated with particular off-diagonal elements, say the 12 element, must be equal to zero. This gives rise to 10 independent conditions.

TABLE 1 (*Continued*)

Derivatives	Coefficients
	$+ (B_2^* A_{11})_{[12,21]}\left\{g_2 \frac{\partial f_{11}}{\partial y} + \beta_2 \frac{\partial g_2}{\partial y} f_{11}\right\}$ $- B_1^* A_{12}\left\{f_{12}\frac{\partial g_1}{\partial y} + (\gamma_{12}+\epsilon_{12})\frac{\partial f_{12}}{\partial y} g_1\right\} = 0.$
$\frac{\partial^2}{dy^2}$	$-(A_{22}^* A_{11})_{[12,21]}\left\{f_{11}\frac{\partial^2 f_{22}}{\partial x^2} + (\beta_{11}+\gamma_{11}+2\epsilon_{11})\frac{\partial f_{11}}{\partial x}\frac{\partial f_{22}}{\partial x} + \epsilon_{11}\frac{\partial^2 f_{11}}{\partial x^2} f_{22}\right\}$ $- (A_{22}^* A_{12})_{[12,21]}\{f_{12}\frac{\partial^2 f_{22}}{\partial x\,\partial y} + (\beta_{12}+\epsilon_{12})\frac{\partial f_{12}}{\partial x}\frac{\partial f_{22}}{\partial y}$ $+(\gamma_{12}+\epsilon_{12})\frac{\partial f_{22}}{\partial x}\frac{\partial f_{12}}{\partial y} + \epsilon_{12}\frac{\partial^2 f_{12}}{\partial x\,\partial y} f_{22}\}$ $-(B_2^*A_{12})_{[12,21]}\left\{f_{12}\frac{\partial g_2}{\partial x} + (\beta_{12}+\epsilon_{12})\frac{\partial f_{12}}{\partial x} g_2\right\}$ $+B_1^*A_{22}\left\{\frac{\partial f_{22}}{\partial x} g_1 + \beta_1 f_{22}\frac{\partial g_1}{\partial x}\right\}$ $-(B_2^*A_{22})_{[12,21]}\left\{(1-\beta_{22}-\gamma_{22}-2\epsilon_{22})\frac{\partial f_{22}}{\partial y} g_2 - (2-\beta_2) f_{22}\frac{\partial g_2}{\partial y}\right\} = 0.$
$\frac{\partial^2}{\partial x\,\partial y}$	$-(A_{12}^*A_{11})_{[12,21]}\left\{f_{11}\frac{\partial^2 f_{12}}{\partial x^2} + (\beta_{11}+\gamma_{11}+2\epsilon_{11})\frac{\partial f_{11}}{\partial x}\frac{\partial f_{12}}{\partial x} + \epsilon_{11}\frac{\partial^2 f_{11}}{\partial x^2} f_{12}\right\}$ $+(A_{22}^*A_{12})_{[12,21]}\left\{\frac{\partial^2 f_{12}}{\partial y^2} f_{22} + (\beta_{22}+\gamma_{22}+2\epsilon_{22})\frac{\partial f_{12}}{\partial y}\frac{\partial f_{22}}{\partial y} + \epsilon_{22} f_{12}\frac{\partial^2 f_{22}}{\partial y^2}\right\}$ $-(B_2^*A_{11})_{[12,21]}\left\{2f_{11}\frac{\partial g_2}{\partial x} + (\beta_{11}+\gamma_{11}+2\epsilon_{11})\frac{\partial f_{11}}{\partial x} g_2\right\}$ $-(B_1^*A_{12})_{[12,21]}\left\{(1-\beta_1)f_{12}\frac{\partial g_1}{\partial x} - (1-\beta_{12}-\epsilon_{12})\frac{\partial f_{12}}{\partial x} g_1\right\}$ $-(B_2^*A_{12})_{[12,21]}\left\{(1-\beta_2)f_{12}\frac{\partial g_2}{\partial y} - (1-\gamma_{12}-\epsilon_{12})\frac{\partial f_{12}}{\partial y} g_2\right\}$ $+(B_1^*A_{22})_{[12,21]}\left\{2f_{22}\frac{\partial g_1}{\partial y} + (\beta_{22}+\gamma_{22}+2\epsilon_{22})\frac{\partial f_{22}}{\partial y} g_1\right\} = 0.$
	$(B_1^*A_{11})_{[12,21]}\left\{f_{11}\frac{\partial^2 g_1}{\partial x^2} + (\beta_{11}+\gamma_{11}+2\epsilon_{11})\frac{\partial f_{11}}{\partial x}\frac{\partial g_1}{\partial x} + \epsilon_{11}\frac{\partial^2 f_{11}}{\partial x^2} g_1\right\}$ $+(B_1^*A_{12})_{[12,21]}\{f_{12}\frac{\partial^2 g_1}{\partial x\,\partial y} + (\beta_{12}+\epsilon_{12})\frac{\partial f_{12}}{\partial x}\frac{\partial g_1}{\partial y}$ $+(\gamma_{12}+\epsilon_{12})\frac{\partial f_{12}}{\partial y}\frac{\partial g_1}{\partial x} + \epsilon_{12}\frac{\partial^2 f_{12}}{\partial x\,\partial y} g_1\}$

TABLE 1 (*Continued*)

Derivatives	Coefficients
$\frac{\partial}{\partial x}$	$+ (B_1^* A_{22})_{[12,21]} \left\{ f_{22} \frac{\partial^2 g_1}{\partial y^2} + (\beta_{22} + \gamma_{22} + 2\epsilon_{22}) \frac{\partial f_{22}}{\partial y} \frac{\partial g_1}{\partial y} + \epsilon_{22} \frac{\partial^2 f_{22}}{\partial y^2} g_1 \right\}$ $+ (C^* A_{11})_{[12,21]} \left\{ 2 f_{11} \frac{\partial h}{\partial x} + (\beta_{11} + \gamma_{11} + 2\epsilon_{11}) \frac{\partial f_{11}}{\partial x} h \right\}$ $+ (C^* A_{12})_{[12,21]} \left\{ f_{12} \frac{\partial h}{\partial y} + (\gamma_{12} + \epsilon_{12}) \frac{\partial f_{12}}{\partial y} h \right\}$ $- (B_2^* B_1)_{[12,21]} \left\{ \frac{\partial g_1}{\partial y} g_2 + \beta_2 g_1 \frac{\partial g_2}{\partial y} \right\} = 0.$
$\frac{\partial}{\partial y}$	$(B_2^* A_{11})_{[12,21]} \left\{ f_{11} \frac{\partial^2 g_2}{\partial x^2} + (\beta_{11} + \gamma_{11} + 2\epsilon_{11}) \frac{\partial f_{11}}{\partial x} \frac{\partial g_2}{\partial x} + \epsilon_{11} \frac{\partial^2 f_{11}}{\partial x^2} g_2 \right\}$ $+ (B_2^* A_{12})_{[12,21]} \Big\{ f_{12} \frac{\partial^2 g_2}{\partial x\, \partial y} + (\beta_{12} + \epsilon_{12}) \frac{\partial f_{12}}{\partial x} \frac{\partial g_2}{\partial y}$ $+ (\gamma_{12} + \epsilon_{12}) \frac{\partial f_{12}}{\partial y} \frac{\partial g_2}{\partial x} + \epsilon_{12} \frac{\partial^2 f_{12}}{\partial x\, \partial y} g_2 \Big\}$ $+ (B_2^* A_{22})_{[12,21]} \left\{ f_{22} \frac{\partial^2 g_2}{\partial y^2} + (\beta_{22} + \gamma_{22} + 2\epsilon_{22}) \frac{\partial f_{22}}{\partial y} \frac{\partial g_2}{\partial y} + \epsilon_{22} \frac{\partial^2 f_{22}}{\partial y^2} g_2 \right\}$ $+ (C^* A_{22})_{[12,21]} \left\{ 2 f_{22} \frac{\partial h}{\partial y} + (\beta_{22} + \gamma_{22} + 2\epsilon_{22}) \frac{\partial f_{22}}{\partial y} h \right\}$ $+ (C^* A_{12})_{[12,21]} \left\{ f_{12} \frac{\partial h}{\partial x} + (\beta_{12} + \epsilon_{12}) \frac{\partial f_{12}}{\partial x} h \right\}$ $+ B_2^* B_1 \left\{ g_1 \frac{\partial g_2}{\partial x} + \beta_1 \frac{\partial g_1}{\partial x} g_2 \right\} = 0.$
$\left(\frac{\partial}{\partial x}\right)^0 \left(\frac{\partial}{\partial y}\right)^0$	$(C^* A_{11})_{[12,21]} \left\{ f_{11} \frac{\partial^2 h}{\partial x^2} + (\beta_{11} + \gamma_{11} + 2\epsilon_{11}) \frac{\partial f_{11}}{\partial x} \frac{\partial h}{\partial x} + \epsilon_{11} \frac{\partial^2 f_{11}}{\partial x^2} h \right\}$ $+ (C^* A_{12})_{[12,21]} \Big\{ f_{12} \frac{\partial^2 h}{\partial x\, \partial y} + (\beta_{12} + \epsilon_{12}) \frac{\partial f_{12}}{\partial x} \frac{\partial h}{\partial y}$ $+ (\gamma_{12} + \epsilon_{12}) \frac{\partial f_{12}}{\partial y} \frac{\partial h}{\partial x} + \epsilon_{12} \frac{\partial^2 f_{12}}{\partial x\, \partial y} h \Big\}$ $+ (C^* A_{22})_{[12,21]} \left\{ f_{22} \frac{\partial^2 h}{\partial y^2} + (\beta_{22} + \gamma_{22} + 2\epsilon_{22}) \frac{\partial f_{22}}{\partial y} \frac{\partial h}{\partial y} + \epsilon_{22} \frac{\partial^2 f_{22}}{\partial y^2} h \right\}$ $+ (C^* B_1)_{[12,21]} \left\{ g_1 \frac{\partial h}{\partial x} + \beta_1 \frac{\partial g_1}{\partial x} h \right\}$ $+ (C^* B_2)_{[12,21]} \left\{ g_2 \frac{\partial h}{\partial y} + \beta_2 \frac{\partial g_2}{\partial y} h \right\} = 0.$

(iii) If all the cosets appearing in $R_p^{(3)}\sigma_p$ are parallel, the following scheme may be used to reduce Eq. (26) to two fourth order equations, or to a fourth order equation and a quasi-fourth order equation.

First apply a linear transformation to the dependent variables, say, $u = \alpha v$ where $\alpha = \alpha_p \sigma_p$. Multiply Eq. (26) on the left by α^*. Since all the cosets are parallel according to Lemma 1, they can be diagonalized or triangularized by the appropriate choice of a nonsingular α. Equation (26) is reduced to the following

$$L_1^{(4)} v_1 = l_1(v_2). \tag{30a}$$

$$L_2^{(4)} v_2 = 0. \tag{30b}$$

If all the cosets appearing in Eq. (26) are parallel, the decoupling of the equation can be achieved in various ways. For example multiplication of Eq. (26) on the left by any power of a parallel coset, say $(B_1^* A_{12})^n$, yields the equation containing the cosets which are again all parallel (see Lemma 2); hence they can be diagonalized or triangularized by a proper choice of α.

Summing we have

THEOREM I. *A system possesses modes of fourth order or quasi-fourth order provided one of the following conditions is satisfied*: (i) *The cosets* $A_{ij}^* A_{kl}$, $A_{ij}^* B_k$, $A_{ij}^* C$ *appearing in Eq.* (28a), *are diagonal or triangular with zeros in the same off-diagonal entry.* (ii) *The cosets* $A_{ij}^* A_{kl}$, $A_{ij}^* B_k$, $A_{ij}^* C$ *are neither diagonal nor triangular, but all the coefficients associated with the derivatives in* $R_p^{(3)}$ *are such that* $R_p^{(3)}\sigma_p$ *is either a diagonal or triangular operator with zeros at the entry. When Eq.* (1) *contains two independent variables,* 18 *independent conditions listed in Table* 1 *must be satisfied for* $R_p^{(3)}\sigma_p$ *to be a diagonal operator, whereas* 10 *independent conditions must be satisfied for* $R_p^{(3)}\sigma_p$ *to be a triangular operator.* (iii) *The cosets* $A_{ij}^* A_{kl}$, $A_{ij}^* B_k$, $A_{ij}^* C$, $\cdots$ *etc. appearing in Eq.* (28a) *are all parallel.*

In the following section we shall investigate the class of Eq. (1) which may be decomposed into two third order operators, or one third order operator and one quasi-third order operator.

Define the following complementary operator

$$L^* = \sum B_i^* \left(\alpha_i g_i \frac{\partial}{\partial x_i} + \beta_i \frac{\partial}{\partial x_i} g_i \right) + C^* h, \tag{30}$$

where α_i and β_i are complex numbers subject to the following restrictions

$$\alpha_i + \beta_i = 1. \tag{31}$$

Multiplying Eq. (30) on the left by Eq. (1) yields

$$\begin{aligned} L^*Lu = {} & \sum_{i,j,k} B_k^* A_{ij} f_{ij} g_k \frac{\partial^3 u}{\partial x_i \, \partial x_j \, \partial x_k} + \sum_{i,j} C^* A_{ij} f_{ij} h \frac{\partial^2 u}{\partial x_i \, \partial x_j} \\ & + \sum_i B_i^* B_i \left\{ g_i^2 \frac{\partial^2 u}{\partial x_i^2} + (1 + \beta_i) g_i \frac{\partial g_i}{\partial x_i} \frac{\partial u}{\partial x_i} \right\} + \sum_{\substack{i,j \\ i \neq j}} (B_i^* B_j + B_j^* B_i) \\ & \times g_i g_j \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_i (B_i^* C + C^* B_i) g_i h \frac{\partial u}{\partial x_i} + h^2 C C^* u \\ & + \sum_{i,j,k} B_k^* A_{ij} \mathfrak{L}_{k,ij} u + \sum_{i,j} C^* A_{ij} \mathfrak{L}_{0,ij} u + \sum_{i,j} (B_i^* B_j + B_j^* B_i) \mathfrak{L}_{i,j} u \end{aligned}$$

$$+ \sum_{i,j} B_j^* B_i (\mathcal{L}_{j,i} - \mathcal{L}_{i,j}) u + \sum_i (B_i^* C + C^* B_i) \mathcal{L}_{i,0} u$$
$$+ \sum_i C^* B_i (\mathcal{L}_{0,i} - \mathcal{L}_{i,0}) u = 0 \tag{32a}$$

where the residual operator are those given in Eqs. (27a) and (27h).

Equation (32a) may be written as

$$L^* L u = \sum_{i,j,k} B_k^* A_{ij} \left\{ f_{ij} g_k \frac{\partial^3 u}{\partial x_i \, \partial x_j \, \partial x_k} + \mathcal{L}_{k,ij} u \right\}$$
$$+ \sum_{i,j} C^* A_{ij} \left\{ f_{ij} h \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \mathcal{L}_{0,ij} u \right\} + (E_0^{(2)} \sigma_0 + R_\rho^{(1)} \sigma_\rho) u,$$

where $E_0^{(2)}\sigma_0$ is the operator containing the diagonal cosets, and $R_\rho^{(1)}\sigma_\rho$ contains the difference of the residual and complementary differential operators.

In analogy to the previous analysis, the following theorem is established.

THEOREM II. *A system* (1) *possesses modes of third order or quasi-third order when one of the following conditions is satisfied*: (i) *The cosets* $B_k^* A_{ij}$, $C^* A_{ij}$, $B_j^* B_i$, *and* $C^* B_i$ *in Eq.* (32a) *are diagonal or triangular with zeros in the same off-diagonal entry.* (ii) $B_k^* A_{ij}$, *and* $C^* A_{ij}$ *are either diagonal or triangular, and all the coefficients associated with the derivatives in* $R_\rho^{(1)}\sigma_\rho$ *are such that* $R_\rho^{(1)}\sigma_\rho$ *is either diagonal or triangular with zeros at the same entry. For* $R_\rho^{(1)}\sigma_\rho$ *to be diagonal,* 4 *independent conditions, for the case of two independent variable x, y, listed in Table* 2 *must be satisfied. For* $R_\rho^{(1)}\sigma_\rho$ *to be triangular,* 3 *independent conditions must be satisfied.* (iii) *The cosets* $B_k^* A_{ij}$, $C^* A_{ij}$, $B_i^* B_i$, *and* $C^* B_i$ *are all parallel.*

The possibility of obtaining two second order irreducible operators occurs when all the matrices appearing in Eq. (1) are parallel. The reduction to two irreducible operators is done by either diagonalizing or triangularizing through a similarity transformation as was done previously.

THEOREM III. *A system* (1) *possesses fundamental modes of second order, or modes of quasi-second order provided all the matrices are parallel.*

TABLE 2.

Conditions for the Vanishings of the Coefficients Associated with the Derivatives in $R_\rho^{(1)}\sigma_\rho$.

Derivatives	Coefficients
$\frac{\partial}{\partial x}$	$\frac{\partial g_1}{\partial y} g_2 + \beta_2 g_1 \frac{\partial g_2}{\partial y} = 0.$
$\frac{\partial}{\partial y}$	$g_1 \frac{\partial g_2}{\partial x} + \beta_1 \frac{\partial g_1}{\partial x} g_2 = 0.$
$\left(\frac{\partial}{\partial x}\right)^0 \left(\frac{\partial}{\partial y}\right)^0$	$(C^*B_1)_{[12,21]} \left\{ g_1 \frac{\partial h}{\partial x} + \beta_1 \frac{\partial g_1}{\partial x} h \right\} + (C^*B_2)_{[12,21]} \left\{ g_2 \frac{\partial h}{\partial y} + \beta_2 \frac{\partial g_2}{\partial y} h \right\} = 0.$

By linear independence of two solutions u_1 and u_2, we mean, as usual, $c_1u_1 + c_2u_2 = 0$ if, and only if, $c_1 = c_2 = 0$. In general, linearly independent solutions which are modes of second, third and fourth order do not exist if those conditions prescribed in the previous three theorems are not fulfilled. However, the linearly dependent solutions which are modes of second order or third order may exist. The linearly dependent solutions of modes of second order are discussed in Theorem IV.

THEOREM IV. *The necessary condition that the linear system* (1) *with constant* f_{ij}, g_i *and* h, *say unity, possesses linearly dependent solutions of the modes of second order, is that there exists a similarity transformation*

$$(A_{ij}, B_i, C) \rightarrow (A'_{ij}, B'_i, C') = (SA_{ij}, SB_i, SC), \tag{33}$$

such that the following conditions are satisfied:

$$\pi_1(a_{jk})'_1 + i\pi_2(a_{jk})'_2 + (a_{jk})'_3 = 0, \tag{34a}$$

$$\pi_1(b_j)'_1 + i\pi_2(b_j)'_2 + (b_j)'_3 = 0, \tag{34b}$$

$$\pi_1(c_1)'_1 + i\pi_2(c)'_2 + (c)'_3 = 0, \tag{34c}$$

where

$$\pi_1(a)'_1 = \frac{a_{00} - a'_3}{a'_1 + ia'_2} - \frac{a'_1 + ia'_2}{a_{00} - a'_3}, \qquad \pi_2(a)'_2 = \frac{a_{00} - a'_3}{a'_1 + ia'_2} + \frac{a'_1 + ia_2}{a_{00} - a'_3},$$

and a'_l *are the components of* A *and* $a_{00} = \pm (a_k a_k)^{1/2}$.

Proof. Apply the linear transformation $u = \alpha v$ to Eq. (1) and multiply the equation by α^* from the left. This yields:

$$\alpha^* Lu = \sum_{ij} A'_{ij} \frac{\partial^2 v}{\partial x_i \, \partial x_j} + \sum B'_i \frac{\partial v}{\partial x_i} + C'v = 0, \tag{35a}$$

where

$$A_{ij} \rightarrow A'_{ij} = \alpha^* A_{ij} \alpha = SA_{ij}. \tag{35b}$$

If A is a nonsingular matrix, so is A'_{11}. Write $A'_{11} = a'_{01}\sigma_0 + A^{(0)\prime}_{11}$ where $A^{(0)\prime}_{11} = (a_0 - a'_{01})\sigma_0 + A' = a_{00}\sigma_0 + A'$ and a'_{01} is so chosen such that $a_{00} = \pm(a'_k a'_k)^{1/2} = \pm(a_l a_l)^{1/2}$. Thus, A' is a singular matrix.

Let us apply the following transformation:

$$v = A^{(0)\prime *}_{11} w. \tag{36}$$

Substituting Eq. (36) into Eq. (35a), and noticing that $A^{(0)\prime *}_{11}$ is a singular matrix, we have

$$\alpha^* Lu = a_{01} A^{(0)\prime *}_{11} \frac{\partial^2 w}{\partial x_1^2} + \sum_{i,j} A'_{ij} A^{(0)\prime *}_{11} \frac{\partial^2 w}{\partial x_i \, \partial x_j} + \sum_i B_i A^{(0)\prime *}_{11} \frac{\partial w}{\partial x_i} + C A^{(0)\prime *}_{11} w = 0. \tag{37}$$

Since $A^{(0)\prime *}_{11}$ is a singular matrix, according to Remark 4, cosets with respect to $A^{(0)\prime *}_{11}$ modulo A'_{ij}, B'_i, C' can be triangularized with zero at the same entry by a linear transformation of the following type:

$$w = \beta w' \tag{38a}$$

where

$$\beta = \tfrac{1}{2}\{(a_1' - ia_2' + z)\sigma_0 + (a_{00} - a_3' + x)\sigma_1 + i(a_{00} - a_3' - x)\sigma_2 + (a_1' - ia_2' + z)\sigma_3\}, \quad (38b)$$

and where x and z are arbitrary constants.

Substituting Eq. (38a) into Eq. (37) yields

$$a_{01}A_{11}'' \frac{\partial^2 w'}{\partial x_1^2} + \sum_{ij} A_{ij}'' \frac{\partial^2 w'}{\partial x_i\, \partial x_j} + \sum_i B_i'' \frac{\partial w}{\partial x_i} + C''w' = 0, \quad (39a)$$

where

$$A_{11}'' = -\tfrac{1}{2}[(a_1' + ia_2')(\sigma_0 - \sigma_3) - (a_{00} - a_3')(\sigma_1 + i\sigma_2)] \quad (39b)$$

$$A_{jk}'' = \tfrac{1}{2}\{(a_{00} - a_3')[(a_{jk})_1' + i(a_{jk})_2'] - (a_1' + ia_2')[(a_{jk})_0' - (a_{jk})_3']\}(\sigma_0 - \sigma_3) + \tfrac{1}{2}\{(a_{00} - a_3')[(a_{jk})_0' + (a_{jk})_3'] - (a_1' + ia_2')[(a_{jk})_1' - i(a_{jk})_2']\}(\sigma_1 + i\sigma_2). \quad (39c)$$

B_i'', C'' are similar to A_{jk}'' except that the $(a_{ik})_1'$ appearing in Eq. (39a) are replaced by $(b_i)_1'$, $(c_1)'$, respectively.

It is seen from Eqs. (39b)–(39c) that σ_0, σ_3 appear in the form $\sigma_0 - \sigma_3$, while σ_1 and σ_2 appear in the form $\sigma_1 + i\sigma_2$. Thus, all the elements of the first column of all the matrices are zero. Hence, Eq. (39a) degenerates into two equations for w_2', i.e.,

$$L_1''w_2' = 0, \quad (40a)$$

$$L_2''w_2' = 0. \quad (40b)$$

Since w_2' appears in both (40a)–(40b), L_1'' and L_2'' must be the same linear operator modulo any scalar multiple and null-space operator. This requires that the matrices appearing in Eqs. (40a) and (40b) associated with the same derivative, are the same. That is,

$$\pi_1(a_{jk})_1' + i\pi_2(a_{jk})_2' + (a_{jk})_3' = 0. \quad (41a)$$

$$\pi_1(b_j)_1' + i\pi_2(b_j)_2' + (b_j)_3' = 0. \quad (41b)$$

$$\pi_1(c)_1' + i\pi_2(c)_2' + (c)_3' = 0. \quad (41c)$$

Notice that the above expressions are not invariant under similarity transformation. Hence, there may exist a similarity transformation, indicated by Eq. (35b), under which Eqs. (41a) and (41c) are simultaneously satisfied.

Similar theorems can be established for the linearly dependent solution of third order modes.

4. Application. The nondimensional, linearized equations for the propagation of two sound waves in liquid helium [4] can be written in the following matrix form:

$$\begin{pmatrix} 1 & \gamma_1 \\ 0 & 1 \end{pmatrix} \nabla^2\psi - \begin{pmatrix} C_1^{-2} & 0 \\ -\gamma_2 & C_2^{-2} \end{pmatrix} \frac{\partial^2\psi}{\partial t^2} = 0. \quad (42a)$$

Equation (42a) is equivalent to the following equation, obtained by multiplying on the left by

$$\begin{bmatrix} 1 & -\gamma_1 \\ 0 & 1 \end{bmatrix}.$$

$$L\psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \nabla^2 \psi - \begin{bmatrix} C_1^{-2} + \gamma_1\gamma_2 & -\gamma_1 C_2^{-2} \\ -\gamma_2 & C_2^{-2} \end{bmatrix} \frac{\partial^2 \psi}{\partial t^2} = 0, \tag{42b}$$

where

$$\psi = \begin{bmatrix} \rho \\ T \end{bmatrix} \quad C_1 = \left(\frac{\partial \tilde{p}}{\partial \tilde{\rho}}\right)_{\tilde{T}}^{-1/2}, \quad C_2 = \tilde{S}\tilde{\rho}_S^{1/2}\left[\tilde{\rho}\tilde{\rho}_n\left(\frac{\partial \tilde{S}}{\partial \tilde{T}}\right)_{\tilde{\rho}}\right]^{-1/2}$$

$$\gamma_1 = -\frac{T_0}{\rho_0}\left(\frac{\partial \tilde{p}}{\partial \tilde{T}}\right)_{\tilde{\rho}}, \quad \gamma_2 = \tilde{\rho}_n(\tilde{S}\tilde{\rho}_S)^{-1}\left[1 - \frac{\tilde{\rho}}{\tilde{S}}\left(\frac{\partial \tilde{S}}{\partial \tilde{\rho}}\right)_{\tilde{T}}\right]. \tag{42c}$$

$\tilde{\rho}_0$, $\tilde{T}_0$, are the density and temperature of the liquid helium at some reference state. $\tilde{\rho}_s$, $\tilde{\rho}_n$ are the density of the super-fluid and the normal fluid and $\tilde{\rho}$ is the sum of $\tilde{\rho}_s$ and $\tilde{\rho}_n$. $\tilde{S}$ is the entropy of the liquid helium.

At absolute zero temperature, the density wave propagates at a velocity approximately $3^{1/2}$ times that of the velocity of the thermal wave. Both waves remain undistorted during their propagation.

At a finite temperature, the nature of the wave propagation is expected to be quite different from the case of zero temperature. To investigate the behavior of two waves propagating at a finite temperature, we propose to consider the propagation of an initially triangular shaped thermal disturbance, and a rectangular shaped density disturbance introduced between $-a \leq x \leq a$, as is shown in Fig. 1. By the method of elimination one of the dependent variables, say ρ or T in Eq. (42a), may be eliminated. The elimination is mathematically equivalent to multiplying L on the left by the complementary operator L^*.

$$L^*L\psi = \left(\sigma_0\nabla^2 - A^*\frac{\partial^2}{\partial t^2}\right)\left(\sigma_0\nabla^2 - A\frac{\partial^2}{\partial t^2}\right)\psi$$

$$= \sigma_0\nabla^4\psi - (A + A^*)\nabla^2\frac{\partial^2\psi}{\partial t^2} + A^*A\frac{\partial^4\psi}{\partial t^4} = 0, \tag{43}$$

where A is the second matrix appearing in Eq. (42b). Notice that $A + A^*$, and A^*A are matrices proportional to σ_0.

Equation (43) can be decomposed into two second-order operators:

$$\left(\sigma_0\nabla^2 - \lambda_1\sigma_0\frac{\partial^2}{\partial t^2}\right)\left(\sigma_0\nabla^2 - \lambda_2\sigma_0\frac{\partial^2}{\partial t^2}\right)\psi = 0. \tag{44}$$

Here λ_1, λ_2, are inversely proportional to the square of the two sound wave velocities U_1 and U_2 as follows:

$$U_{1,2}^2 = \frac{1}{\lambda_{1,2}}$$

$$= \tfrac{1}{2}\{C_1^2 + C_2^2 + \gamma_1\gamma_2C_1^2C_2^2 \pm ([(C_1 + C_2)^2 + \gamma_1\gamma_2C_1^2C_2^2][(C_1 - C_2)^2 + \gamma_1\gamma_2C_1^2C_2^2])^{1/2}\}. \tag{45}$$

$+$ and $-$ signs correspond to U_1 and U_2, or (λ_1, λ_2), respectively.

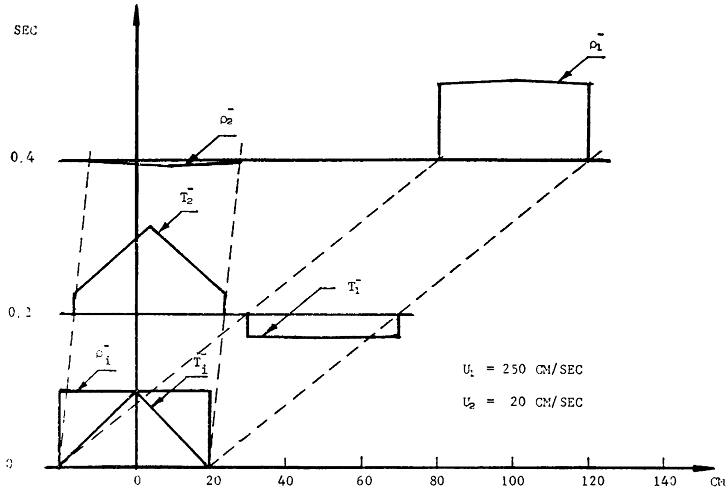

FIG. 1. Two Wave Propagation in Liquid Helium

Equation (44) implies that the density wave is, in fact, two distinct waves propagating at different velocities. Therefore, it is expected that the initially rectangular density disturbance will split into two distinct waves. Unfortunately, the splitting of the rectangular density disturbance into two waves cannot be predicted from Eq. (44). It is essential, therefore, to prescribe two density waves at some later time as initial data, so that the wave patterns at subsequent times can be predicted.

The necessity of prescribing two initial density waves is attributed to the fact that Eq. (44) is of fourth order. An obvious way to avoid this apparently inconvenient situation is to use the method of irreducible operators, as is shown in the following analysis.

To obtain the fundamental modes of second order for Eq. (42b), we diagonalize the second matrix of Eq. (42b). According to Theorem III, ψ is transformed as $\psi = \alpha\varphi$, where $\alpha = \alpha_\rho \sigma_\rho$ and

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

Then multiplying Eq. (42b) on the left by α^{-1}, the two-by-two matrix α is so chosen that $\alpha^{-1}A\alpha$ is diagonalized. The appropriate elements of α are given as follows:

$$\alpha_0 = 1, \qquad \alpha_3 = 0, \tag{46a}$$

$$\alpha_1 = \tfrac{1}{4}(-\gamma_2^{-1} + C_2^2\gamma_1^{-1}) \cdot \{C_1^{-2} - C_2^{-2} + \gamma_1\gamma_2 - [(C_1^{-2} - C_2^{-2} + \gamma_1\gamma_2)^2 + 4\gamma_1\gamma_2C_2^{-2}]^{1/2}\}, \tag{46b}$$

$$\alpha_2 = \frac{i}{4}(-\gamma_2^{-1} - C_2^2\gamma_1^{-1}) \cdot \{C_1^{-2} - C_2^{-2} + \gamma_1\gamma_2 - [(C_1^{-2} - C_2^{-2} + \gamma_1\gamma_2)^2 + 4\gamma_1\gamma_2C_2^{-2}]^{1/2}\}. \tag{46c}$$

The density and temperature waves can be expressed as linear combinations of two fundamental modes φ_1, φ_2 in the following manner:

$$\rho = \varphi_1 + (\alpha_1 - i\alpha_2)\varphi_2, \qquad T = (\alpha_1 + i\alpha_2)\varphi_1 + \varphi_2. \tag{47}$$

Notice that φ_1 is the first density wave, whereas $(\alpha_1 - i\alpha_2)\varphi_2$ is the second density wave; similarly $(\alpha_1 + i\alpha_2)\varphi_1$ is the first thermal wave, and φ_2 is the second thermal wave. φ_1 and φ_2 are governed by the following wave equations:

$$\nabla^2\varphi_1 - \lambda_1 \frac{\partial^2\varphi_1}{\partial t^2} = 0, \tag{48a}$$

$$\nabla^2\varphi_2 - \lambda_2 \frac{\partial^2\varphi_2}{\partial t^2} = 0. \tag{48b}$$

λ_1, λ_2 are those given in Eq. (45). Specializing to the case of one-dimensional wave propagation in the x-direction, we have

$$\varphi_1 = \varphi_1^-(x - U_1 t) + \varphi_1^+(x + U_1 t), \tag{49a}$$

$$\varphi_2 = \varphi_2^-(x - U_2 t) + \varphi_2^+(x + U_2 t). \tag{49b}$$

Noticing that φ_1^-, φ_1^+, φ_2^-, φ_2^+ remain invariant along their characteristics, one can now construct right- and left-running density and thermal waves, if φ_1^-, φ_1^+, φ_2^- and φ_2^+ are prescribed initially.

Let the amplitudes of the initial density and thermal waves be given by $2\rho_i(x)$, and $2T_i(x)$. Since half of the initial density and thermal waves propagate along negative characteristics, $x - U_1 t$, $x - U_2 t$, one can construct right-running waves from the initial value $\varphi_1^-(x, 0)$ and $\varphi_2^-(x, 0)$ computed from Eq. (47):

$$\varphi_1^-(x, 0) = \frac{\rho_i(x) - (\alpha_1 - i\alpha_2)T_i(x)}{1 - \alpha_1^2 - \alpha_2^2}, \tag{50a}$$

$$\varphi_2^-(x, 0) = \frac{T_i(x) - (\alpha_1 + i\alpha_2)\rho_i(x)}{1 - \alpha_1^2 - \alpha_2^2}. \tag{50b}$$

For an initially rectangular shaped density wave and a triangular shaped thermal wave as shown in Fig. 1, two density waves and two thermal waves are calculated at $t = 0.2$ sec. and $t = 0.4$ sec., respectively.

It is seen that the first density wave is compressive, whereas the second is a rarefaction wave.

Finally, it can be shown from Eqs. (46b) and (46c) that both $\alpha_1 - i\alpha_2$ and $\alpha_1 + i\alpha_2$ vanish as the temperature approaches zero, thus agreeing with the already established fact that the density wave propagates at the first speed of sound, and the thermal wave propagates at the second speed of sound.

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References

[1] R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. II, Interscience, New York, 1962
[2] F. H. Clauser, Phys. of Fluids. **6**, 231 (1963)
[3] I. M. Gel'fand, *Lectures on linear algebra*, Interscience, New York, 1963
[4] K. Huang, *Statistical mechanics*, Academic Press, New York, 1962