A VARIATIONAL PRINCIPLE FOR THE FIRST EIGENVALUE OF A SEMIFREE MEMBRANE*

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It is well known that the eigenvalues of

$$-\Delta u + cu = \lambda u \text{ in } G, \qquad u = 0 \text{ on } \partial G, \tag{1}$$

vary inversely with the domain G, in the sense that a decrease in the domain G will increase the eigenvalues of (1). Physically, we may think of (1) as determining the standing waves of an elastic membrane embedded in a medium which exerts a force $-c(x)u$ on each point of the membrane. If the membrane is tied down on ∂G, then a decrease in G will increase the fundamental frequency of the membrane.

If the membrane is not tied down on ∂G but satisfies some other selfadjoint boundary condition then the above results need not carry over. Specifically, consider the case of a free membrane

$$-\Delta u + cu = \lambda u \text{ in } G, \qquad \partial u/\partial\nu = 0 \text{ on } \partial G, \tag{1'}$$

where $\underline{c} = \inf_{x \in G} c(x)$ and $\bar{c} = \sup_{x \in G} c(x)$. Then the first eigenvalue of (1′) satisfies $\underline{c} \leq \lambda_1 \leq \bar{c}$, with equality iff $c(x)$ is constant. If in a certain neighborhood of ∂G $\lambda_1 > c(x)$, then the elastic forces within the membrane dominate the external force $-cu$, and we can expect that removing such a piece of the free membrane will increase the fundamental frequency. If, on the other hand, $\lambda_1 < c(x)$ in a certain neighborhood of ∂G, then the elastic forces within the membrane are dominated by the external force, and we can expect to decrease the fundamental frequency by removing a portion of the free membrane in which $\lambda_1 < c(x)$.

In order to verify these heuristic conclusions (in a slightly more general setting) we shall consider the dependence of the first eigenvalue of

$$\begin{aligned} -\Delta u + cu &= \lambda u \quad \text{in} \quad G, \\ \partial u/\partial\nu &= 0 \quad \text{on} \quad \Gamma_1 \subset \partial G, \\ \partial u/\partial\nu + \sigma u &= 0 \quad \text{on} \quad \Gamma_2 \equiv \partial G - \bar{\Gamma}_1, \qquad -\infty < \sigma(x) \leq +\infty, \end{aligned} \tag{2}$$

with changes in G brought about by small variations in Γ_1. Specifically, we consider the smaller domain $G^* \subset G$ such that $\partial G^* \cap \partial G = \Gamma_2$ and $\partial G^* \cap G = \Gamma_1^*$ and seek a relation between the first eigenvalue of (2) and

$$\begin{aligned} -\Delta u + cu &= \lambda^* u \quad \text{in} \quad G^*, \\ \partial u/\partial\nu &= 0 \quad \text{on} \quad \Gamma_1^*, \\ \partial u/\partial\nu + \sigma u &= 0 \quad \text{on} \quad \Gamma_2 . \end{aligned} \tag{2'}$$

We shall show that if λ_1 and λ_1^* are the first eigenvalues of (2) and (2′) respectively, and if $c(x) - \lambda_1^* < 0$ in $G - \overline{G^*}$, then $\lambda_1 < \lambda_1^*$ (as in the case of Dirichlet boundary conditions). However, if $c(x) - \lambda_1 \geq 0$ in $G - G^*$, then $\lambda_1 > \lambda_1^*$.

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It is assumed throughout that the coefficient c is continuous and that the domains G and G^* are of bounded curvature so that the classical variational theory for eigenvalues [1] is valid and the extremal functions so obtained are solutions of the related Euler-Lagrange equations in the classical sense. The positive normalized eigenfunctions corresponding to λ_1 and λ_1^* will be denoted by $v(x)$ and $v^*(x)$, respectively. The exterior normal derivative is denoted by $\partial/\partial\nu$.

THEOREM 1. *If* $c(x) - \lambda_1^* < 0$ *in* $G - \overline{G^*}$, *then* $\lambda_1 < \lambda_1^*$.

Proof. By means of the usual variational formula we obtain

$$\lambda_1 - \lambda_1^* = \min_{u \in \Phi} \frac{\iint_G [|\nabla u|^2 + (c - \lambda_1^*)u^2]\, dx + \int_{\Gamma_2} \sigma u^2\, ds}{\iint_G u^2\, dx}$$

where the class Φ consists of nontrivial functions which are continuous in G, have piecewise continuous first partial derivatives in G, and vanish whenever $\sigma(x) = +\infty$ (such points being excluded from the boundary integral over Γ_2). We shall extend $v^*(x)$ into G as follows: consider the first order partial differential equation

$$|\nabla u|^2 + \frac{c - \lambda_1^*}{2} u^2 = 0, \qquad u = v^* \quad \text{on} \quad \Gamma_1^*. \tag{3}$$

Since $c - \lambda_1^* < 0$, this equation makes sense. By choosing the proper square roots of the $(\partial u/\partial x_i)^2$, we can assume locally that Γ_1^* is not a characteristic for the differential equation. Finally, for sufficiently small exterior displacements Γ_1 of Γ_1^*, classical theorems [1] guarantee the local existence of a continuously differentiable solution of (3) in $G - G^*$. Denoting such a solution by $\tilde{v}(x)$, we note that

$$\begin{aligned} w(x) &= v^*(x) \quad \text{in} \quad G^* \\ &= \tilde{v}(x) \quad \text{in} \quad G - G^* \end{aligned}$$

belongs to the class Φ, and that

$$\lambda_1 - \lambda_1^* \le \frac{\iint_G [|\nabla w|^2 + (c - \lambda_1^*)w^2]\, dx + \int_{\Gamma_2} \sigma w^2\, ds}{\iint_G w^2\, dx}$$

$$\lambda_1 - \lambda_1^* \le \frac{\iint_{G^*} [\,|\nabla v^*|^2 + (c - \lambda_1^*)v^{*2}]\, dx + \int_{\Gamma_2} \sigma v^{*2}\, ds}{\iint_G w^2\, dx} + \frac{\iint_{G-G^*} \frac{(c - \lambda_1^*)}{2} \tilde{v}^2\, ds}{\iint_G w^2\, dx}. \tag{4}$$

By Green's Theorem and (2′) the first term on the right side of (4) is zero. Since $(c - \lambda_1^*) < 0$ in $G - G^*$, the second term is negative and $\lambda_1 < \lambda_1^*$.

In order to prove the converse relationship we shall make an additional assumption regarding the regularity of (2): for sufficiently small displacements along an interior direction normal to Γ_1, $\partial v/\partial\nu$ is to be a monotone function of distance from Γ_1. With this we can prove

THEOREM 2. *If* $c(x) - \lambda_1 \ge 0$ *in* $G - \overline{G^*}$, *then* $\lambda_1 > \lambda_1^*$.

Proof. Let Γ_0 be the subset of Γ_1 for which $\partial v/\partial \nu$ is a strictly decreasing function of distance from Γ_1 (measured along an interior normal direction). We shall show that Γ_0 is empty. For if x_0 is an isolated point of Γ_0, then the level curve $v = v(x_0)$ must form a cusp with vertex at x_0 and with the interior normal to Γ_1 at x_0 lying inside the cusp. Since $\partial v/\partial \nu \uparrow 0$ at boundary points near x_0, an examination of level curves near $v = v(x_0)$ shows that $v(x)$ has a positive maximum along the level curve $v = v(x_0)$. But since $c(x) - \lambda_1 \geq 0$ near Γ_1, this contradicts the maximum principle and shows that Γ_0 has no isolated points. If x_0 is an interior point of Γ_0 then we can construct a sphere S_0 which lies in $G - \overline{G^*}$, is tangent to Γ_1 at x_0, and is sufficiently small so that $v(x_0) \geq v(x)$ for all $x \in S_0$. But by Hopf's second lemma [2], this implies $\partial v/\partial \nu > 0$ at x_0, which contradicts (2) and shows that Γ_0 is empty.

Therefore $\partial v/\partial \nu \uparrow 0$ as $x \to \Gamma_1$ along an interior normal direction, and for a sufficiently small interior displacement from Γ_1 to Γ_1^*, we have $\partial v/\partial \nu \leq 0$ on Γ_1^*, $\partial v/\partial \nu \not\equiv 0$. Therefore, in G^*, $v(x)$ will satisfy

$$\begin{aligned} -\Delta v + cv &= \lambda_1 v \quad \text{in} \quad G^*, \\ \partial v/\partial \nu + \sigma^* v &= 0 \quad \text{on} \quad \Gamma_1^*, \\ \partial v/\partial \nu + \sigma v &= 0 \quad \text{on} \quad \Gamma_2, \end{aligned} \tag{5}$$

where $\sigma^* \geq 0$ on Γ_1^*, $\sigma^* \not\equiv 0$. By classical variational principles, the first eigenvalue of (5) is larger than the first eigenvalue of (2′). Since $v(x)$ is a nonnegative eigenfunction of (5) corresponding to λ_1, we have $\lambda_1 > \lambda_1^*$.

References

[1] R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. 1, Interscience, New York

[2] E. Hopf, *A remark on linear elliptic differential equations of second order*, Proc. Amer. Math. Soc. **7**, 791-793 (1952)