

INTEGRAL EQUATIONS FORMULATION OF CLASSICAL ELASTICITY*

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1. Introduction. The strong analogy between the potential theory and classical elastostatics is rather well known and has been extensively exploited for various boundary value problems [1]–[9]. In this paper we stress the strong analogy between the Stokes flow equations of hydrodynamics and the equations of classical elasticity—statical as well as dynamical. The general integral formulation of Stokes flow problems was initiated by Lorentz and forms the basis for Oseen’s treatise on low Reynolds number hydrodynamics [10]. This method has been recently extended by Williams [11]. It can be applied rather effectively to the corresponding boundary value problems of elastostatics and elastodynamics.

In this paper we discuss three cases. The first case is that of finding the displacement field as well as tractions on a rigid obstacle of an arbitrary shape when translated within an unbounded elastic medium. There results a simple set of integral equations to solve. Since the problems of this nature have been solved extensively, we assume that these integral equations have a known solution. We next consider the case when there is a rigid boundary in the neighborhood of the obstacle. This problem again reduces to that of solving a set of integral equations similar to the first set but more complicated in nature. It is shown that the approximate solution of these equations can be carried out completely, provided the solution for the corresponding case of the infinite medium is known.

The rest of the paper deals with the problems of dynamic elasticity. We have recently studied the vibrations of axially symmetric bodies in an unbounded elastic medium by the method of matched asymptotic expansions [12]. In the present paper we prove that the integral equation approach can be effectively employed to study the same problem and furthermore to generalize it to the obstacles of arbitrary shapes. The paper ends with the example of a thin rigid circular disk which is attached to the free surface of an elastic half space and excited by a periodic force along a direction which makes an angle α with its axis of symmetry.

The main advantage of the present approach is that it permits a unified treatment of a large variety of boundary value problems. In many cases the first order corrections to the known displacement field in an infinite elastic space may be obtained without detailed calculations or the use of the theory of special functions. Furthermore, it is possible to treat obstacles of arbitrary geometries as effectively as it is to treat a body having the spherical symmetry.

2. Elastostatics. The nondimensionalized Navier-Cauchy equations of elastostatics are

$$(\lambda + \mu) \operatorname{grad} \theta + \mu \nabla^2 \mathbf{u} = 0, \quad \theta = \operatorname{div} \mathbf{u}, \quad (1)$$

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where \mathbf{u} (u_i , $i = 1, 2, 3$) is the displacement vector and λ and μ are Lamé's constants of the medium. The displacement vector \mathbf{u} is considered to be a function of Cartesian coordinates x_i . The above equations have been nondimensionalized by introducing a characteristic geometric length 'a' in the problem. We can derive integral equations for the solution of Eqs. (1) which are analogous to Green's formulas in potential theory. In order to present these formulas we define certain tensor functions which are the appropriate generalizations of the Green's function of potential theory

Let us consider a light rigid body B and let S be its bounding surface. We further assume that the differential equations (1) have a regular solution in the elastic medium surrounding B . We shall designate the domain exterior to B as D . Let $P \equiv x_i(x_1, x_2, x_3)$, be any specified point in D . To obtain the field caused by the uniform translation of B , we calculate the contribution of each elementary point P^0 at P and then add all these contributions. Let us also keep in mind that, due to the mathematical nature of the differential equations (1), the condition of \mathbf{u} at ∞ plays an important role in the solution of these equations. We can now define the above-mentioned tensor function $\mathbf{T}(T_{ij})$ as a function of \mathbf{r} and \mathbf{r}_0 , where \mathbf{r} and \mathbf{r}_0 are the position vectors of the points P and P^0 respectively

$$\mathbf{T} = \frac{1}{8\pi} \left[\mathbf{U} \nabla^2 \phi - \frac{\lambda + \mu}{\lambda + 2\mu} \text{grad grad } \phi \right], \quad (2)$$

or

$$T_{ij} = \frac{1}{8\pi} \left[\delta_{ij} \frac{\partial^2 \phi}{\partial x_k \partial x_k} - \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right],$$

where the function ϕ is a function of $|\mathbf{r} - \mathbf{r}_0|$ only and $\mathbf{U} = \delta_{ij}$ are the components of the Kronecker delta.

Let us now suppose that ϕ satisfies the biharmonic equation

$$\nabla^4 \phi = 0. \quad (3)$$

It then follows by substitution in (1) that

$$(\lambda + \mu) \text{grad } \theta + \mu \nabla^2 \mathbf{T} = 0, \quad (4)$$

or

$$(\lambda + \mu) \frac{\partial \theta_i}{\partial x_i} + \mu \frac{\partial^2 T_{ij}}{\partial x_k \partial x_k} = 0,$$

where

$$\theta_i = \partial T_{ik} / \partial x_k. \quad (5)$$

This means that for each i (1, 2 or 3), the vector whose components are T_{i1} , T_{i2} , T_{i3} and the dilation θ_i constitute solutions of the Eqs. (1). We are thus left with the biharmonic equation (3) to solve.

The quantity ϕ is a function of $(\mathbf{r} - \mathbf{r}_0)$ only. Therefore the biharmonic equation (3) written in spherical coordinates becomes

$$\nabla^2 \nabla^2 \phi = 0, \quad \nabla^2 = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} \right). \quad (6)$$

Since we are looking for such a function \mathbf{T} which tends to zero as $|\mathbf{r} - \mathbf{r}_0|$ tends to infinity

(to be precise \mathbf{T} is $O(1/r)$ at ∞), the solution appropriate to our problem is $\phi = (\mathbf{r} - \mathbf{r}_0)$. This yields

$$\nabla^2 \phi = \frac{2}{|\mathbf{r} - \mathbf{r}_0|}, \quad (7)$$

$$T_{ij} = \frac{1}{8\pi} \left[\frac{\lambda + 3\mu}{\lambda + 2\mu} \frac{\delta_{ij}}{|r - r_0|} + \frac{\lambda + \mu}{\lambda + 2\mu} \frac{(x_i - x_i^0)(x_j - x_j^0)}{|r - r_0|^3} \right], \quad (8)$$

$$\theta_i = -\frac{\mu}{4\pi(\lambda + 2\mu)} \frac{x_i - x_i^0}{|r - r_0|^3}. \quad (9)$$

This tensor field T_{ij} was introduced by Somigliana [2] and has appeared in literature thereafter. For instance, it has been recently used by Diaz and Greenberg [13] in connection with the evaluation of the bounds for the solution of the boundary value problems of elasticity.

It is now possible to write down an expression for $\mathbf{u}(P)$, i.e., the solution of Eqs. (1) at an arbitrary point P in the region D by the method which closely resembles the one used in the potential theory. In fact we surround Π with an infinitesimal sphere of radius $r = \sigma$ such that this sphere lies entirely outside S . The boundary surface now consists of two parts—the surface S of B and the spherical surface S_σ given as $r = \sigma$. Let D_σ denote the volume covered by S_σ . By starting with Green's second identity

$$\int_V (f \nabla^2 g - g \nabla^2 f) dV = \int_{\bar{S}} \left(f \frac{dg}{dn} - g \frac{df}{dn} \right) dS, \quad (10)$$

where $V = D - D_\sigma$, $\bar{S} = S + S_\sigma$ and \mathbf{n} is the unit vector drawn outward and normal to \bar{S} ; setting $f = u_i$, $g = \Psi_{ij}$, it can be shown, in precisely the way it is done in the potential theory, that [2], [14], [16]

$$\mathbf{u}(P) = -\int_S \left[\left\{ \mu \frac{d\mathbf{u}}{dn} + (\lambda + \mu) \theta \mathbf{n} \right\} \cdot \mathbf{T} - \mathbf{u} \cdot \left\{ \mu \frac{d\mathbf{T}}{dn} + (\lambda + \mu) \theta \mathbf{n} \right\} \right] dS. \quad (11)$$

By the help of Eq. (4) and Green's theorem, it follows that if \mathbf{u} is constant on S , then Eq. (11) reduces to

$$\mathbf{u}(P) = -\int_S \mathbf{f} \cdot \mathbf{T} dS, \quad (12)$$

where

$$\mathbf{f} = \mu d\mathbf{u}/dn + (\lambda + \mu) \theta \mathbf{n}. \quad (13)$$

The above formulas may be applied to the problem of the displacement field generated by the light rigid obstacle B which is embedded in an unbounded elastic space and is given a uniform translation d_0 (d_0/a in the nondimensional units). Therefore the boundary conditions are

$$\mathbf{u} = (d_0/a) \mathbf{I}, \text{ on } S, \quad \mathbf{u} \rightarrow \mathbf{0} \text{ as } \mathbf{r} \rightarrow \infty, \quad (14)$$

where \mathbf{I} is the unit direction along the translation of B .

Clearly, the displacement field is given by Eq. (12). Furthermore, it follows from the definition of the traction field t that

$$t_i = \lambda \theta n_i + \mu n_j (u_{,ij} + u_{i,j}), \quad (15)$$

and from the definition of the stress tensor that the total static force \mathbf{F} acting on the body B is given by the formula:

$$\mathbf{F} = \int_s \mathbf{f} dS. \quad (16)$$

Thus the boundary value problem for the displacement field created in D , reduces, by the help of relation (12), to the solution of the integral equation

$$\frac{d_0}{a} \mathbf{I} = - \int \mathbf{f} \cdot \mathbf{T} dS. \quad (17)$$

Since the solutions of boundary value problems of this nature are known for various configurations of B , we shall assume that we know the solution of the integral equation (17). With the help of this information we shall proceed to solve more complicated cases.

3. Boundary effects. In this section we consider the first order effects of boundaries on the tractions experienced by a light rigid body which is given a uniform translation. We now have two surfaces S_1 and S_2 —the surface S_1 of the displaced body B and the surface S_2 of the rigid boundary such as a cylinder enclosing B . In the region enclosed by any given closed surface S_2 we define a tensor \mathbf{T} which satisfies Eq. (1) with the additional condition that $\mathbf{T} = 0$ on S_2 . If both P and P^0 are within S_2 , then $\mathbf{e} \cdot \mathbf{T}$ may be interpreted as the displacement field at P in the region enclosed by the rigid surface S_2 when a concentrated force parallel to the unit direction \mathbf{e} is acting at P^0 . Furthermore, \mathbf{T} may be written as $\mathbf{T}_1 + \mathbf{T}_2$, where \mathbf{T}_1 is the Green's tensor defined in the previous section, satisfies Eq. (1) and has Green's function type singularity at infinity:

$$\mathbf{T}_1 = \frac{1}{8\pi} (\mathbf{U}\nabla^2 |\mathbf{r} - \mathbf{r}_0| - \frac{(\lambda + \mu)}{(\lambda + 2\mu)} \text{grad grad } |\mathbf{r} - \mathbf{r}_0|). \quad (18)$$

The tensor \mathbf{T}_2 also satisfies (1) but is regular in the region under consideration. We assume that \mathbf{T}_2 may be determined explicitly for any particular surface S_2 .

Now, proceeding formally as in the previous section, we obtain the same expression for $\mathbf{u}(P)$ as given by the relation (11) for any point P in the region bounded by S_2 and S_1 , with S_1 completely enclosed within S_2 :

$$\mathbf{u}(P) = - \int_{s_1} \left[\left\{ \mu \frac{d\mathbf{u}}{dn} + (\lambda + \mu)\theta\mathbf{n} \right\} \cdot \mathbf{T} - \mathbf{u} \cdot \left\{ \mu \frac{d\mathbf{T}}{dn} + (\lambda + \mu)\theta\mathbf{n} \right\} \right] dS, \quad (19)$$

where P^0 is a variable point on S_1 and, as before, \mathbf{n} denotes the outward drawn normal to S_1 . The additional condition of \mathbf{u} in the present case is that it vanishes on S_2 . Moreover, when S_2 recedes to infinity, we rederive the result (11) by putting $\mathbf{T} = \mathbf{T}_1$. For the case when \mathbf{u} is constant on S_1 , we get the result similar to (12) with the total \mathbf{T} :

$$\mathbf{u}(P) = - \int_{s_1} \mathbf{f} \cdot \mathbf{T} dS = - \int_{s_1} \left\{ \mu \frac{d\mathbf{u}}{dn} + (\lambda + \mu)\theta\mathbf{n} \right\} \cdot \mathbf{T} dS. \quad (20)$$

The above formula is applicable to the problem of the displacement field generated by the translation of the light rigid particle B which is given a uniform displacement \mathbf{v} in the region bounded by the rigid boundary S_2 . The formula (20) gives the required displacement field, while the formula

$$\mathbf{F} = \int_{S_1} \mathbf{f} \, dS \quad (21)$$

gives the corresponding static force \mathbf{F} acting on B . Since we are going to relate this force with the one experienced by B in an infinite elastic space, let us denote the static force \mathbf{F} which B would experience in an unbounded medium by F_∞ .

A precisely similar situation arises in slow viscous flow problems. There the concept of a resistance tensor is very handy. Analogously, we define a quantity which we shall call the traction tensor. For the case of an infinite elastic space we shall denote it as Φ_∞ ($\Phi_{\infty ij}$). It has the property that the total static force exerted on a body which has been given a uniform translation \mathbf{v} within an infinite elastic space has the value $-(d_0/a) \Phi_\infty \cdot \mathbf{v}$. For the present case, when the elastic space is bounded by a rigid surface S_2 , the corresponding traction tensor is Φ (Φ_{ij}) and the corresponding static force \mathbf{F} is equivalent to $-(d_0/a) \Phi \cdot \mathbf{v}$.

There arises a parameter ϵ in this problem. This is equal to the ratio of a , the characteristic dimension of B , to the minimum distance between a point of S_1 and a point of S_2 . We take this quantity to be small in the present analysis. We further observe that \mathbf{T} and \mathbf{T}_1 are both functions of \mathbf{r} and \mathbf{r}_0 and since $\mathbf{T}_2 = \mathbf{T} - \mathbf{T}_1$, \mathbf{T}_2 is also a function of \mathbf{r} and \mathbf{r}_0 . If the origin O is taken to be some point of S_1 and P and P^0 are also both on S_1 , then it follows that

$$\mathbf{T}_2 = \mathbf{T}_2^0 + \mathbf{r} \cdot [\text{grad } \mathbf{T}_2]_{\mathbf{r}=\mathbf{r}_0} + \mathbf{r}_0 \cdot [\text{grad}^0 T_2]_{\mathbf{r}=\mathbf{r}_0} + O(\epsilon^3), \quad (22)$$

where $\mathbf{T}_2^0 = \mathbf{T}_2(0, 0)$ and the term grad^0 implies the differentiation with respect to the components of \mathbf{r}_0 .

The boundary value problem for the displacement field generated by the translation of B gives rise to an integral equation similar to the one obtained in the previous section. In fact the corresponding formula in the present case is

$$\frac{d_0}{a} \mathbf{I} = - \int_{S_1} \mathbf{f} \cdot (\mathbf{T}_1 + \mathbf{T}_2) \, dS, \quad (23)$$

with P and P^0 both on S_1 . If we neglect the contributions of the second and third terms on the right-hand side of Eq. (22) which are of $O(\epsilon^2)$, i.e., if we replace T_2 by T_2^0 in Eq. (23), it follows that

$$\frac{d_0}{a} \mathbf{I} + \mathbf{F} \cdot \mathbf{T}_2^0 = - \int_{S_1} \mathbf{f} \cdot \mathbf{T}_1 \, dS. \quad (24)$$

This equation is the required integral equation for the displacement vector in the space enclosed between S_2 and S_1 . But this same integral equation can be considered to give the displacement vector in an unbounded medium with B having been given the uniform displacement $(d_0/a) \mathbf{I} + \mathbf{F} \cdot \mathbf{T}_2^0$. Therefore Eqs. (21) and (24) give

$$\mathbf{F} = - \left(\frac{d_0}{a} \mathbf{I} + \mathbf{F} \cdot \mathbf{T}_2^0 \right) \cdot \Phi_\infty, \quad (25)$$

and the solution of Eq. (25) is

$$\mathbf{F} = - \frac{d_0}{a} \mathbf{I} \cdot [\Phi_\infty^{-1} + \mathbf{T}_2^0]^{-1}. \quad (26)$$

Furthermore, it readily follows from Eq. (25) that if \mathbf{F} on the right-hand side is replaced by F_∞ , then the formula becomes

$$\mathbf{F} \cong -\left(\frac{d_0}{a} \mathbf{I} + F_\infty \cdot T_2^0\right) \cdot \Phi_\infty, \quad (27)$$

and the error in Eq. (27) is of order $O(\epsilon^2)$.

We now consider the simplified case when \mathbf{I} is parallel to one of the principal axes of the tractions on S_1 ; these axes being defined so that, in translation through an unbounded medium parallel to one of them, the total static force on B is in the direction of the translation. In this case Eq. (25) takes a simple form. In fact, let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ stand for the unit eigenvectors of Φ_∞ such that

$$\Phi_\infty = \Phi_{\infty 1} \mathbf{e}_1 \mathbf{e}_2 + \Phi_{\infty 2} \mathbf{e}_2 \mathbf{e}_2 + \Phi_{\infty 3} \mathbf{e}_3 \mathbf{e}_3, \quad (28)$$

and decompose the tensor T_2^0 into the components T_{2ij}^0 with respect to these three unit vectors as the basis. Also let $\mathbf{I} = \mathbf{e}_1$. Substituting these expressions in Eq. (25), we derive the formula

$$F/F_\infty = 1/(1 - \lambda F_\infty), \quad (29)$$

where λ is independent of the form of S_1 . Similarly, when B is moving parallel to a principal axis of the traction, it follows from Eq. (27) that

$$F/F_\infty \cong 1 + \lambda F_\infty, \quad (30)$$

where λF_∞ is $O(\epsilon)$; hence, neglecting the term $O(\epsilon^2)$, Eqs. (29) and (30) are identical, as are Eq. (26) and (27).

4. Elastodynamics. We have recently derived formulas [12] which give the dynamical displacement field in an infinite elastic space wherein is embedded a light rigid axially symmetric particle. This particle is excited by a periodic force acting in the direction of its axis of symmetry. In this section we discuss the generalization of that problem for a particle of an arbitrary shape. This particle is depressed by an amount $d\mathbf{I} = d_0 e^{i\omega t} \mathbf{I}$, where \mathbf{I} is a unit vector, into the unbounded medium by an exciting force of the same frequency ω .

In the equations of elastodynamics,

$$(\lambda + \mu) \text{grad } \theta + \mu \nabla^2 \mathbf{u} - \rho \partial^2 \mathbf{u} / \partial t^2 = 0, \quad (31)$$

we set $\mathbf{u} = \mathbf{u}_0 e^{i\omega t}$ and drop the zero subscript, obtaining thereby the equation

$$(\lambda + \mu) \text{grad } \theta + \mu \nabla^2 \mathbf{u} + \rho \omega^2 \mathbf{u} = 0, \quad (32)$$

where ρ is the density of the medium. If we now nondimensionalize this equation by the help of the characteristic length a of the body, it takes the form

$$(\lambda + \mu) \text{grad } \theta + \mu \nabla^2 \mathbf{u} + \rho \omega^2 a^2 \mathbf{u} = 0. \quad (33)$$

In our previous work [12] on axially symmetric bodies, we defined two dimensionless parameters:

$$M^2 = \rho \omega^2 a^2 / (\lambda + 2\mu); \quad m^2 = \rho \omega^2 a^2 / \mu. \quad (34)$$

Furthermore, a material constant τ was defined as

$$\tau = M/m = (\mu)^{1/2} / (\lambda + 2\mu). \quad (35)$$

These quantities are helpful in the present analysis as well.

The next step is to derive an appropriate integral representation for \mathbf{u} . In fact, if we repeat the analysis as given in Sec. 2 for the case of static elasticity with present differential equations, we find that the value of \mathbf{u} at an arbitrary point P in the medium exterior to S is given by the same expression:

$$\mathbf{u} = - \int_S \mathbf{f} \cdot \mathbf{T} - \mathbf{u} \cdot \left(\mu \frac{d\mathbf{T}}{dn} + (\lambda + \mu) \theta \mathbf{n} \right) dS, \tag{36}$$

where \mathbf{f} , as before, is equal to $\mu \, du/dn + (\lambda + \mu) \theta \mathbf{n}$. The tensor \mathbf{T} is now the Green's tensor for Eq. (33) and we assume that it has the expression

$$\mathbf{T} = \mathbf{U}(\nabla^2 + M^2)\phi - \frac{\lambda + \mu}{\lambda + 2\mu} \text{grad grad } \phi, \tag{37}$$

or

$$T_{,ij} = \delta_{,ij}(\nabla^2 + M^2)\phi - \frac{\lambda + \mu}{\lambda + 2\mu} \frac{\partial^2 \phi}{\partial x_i \partial x_j}.$$

The function ϕ satisfies the differential equation

$$(\nabla^2 + m^2) (\nabla^2 + M^2) \phi = 0. \tag{38}$$

An appropriate solution of Eq. (38) is

$$\phi = \frac{1}{4\pi} \frac{1}{(m^2 - M^2)} \left[\frac{\exp(-iM |\mathbf{r} - \mathbf{r}_0|)}{|\mathbf{r} - \mathbf{r}_0|} - \frac{\exp(-im |\mathbf{r} - \mathbf{r}_0|)}{|\mathbf{r} - \mathbf{r}_0|} \right]. \tag{39}$$

Let us note that the parameters M and m are of the same order of magnitude because their ratio τ is just a material constant. When we substitute (39) in (37) and then the resulting value of \mathbf{T} in (36), we observe that \mathbf{u} is a complex quantity. Furthermore, if we expand this value of \mathbf{T} in powers of m , we get

$$\mathbf{T} = \mathbf{T}_0 - \frac{i}{12\pi} (\tau^3 + 2)m\mathbf{U} + O(m^2), \tag{40}$$

where \mathbf{T}_0 , as derived in Sec. 2, is

$$\mathbf{T}_0 = \frac{1}{8\pi} \left\{ \mathbf{U} \nabla^2 |\mathbf{r} - \mathbf{r}_0| - \frac{\lambda + \mu}{\lambda + 2\mu} \text{grad grad } |\mathbf{r} - \mathbf{r}_0| \right\}. \tag{41}$$

The elastodynamic boundary value problems can now be solved in the same fashion as the previous elastostatic problems. Let us first observe that the boundary conditions are $\mathbf{u} = (d_0/a) \mathbf{I}$ on S and \mathbf{u} satisfies the radiation condition at ∞ . The integral equation governing the displacement field is obtained by taking P in Eq. (36) to be on S and setting $\mathbf{u} = (d_0/a) \mathbf{I}$ on S . As \mathbf{u} is constant on S , it may be taken outside the integral in Eq. (36). It now follows from the relation

$$-m^2 \mathbf{T} = (\lambda + \mu) \text{grad } \theta + \mu \nabla^2 \mathbf{T} \tag{42}$$

and Green's theorems that

$$\int_S \left(\mu \frac{d\mathbf{T}}{dn} + (\lambda + \mu) \theta \mathbf{n} \right) dS = -m^2 \int_V \mathbf{T} dV, \tag{43}$$

where V denotes the interior of B . Recalling the formula for the total force \mathbf{F} ,

$$\mathbf{F} = \int_S \mathbf{f} dS, \tag{44}$$

we readily obtain from the relations (36), (40), and (43) the result

$$\frac{d_0}{a} \mathbf{I} - \frac{i}{12\pi} (\tau^3 + 2)m\mathbf{F} = - \int_s \mathbf{T}_0 \cdot \mathbf{f} dS + O(m^2). \quad (45)$$

When $m = 0$, the above equation reduces to Eq. (17).

By following the same steps as we did in deriving relation (27), we get the corresponding force formula as

$$\mathbf{F} = \Phi_\infty \cdot \left\{ \frac{d_0}{a} \mathbf{I} - \frac{i}{12\pi} (\tau^3 + 2)m \left(-\frac{d_0}{a} \Phi_\infty \cdot \mathbf{I} \right) \right\} + O(m^2). \quad (46)$$

For a body displaced parallel to one of its principal axes of traction which, as defined before, are the directions such that $-(d_0/a) \Phi_\infty \cdot \mathbf{u} = \mathbf{F}_\infty = -F_\infty \mathbf{u}$, Eq. (46) takes the simple form

$$\mathbf{F} = -F_\infty \{ 1 + (a/d_0) (F_\infty i / 12\pi) (\tau^3 + 2)m \} \mathbf{I} + O(m^2). \quad (47)$$

If we set $F_\infty = X_0/\mu a^2$, where X_0 is given in the physical units, the formula (47) reduces to the one derived for the axially symmetric bodies by us [12]. In our previous work we had mistakenly¹ assumed that the result (47) is applicable also to nonaxially symmetric configurations.

As an application of the general formula (47), we consider the case of a thin rigid circular disk of radius a attached to the free surface of an elastic half space and excited by a periodic force of frequency ω acting at angle α to its axis of symmetry. Let \mathbf{e}_1 be a unit vector in the plane of the disk and let \mathbf{e}_2 be unit vector along the axis of symmetry. The force formula for a translation of amount d_0 of the disk along the directions \mathbf{e}_2 is known and when written in physical units is [15]

$$X_0 = 8\pi\mu d_0(1 - \tau^2) a. \quad (48)$$

Similarly, the static force for a uniform translation of amount d_0 along the direction \mathbf{e}_1 is

$$X_0 = 16\pi\mu d_0(\tau^2 + 3)^{-1} a. \quad (49)$$

When these formulas are substituted in Eq. (46), we derive the required force in physical units:

$$\begin{aligned} \mathbf{X} = & -\pi\mu a d_0 e^{i\omega t} [8(1 - \tau^2) \cos \alpha \mathbf{e}_2 \{ 1 + (2i/3\pi)(1 - \tau^2)(\tau^3 + 2)m \} \\ & + 16(\tau^2 + 3)^{-1} \sin \alpha \mathbf{e}_1 \{ 1 + (4i/3\pi)(\tau^2 + 3)^{-1}(\tau^3 + 2)m \}]. \quad (50) \end{aligned}$$

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¹Note added in proof: The author's attention has been drawn to the paper by W. E. Williams, *Quart. J. Mech. Appl. Math.* **19**, 414 (1966), wherein this error in his previous work is pointed out.

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