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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\surd$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos(\pi x/2b)}{\cos(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u \, du \text{ is preferable to } \int \frac{\sin u}{u} \, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t \, (a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372 (1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

CONTENTS

BOOK REVIEW SECTION

Foundations of optimal control theory. By E. B. Lee and L. Markus. John Wiley and Sons, Inc., New York—London—Sydney, 1967. x + 567 pp. $17.95.

This book is a very good exposition of the subject. The authors, well known for their own research, are, of course, in an excellent position to give an account of the latest developments in this field which has grown so fast in the last decade. In spite of this, the book is not exclusively for the researcher or even the higher mathematician (but for these, too, it is of much interest). The basic knowledge assumed is on the level of good courses in advanced calculus and differential equations. Other mathematical tools which are used in the book, like convex sets, functional analysis and so, are explained sufficiently well in additional appendices. On the other hand, this book is not overloaded with such complementary material. This way, the reader with less mathematical background is easily introduced into the main language generally used in research papers on control theory.

The first chapter gives examples of optimal control problems and, in appendices, a short account of the geometric theory of differential equations and the algebraic (matrix) theory of linear systems. Chapter 2 treats linear control systems, with an appendix on convex sets. In chapter 3, optimal control problems with integral convex cost criteria are discussed. Chapters 4 and 5 bring the maximum principle and related material. Chapter 6 is concerned with controllability, observability and stability in general (non-linear) systems, these properties having been already discussed in chapter 2 for linear systems. Chapter 7 is about the problem of synthesis of optimal controllers. An appendix about steepest descent and computational methods takes care of the interest of numerical analysts, and another appendix about bibliography on optimal control processes governed by ordinary and partial functional differential equations opens the door to these much harder problems. The bibliographical reference list of over 700 original papers is a good proof of the interest which this field excited among researchers in mathematics and theoretical engineering.

EMILIO ROXIN (*Kingston, R. I.*)

Maximum principles in differential equations. By Murray H. Protter and Hans F. Weinberger. Prentice-Hall Inc., Englewood Cliffs, N. J., 1967. x + 261 pp. $8.00.

This book is an excellent addition to the literature on differential equations. The authors begin at an elementary level with an illuminating chapter on maximum principles for ordinary differential equations. They then proceed, in succeeding chapters, to treat maximum principles for elliptic, parabolic, and hyperbolic second order partial differential equations.

The use of maximum principles in finding numerical approximations is stressed along with the qualitative use of these principles. Physical applications are included and there is some material on nonlinear operators. There are many exercises, and a number of remarks which assist the reader in thoroughly analysing the material are scattered throughout the text. Boldface type is used for statements of theorems and both boldface and italics are used to highlight definitions written into the text material. Several well-drawn figures add to the presentation.

There is new material in the book, each chapter ends with a section of biographical notes, and there is a rather extensive bibliography which includes references to quite recent work. This makes the book particularly useful in acquainting the mathematics student with the role of maximum principles both in more or less classical contexts and in contemporary research. However, completion of an advanced calculus course is more than adequate preparation for this text. In fact, Green's theorem and a knowledge of basic properties of continuous and differentiable functions is sufficient for an understanding of most of the material. Thus this book makes a very useful tool accessible to a wide audience of scientists and engineers.

W. M. GREENLEE (*Evanston, Ill.*)

A handbook of numerical matrix inversion and solution techniques. By Joan R. Westlake. John Wiley & Sons, Inc., New York, London, Sydney, 1968. viii + 171 pp. $10.95.

In a field that is so fundamental as the numerical inversion of matrices and the solution of linear equations—a field containing thousands of ideas and extending back several thousands of years—it is essential to make available summaries of the best thinking and the best experience. Joan R. Westlake's handbook presents brief but clear descriptions of the principle methods in current use together with discussion of the relative advantages and disadvantages of these methods. It should therefore be useful and appreciated by programmers, researchers, and teachers of numerical analysis and linear algebra.

Contents: direct methods, interative methods, ill-conditioning and measures of condition, error measures, scaling, work required—number of operations, comments and comparisons. Appendices contain a glossary of matrix terminology, theorems on matrix algebra, and a tabulation of test matrices. There is a bibliography of 126 items.

No computer programs have been included.

PHILIP J. DAVIS (*Providence, R. I.*)

Graphs, dynamic programming, and finite games. By A. Kaufmann. Translated from the French by H. Sneyd. Volume 36 of the series *Mathematics in science and engineering.* Academic Press, New York and London, 1967. xvii + 484 pp. $14.50.

This volume, a translation of the book "Methodes et Modeles de la Récherche Operationnelle, Tome II" published in French in 1964, is an introduction to the three topics named in its English title. The book has the strengths and weaknesses one might expect of the product of as prolific an author as Professor Kaufmann. The exposition, aims and organization are excellent. However, the technical content is uneven, largely borrowed, and occasionally wrong. It is my opinion that the material seems hastily assembled and does not show the care demanded by a good expository or introductory book.

More specifically, the pedagogical philosophy is to introduce each of the three topics by means of a chapter emphasizing interesting problems, completely worked examples, and discussion of practical application. There follow three "reader's-digest-type" chapters largely based, respectively, on books by Berge, Howard, and MacKinsey containing more advanced mathematical developments. While the first three chapters contain valuable introductory material, it is doubtful whether the latter, and rather dated, chapters will please either the dilettante or specialist.

The advanced chapter on graphs defines more than a hundred properties of graphs and gives illustrative examples. A few problems are stated and algorithms given. The procedure given for the shortest path problem is an extremely inefficient one. The bulk of the chapter on dynamic programming treats in detail the infinite duration, stationary, Markov decision process model and algorithms of R. Howard. The chapter on game theory emphasizes the fundamental minimax theorems and methods of computing solutions of zero-sum two-person matrix games.

I would gladly recommend the book merely on the basis of the first three chapters were I convinced that they are as accurate as they are varied and interesting. However, a careful reading of Chapter II on dynamic proprammming, my area of specialization and also the subject of another book of Professor Kaufmann, revealed too many errors. Most serious, the discussion of Bayesian adaptive control incorrectly applies Bayes' Law (by computing an alleged updated *value*, rather than density, of an unknown quantity p) and also misrepresents the philosophy and realization of optimal Bayesian adaptive behavior. (Interestingly, sections of Chapters III and VI on statistical decison theory attributed to G. d'Herbemont correctly describe Bayesian adaptation in the context of a game against nature.) Further, it is erroneously asserted elsewhere in the chapter that if the current decision in an n-stage decison process is chosen as optimal with respect to an m-stage horizon ($m < n$), the larger the value of m, the better the solution. While this sounds plausible, counter-examples are, unfortunately, well known and easily constructed. Several other inaccuracies were noted, disqualifying, I feel, the chapter for teaching or self-study.

In summary, I applaud the completely worked and well-documented examples, the varied problems and applications, and lively expository style of the first three chapters of this book. However, there are too many inaccuracies, at least in Chapter II, and the last half of the book is too uneven and dated, to merit recommendation.

S. E. DREYFUS (*Berkeley, Calif.*)

Stability of motion. By Wolfgang Hahn. Translated from the German by Arne P. Baartz. Volume 138 of *Grundlehren der mathematischen wissenschaften.* Springer-Verlag New York, Inc., New York, 1967. xii + 446 pp. $18.00.

This is an excellent book on stability theory. The suthor has managed to include the relevant aspects of the current topics of interest in this subject and at the same time not to become excessively overburdened with generality. Some of the topics covered are special procedures for determining the stability of autonomous linear systems, the direct method of Liapunov including Popov's criterion and other recent extensions, the converse theorems of Liapunov, the general theory of linear time-varying systems and the general Liapunov expansion theorem near an equilibrium point. There is also a long chapter on periodic and almost periodic solutions of differential equations with the modern perturbation techniques discussed. Although most of the book is devoted to ordinary differential equations, some attention is devoted to difference equations, differential-difference equations and partial differential equations. This book should be useful to persons interested in either the practical or theoretical aspects of stability theory.

JACK K. HALE (*Providence, R. I.*)

Engineering plasticity. Edited by J. Heyman and F. A. Leckie. Cambridge University Press, Cambridge, 1968. 706 pp. $22.50.

These are the proceedings of a conference held in Cambridge, England in March 1968 to honor the contributions of Professor Sir John Baker, Prefessor of Mechanical Sciences at the University of Cambridge and Fellow of Clare College.

The volume contains the following papers:

The effect of change of geometry on the limit pressure of a flush nozzle in a spherical pressure vessel, by D. F. Allman & S. S. Gill.
On the limit analysis of I-beams with warping restraints, by G. Augusti.
Ductility as a basis for steel design, by Lynn S. Beedle.
Deformation of rate-sensitive structures under impulsive loading, by S. R. Bodner.
Simple ideas in the large-deflection plastic theory of plates and slabs, by C. R. Calladine.
On the use of simple discontinuous fields to bound limit loads, by D. C. Drucker and W. F. Chen.
The existence of exact solutions in limit analysis for homogeneous isotropic plates of rigid perfectly-plastic material, by E. N. Fox.
Dynamic buckling of rectangular plates in sustained plastic compressive flow, by F. N. Goodier.
A more rational approach to strain-hardening data, by R. M. Haythornthwaite.
Bending moment distributions in collapsing frames, by Jacques Heyman.
Numerical applications of minimum principles in plasticity, by Philip G. Hodge, Jr.
Creep buckling of rectangular plates under uniaxial compression, by Nicholas F. Hoff.
The method of charateristics and its application to the theory of stress waves in solids, by H. G. Hopkins.
Failure loads of biaxially loaded I-section columns restrained about the minor axis, by M. R. Horne.
An experimental determination of the contact stresses between plastically deformed cylinders and spheres, by K. L. Johnson.
High-speed blanking of steel, by W. Johnson and F. W. Travis.
Plastic instability of a spherical shell, by F. A. Leckie and R. K. Penny.
Analysis of plastic spherical shell, by L. C. Lee and E. T. Onat.
General theory of elasto-plastic membrane-plates, by Ch. Massonnet.
Limit load of a cantilever in plane stress, by B. G. Neal.
The inelastic behaviour of aluminium alloy tension members when subjected to heating on one face, by E. W. Parkes.
Bounds on bending moments caused in indeterminate beams by bounded inelastic curvatures, by William Prager.
On the generalized stress-strain behaviour of 'wet' clay, by K. H. Roscoe and J. B. Burland.
Some aspects of minimum-weight design, by Marcel A. Save.
Plastic design and trussed frames, by L. K. Stevens.
Plastic shear deformations in dynamic load problems, by P. S. Symonds.
Some controversial and curious developments in the plastic theory of structures, by R. H. Wood.

Computer solution of linear algebraic systems. By George E. Forsythe and Cleve B. Moler. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1967. xi + 148 pp. $6.75.

This book, which is one of the Prentice-Hall series on automatic computation, resulted from "teching a senior graduate course in numerical analysis at Stanford University from a textbook which devoted almost no attention to matrix problems. The [textbook's] author advised the reader to expect suitable programs at the computation center .....".

The book is dedicated to J. H. Wilkinson. A "major portion [of it] is devoted to explaining and motivating some of the algorithms and analyses put forward by Wilkinson", as "it is not easy for the less advanced student to find his way through the very substantial material in [Wilkinson's] books". A beginning course in linear algebra is presupposed, as is an acquaintance with computer programming FORTRAN, ALGOL 60 or PL/I), and "the book is intended to bring a reader with such a background as quickly as possible to an understanding of a few good programs for solving linear equation systems and of the fundamental concepts of error involved".

The book, the production of which is well up to Prentice-Hall's usual high standard, consists of 25 short chapters (13 are four pages or less in length), an appendix (which discusses two of the exercises), a bibliography and author index, and a subject index. The chapters cover a discussion of norms and diagonalization (developed in terms of the transformation U^TAV, U and V being orthogonal); types and sources of problems in linear algebra and type of matrices encountered; conditioning; Gaussian elimination with Crout and Doolittle variants and improvement techniques; scaling (leaning heavily on F. L. Bauer); nearly singular matrices; various computer programs; matrix inversion; error analysis (following Wilkinson); interative processes; band and positive definite matrices; and (in five pages) nonlinear systems.

The treatment is crisp and sometimes disconcertingly elliptic. Three examples of this ellipsis which are likely to leave at least the reader who has only a first course in linear algebra with unanswered queries are the following. (The italics are mine. The context adds nothing to the sense of these examples, except for a reference in one case.)

(1)For a square matrix A and suitable permutation matrix P, a lower unit triangular matrix L and an upper triangular matrix L, $LU = PA$. But "L and U are *not always* uniquely determined by P and A" (p. 36);

(2)there are sometimes advantages in chopped arithmetic "*because* the sign of the chopping error is known" (p. 91); and

(3)"..... the use of normal equations is not the most efficient or accurate method of solving least-squares problems" (p. 17).

The use of "it can be shown that ....", or equivalent, occurs a number of times without a suitable reference, and references to books are usually (though not always) unaccompanied by chapter or page details. Occasionally statements which have been fully discussed elsewhere are "pulled out of a hat" with no reference, so that they seem to be more empirical in character than is in fact the case (e.g., the discussion of partial pivoting on p. 35).

The choice of topics not treated is interesting. Discussion of computational techniques does not extend to an eigenvalue problem. Linear programming is omitted on the grounds that it "requires quite different attack". (Does it really?) The role of the right-hand side (which, in near-singular cases, may save the day by having neligible components in "bad" directions) is not discussed. Saving space by storing only half of a symmetric matrix is deprecated perhaps a little hastily on the grounds of programming complication and consequent loss of time (but the cost with large matrices of additional transfers from a backing store is not considered). Perhaps this is why the inversion of symmetric $n \times n$ matrices within $n(n + 1)/2$ locations (e.g., by Efroymson's method) is not mentioned.

I would hesitate to try some of the exercises on my own students without giving them a hint or two. For example, with next to no preamble, proving that pivoting is unnecessary for irreducible dominant matrices is something of a poser.

To sum up, I found the book stimulating and would consider using it to accompany a lecture course of the type envisaged by the authors (but not as a "guide to Wilkinson"; in fact, on a number of points the reverse is true). However, unless the reader has a wider background than that proposed, he will not find it an ideal book for home study.

John M. Bennett (*Sydney*)