ON BEST QUADRATURE OF ANALYTIC FUNCTIONS*

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1. Introduction. We consider the error $R(f)$ of the interpolatory quadrature formula

$$R(f) = \int_{-1}^{1} f(x_0 + x)\,dx - 2 \sum_{j=1}^{N} d_j f(x_0 + c_j) \tag{1}$$

for $-1 \le c_j \le 1$ and $d_j \ne 0$, $j = 1, 2, \cdots, N$. We assume that the function f is analytic in a region G containing the circle $|z - x_0| = r$, $r > 1$. It is well known (Davis [1953]) that this condition is sufficient for an error estimate of the form

$$|R(f)|^2 \le \left\{\frac{1}{2\pi} \sum_{n=0}^{\infty} r^{-2n} |R(x^n)|^2\right\}\left\{\int_0^{2\pi} |f(x_0 + re^{i\phi})|^2 \,d\phi\right\} = \sigma^2 \,||f||^2. \tag{2}$$

Putting $t(n) = (1/2)(1 + (-1)^n)$ and $A = (1/r)(1 < r \Rightarrow A < 1)$, we find for σ^2 the expression

$$\sigma^2 = \frac{2}{\pi} \sum_{n=0}^{\infty} A^{2n}\left(\frac{t(n)}{n+1} - \sum_{j=1}^{N} d_j c_j^n\right)^2. \tag{3}$$

Obviously σ^2 increases with A and depends on the variable A and the quadrature formula only.

The purpose of this paper is to give conditions which are necessary for minimizing σ^2 as a function of c_i and d_i. This has been previously treated by Urabe [1955] for equally spaced c_i, by Wilf [1964], and others. It will be shown that for $N = 1$ the midpoint rule minimizes (3). Furthermore, we will establish the relation $\sigma = O(N^{-2})$ for $N \to \infty$; this result will also be derived for functions of the class $L^2(E_\rho)$.

2. Minimum conditions. In this part we consider the question how the values of $c_1, \cdots, c_N, d_1, \cdots, d_N$ are to be determined in order to minimize σ^2 as a function of these variables. Let the minimum with respect to N be denoted by

$$\sigma_N^2 = \min_{c_j, d_j} \sigma^2(c_1, \cdots, c_N, d_1, \cdots, d_N). \tag{4}$$

The conditions necessary for a minimum

$$\partial\sigma^2/\partial d_k = 0, \qquad \partial\sigma^2/\partial c_k = 0 \quad (k = 1, 2, \cdots, N)$$

are of the form (dropping a constant factor)

$$\begin{aligned} &\sum_{n=0}^{\infty} c_k^n A^{2n}\left(\frac{t(n)}{n+1} - \sum_{j=1}^{N} d_j c_j^n\right) = 0, \\ &\sum_{n=0}^{\infty} n c_k^{n-1} A^{2n}\left(\frac{t(n)}{n+1} - \sum_{j=1}^{N} d_j c_j^n\right) = 0, \end{aligned} \tag{5}$$

$k = 1, \cdots, N$. These $2N$ simultaneous equations do not seem to have a solution in the form of a simple analytic expression. However, because of

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$$R(x^n) = 2\left(\frac{t(n)}{n+1} - \sum_{i=1}^{N} d_i c_i^n\right)$$

the conditions (5) are equivalent to

$$R\left(\frac{1}{(1 - A^2 c_k x)^l}\right) = 0, \tag{6}$$

$l = 1, 2$ and $k = 1, 2, \cdots, N$. In other words: the c_k's and d_k's are to be determined such that the integrals of the $2N$ functions $(1 - A^2 c_k x)^{-l}$ are given exactly by the quadrature rule.

3. Case $N = 1$. The case $N = 1$ allows the exact analytic solution of (6).

THEOREM.

$$\sigma_1^2 = \min_{c_1, d_1} \sigma^2(c_1, d_1) = \sigma^2(0, 1). \tag{7}$$

Proof. Obviously the choice $c_1 = 0$ and $d_1 = 1$ satisfies the equations (6) which represent necessary conditions for the c_k's and d_k's to yield the minimal error in the quadrature formula defined by (1). Moreover, it can be shown easily that these conditions are also sufficient in the case $N = 1$. Next we prove that these values of c_1 and d_1 are unique. It follows from (6) that

$$\begin{aligned} \frac{1}{2A^2 c_1} \log \frac{1 + A^2 c_1}{1 - A^2 c_1} &= \frac{d_1}{1 - A^2 c_1^2}, \\ \frac{1}{1 - A^4 c_1^2} &= \frac{d_1}{(1 - A^2 c_1^2)^2}. \end{aligned} \tag{8}$$

Because of the symmetry property $\sigma(c_1, d_1) = \sigma(-c_1, d_1)$ we restrict our investigations of c_1 to the interval (0, 1]. No solution of (8) exists for this interval. Also $c_1 = 0$ (and hence $d_1 = 1$) is the unique solution for $N = 1$, i.e., the midpoint rule.

REMARK. The result does not depend on A. It follows directly from (3) that

$$\sigma_1 = \left(\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{A^{4n}}{(2n+1)^2}\right)^{1/2} \leqq \left(\frac{\pi}{4} - \frac{2}{\pi}\right)^{1/2} A^2 < 0.3858\, A^2.$$

4. Magnitude of σ_N. For arbitrary N the equation system (6) has a solution dependent on A (cf. Valentin [1965]). The study of the magnitude of σ_N as a function of N leads to the following

THEOREM.

$$\sigma_N = O(N^{-2}). \tag{9}$$

Proof. We divide the interval $[-1, 1]$ in k subintervals of the equal length $2h$ ($kh = 1$), and integrate over each subinterval using the midpoint rule. Since the repeated midpoint rule applied to a function f has the general error expression $R(f) = (h^2/3)f''(\xi)$, $-1 \leq \xi \leq 1$, we obtain for $f = x^n$, $n \geq 2$,

$$|R(x^n)| \leq \tfrac{1}{3} n(n-1) k^{-2}.$$

Therefore, it follows that

$$\sigma^2 = \frac{2}{\pi} \sum_{n=2}^{\infty} |R(x^n)|^2 A^{2n}$$

$$\leq \frac{2}{9\pi} k^{-4} \sum_{n=2}^{\infty} \{n(n-1)A_*^n\}^2 = O(k^{-4}).$$

It is clear that σ_N is not greater than the value of σ we obtain when dividing the interval into N equal parts and integrating as above. Thus

$$\sigma_N = O(N^{-2}).$$

5. Ellipse E_ρ. If f belongs to the Hilbert space $L^2(E_\rho)$, i.e., f is analytic inside the ellipse E_ρ, $\rho > 1$, and $\int_{E_\rho} \int |f(z)|^2 \, dx \, dy$ exists, where E_ρ has the foci ± 1 and the axes $a = \frac{1}{2}(\rho + \rho^{-1})$ and $b = \frac{1}{2}(\rho - \rho^{-1})$, the error bound corresponding to (2) is (e.g., Davis [1963])

$$\text{(10)} \quad \begin{aligned} |R(f)|^2 &= \frac{4}{\pi} \sum_{n=0}^{\infty} (n+1) \frac{|R(U_n)|^2}{\rho^{2n+2} - \rho^{-2n-2}} \int_{E_\rho} \int |f(z)|^2 \, dx \, dy \\ &= \sigma_E^2 \, \|f\|_E^2 , \end{aligned}$$

where $U_n(z) = (1 - z^2)^{-1/2} \sin [(n+1) \arccos z]$ is the Čebyšev polynomial of the second kind. By an inequality due to Markoff

$$|U_n''(x)| \leq (n+1)n^2(n-1)^2, \qquad -1 \leq x \leq 1$$

the repeated midpoint rule (notation as above) yields

$$\sigma^2 \leq \frac{4}{9\pi} \sum_{n=2}^{\infty} \frac{(n+1)^3 n^4 (n-1)^4}{\rho^{2n+2} - \rho^{-2n-2}} k^{-4} = O(k^{-4}).$$

Thus, the result

$$\text{(11)} \qquad \sigma_E = O(N^{-2}), \qquad N \to \infty,$$

holds for the optimal choice in this case too.

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