CLOSE-IN ORBITS IN THE RESTRICTED PROBLEM OF THREE BODIES*

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Abstract. Utilizing a transformation due to Birkhoff [3], we establish for the restricted problem of three bodies the existence of conditionally periodic orbits that move in a small neighborhood of one of the primaries. These orbits result from perturbation of elliptic orbits and are valid for all mass ratios of the two primaries.

Introduction. In a recent paper, R. Arenstorf [1] has shown the existence of a new class of periodic obrits in the restricted problem of three bodies. These orbits are valid for all mass ratios of the two primaries; they move in a small neighborhood of one of the primaries and result from the perturbation of elliptic orbits. In the present paper, we prove the existence of a very similar class of orbits, but instead of being periodic orbits our orbits are conditionally periodic, i.e., the two fundamental frequencies λ_1, λ_2 are not commensurable. In an earlier paper Conley [4] asserted the existence of a class of similar orbits, but did not publish a proof.

The method of proof of the present paper is very simple. We use a similarity transformation due to Birkhoff [3] to transform the differential equation into a form so that we may apply a general theorem on the existence of conditionally periodic orbits. We state the general theorem in a form proved by Barrar [2]. Other forms of the theorem are due to Kolmogorov, Arnold, and Moser.

Statement of problem. If in the restricted planar problem of three bodies we choose P_1 and P_2 to be the two bodies of finite masses, m_1 and m_2 respectively, normalized so that $m_1 + m_2 = 1$ and choose uniformly rotating coordinates (x, y) centered at P_1 (so that P_2 is always at the point $(1, 0)$), then the Hamiltonian determining the equation of motion of the third particle P in this rotating coordinate system is (see Wintner [7, pp. 7, 8]):

$$H = \frac{m_1}{r} - \frac{v^2}{2} + \frac{m_2}{(1 - 2r\cos\gamma + r^2)^{1/2}} - m_2 r\cos\gamma, \tag{1}$$

where $r = (x^2 + y^2)^{1/2}$, γ is the angle between the x axis and the particle P, and v is the velocity.

If we now solve the two-body problem resulting from the first two terms in the above Hamiltonian in the standard way by using the canonical variables

$$\begin{aligned} L &= (m_1 a)^{1/2} & l &= n(t - \beta) \\ G &= (m_1 a(1 - e^2))^{1/2} & &\omega \end{aligned}$$

with a = semi-major axis, e = eccentricity, ω = pericenter, $m_1 = n^2a^3$, β the time of pericenter passage, the Hamiltonian becomes

$$H = \frac{(m_1)^2}{2L^2} + \frac{m_2}{(1 - 2r\cos\gamma + r^2)^{1/2}} - m_2 r\cos\gamma. \tag{2}$$

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Finally, since ω and t only occur in the combination $\omega - t$ in H, if we introduce the variable $g = \omega - t$, t will no longer appear explicitly, and the system will still be canonical with Hamiltonian (see Wintner [8, p. 8]):

$$H = \frac{(m_1)^2}{2L^2} + G + m_2R^*(L, G, l, g),$$
$$R^* = \frac{1}{(1 - 2\bar{r}\cos\gamma + r^2)^{1/2}} - r\cos\gamma. \tag{3}$$

G. Birkhoff [3, §12] introduced a transformation $x = \epsilon^2x'$, $t = \epsilon^3T$ that was very useful for studying orbits with a large Jacobi constant. This transformation is also helpful in the present problem. Using Delaunay variables this transformation is realized by

$$L' = L/\epsilon, \qquad G' = G/\epsilon, \qquad H' = \epsilon^2H, \qquad T = t/\epsilon^3,$$
$$l' = l, \qquad g' = g, \tag{4}$$

where the primed quantities represent the new variables. It can be directly verified, and also results from the formula

$$(1/\epsilon)[Ldl + Gdg - Hdt] = L'dl' + G'gd' - H'dT,$$

that the new system is canonical.

We now note that

$$H' = \epsilon^2H = (m_1)^2/2(L')^2 + \epsilon^3G' + m_2\epsilon^2R^*. \tag{5}$$

The aim of the present paper is to prove the existence of conditionally periodic solutions of (5) for arbitrary but fixed m_1, m_2, and for sufficiently small ϵ. From the way the transformation (4) arose, it follows that the resulting orbit will remain in a small neighborhood of P_1.

Solution of problem. Since $a = L^2/m_1$ and $r = a(1 - e\cos E)$, we have

$$r = \epsilon^2(L')^2(1 - e\cos E)/m_1 .$$

Furthermore, for $|r| < 1$, R^* may be expanded in a power series in r, the constant term of which, since it does not affect the differential equation, may be neglected. Next the term $r\cos\gamma$ will drop out. Since the remaining terms are $O(r^2)$, it follows that after neglecting constant terms $m_2\epsilon^2R^* = O(\epsilon^6)$.

Letting $\mu = \epsilon^3$ in (5), we wish to apply the following theorem (see Barrar [2]):

EXISTENCE THEOREM. *Consider a Hamiltonian of the form:*

$$H = H_0(p_1) + \mu[H_1(p_1, p_2) + H_2(p_1, p_2, q_1, q_2, \mu)] \quad \text{with} \quad 0 < \mu \leq 1, \tag{H1}$$

and corresponding differential equations

$$dp_i/dt = \partial H/\partial q_i, \qquad dq_i/dt = -\partial H/\partial p_i, \qquad i = 1, 2, \tag{H2}$$

where all functions are assumed to be analytic, and H_2 is periodic of period 2π in q_1, and q_2.

Now for two fixed values p_1^0, p_2^0, let

$$\lambda_1 = \left.\frac{\partial(H_0 + \mu H_1)}{\partial p_1}\right|_{p_i^0} \qquad \lambda_2 = \left.\frac{\partial H_1}{\partial p_2}\right|_{p_i^0}$$

satisfy, for some $\delta > 0$,

$$\left| m_1 + m_2 \frac{\mu\lambda_2}{\lambda_1} \right| \geq \frac{\delta\mu}{(m_2)^2} \quad \textit{for all integer} \quad m_1 \quad \textit{and} \quad m_2 \quad \textit{with} \quad m_2 \neq 0. \tag{H3}$$

Further, for all μ *in some neighborhood of* μ_0 , *with* $0 \leq \mu_0 \leq 1$, *let*

$$\det \left| \frac{\partial^2 (H_0 + \mu H_1)}{\partial p_i \, \partial p_j} \right|_{p_i^0} = 2N\mu \neq 0. \tag{H4}$$

Then for H_2 *sufficiently small, and for* μ *in the above neighborhood, there is a conditionally periodic solution of* (H1) *of the form:*

$$q_1 = \lambda_1 (t - \beta_1) + \Phi_1 (e^{i\lambda_1 t}, e^{i\mu\lambda_2 t})$$

$$q_2 = \mu\lambda_2 (t - \beta_2) + \Phi_2 (e^{i\lambda_1 t}, e^{i\mu\lambda_2 t})$$

$$p_i = A_i + \psi_i (e^{i\lambda_1 t}, e^{i\mu\lambda_2 t})$$

where A_i , β_i *are constants.*

Note. For $0 \leq (\lambda_2/\lambda_1) \leq M$ the exterior measure of all (λ_2/λ_1) not satisfying (H3) is less than $(4\pi^2/3)\, M\mu\delta$ (see Siegel [6]). Hence (H3) is no essential restriction.

In the present problem, we set $\mu = \epsilon^3$ in (H1), and find

$$\lambda_1 = -(m_1)^2/(L')^3, \qquad \lambda_2 = 1.$$

As presently formulated the condition (H4) is not met, since the determinant vanishes identically. However, if we employ the device of Poincaré [5, §43] of squaring the Hamiltonian, and consider the equivalent Hamiltonian $H'' = (H')^2/2C_1$, where H' has the constant value C_1 ,[1] condition (H4) will be met, and the existence theorem is applicable.

In this fashion we have established the existence of conditionally periodic orbits in a small neighborhood of the primary P_1 , for arbitrary mass ratios of the two primaries P_1 , P_2 .

References

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[1]Birkhoff [3, §12] chose $\epsilon^2 = 1/C$, with C the Jacobi constant, so actually $H' = 1$.