SOME CASES OF BIFURCATION IN ELASTIC-PLASTIC SOLIDS IN PLANE STRAIN*

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Summary. Conditions for initiation of necking, buckling and surface instabilities in an elastic-plastic solid in plane strain are derived. The results constitute a generalization of previous investigations for rigid-plastic and elastic solids.

1. Introduction. This paper deals with some problems of bifurcation in elastic-plastic solids in sustained flow. The phenomena of necking, buckling and surface instabilities occurring from a state of plane strain are investigated and expressions for the critical stress at bifurcation are derived. A constitutive law proposed by Hill [3] is taken to define the material property.

Previous studies of the necking form of instability in plane strain have been confined either to rigid-plastic solids (Lee [4], Wang [6], Onat and Prager [5] and Cowper and Onat [2]) or to elastic solids (Wesolowski [7]). Cowper and Onat also investigated the buckling phenomenon in rigid-plastic solids. A form of instability localized at the surface has been shown to occur in an isotropic elastic solid by Biot [1] and by Wesolowski [7]. This form of instability does not occur in rigid-plastic specimens. Since it is likely that some forms of instability may be excluded by restriction to rigid-plastic behaviour, the present study was undertaken to investigate the possibility of further forms of bifurcation. The results obtained here also constituted a generalization of previous results obtained for elastic and rigid-plastic solids.

2. Formulation of the problem. Consider an incompressible rectangular elastic-plastic solid subject to an externally applied axial load P. The body undergoes finite homogeneous deformation due to P from some initial configuration B^0 to the current configuration B. The shape of the body in the current configuration is supposed rectangular and of dimension $2a \times 2b \times 2c$. The behaviour of the specimen in the transition from B to a neighbouring configuration B', under an infinitesimal incremental load dP, is isolated for study. The deformation of the body is constrained such that the dimension $2c$ of the specimen remains constant.

A fixed coordinate frame x_i coinciding with the axes of the specimen is taken as the reference frame. Wherever convenient the coordinates x_1, x_2, x_3 will be replaced by x, y, z, respectively, and the velocity components v_i, v_2, v_3 by u, v, w, respectively. With respect to this frame, the internal distribution of stress, assumed homogeneous, is supposed given by

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$$\sigma_{ij} = s_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma/2 \end{bmatrix} \tag{1}$$

where σ_{ij} , s_{ij} are, respectively, the components of the true stress and the nominal stress tensors.

The stress distribution (1) satisfies the equations of equilibrium:

$$\sigma_{ij,i} = 0$$

and the boundary conditions:

$$\begin{aligned} T_i &= (\sigma, 0, 0) \quad \text{on the faces} \quad x = \pm a, \\ &= (0, 0, 0) \quad \cdots \quad y = \pm b, \\ &= (0, 0, \sigma/2) \quad \cdots \quad z = \pm c. \end{aligned}$$

The body is now tested for instability by subjecting it to an incremental deformation and corresponding incremental stresses. Under continuing deformation from the current configuration B, the rate of surface traction $\dot{T}_i$ is prescribed in terms of the material derivative $\dot{s}_{ij}$ of the nominal stress tensor (Hill [3]), which is related to the material derivative $\dot{\sigma}_{ij}$ of the true stress tensor by:

$$\dot{s}_{ij} = \dot{\sigma}_{ij} + \sigma_{ij}v_{k,k} - \sigma_{jk}v_{i,k} . \tag{2}$$

Here a comma denotes the partial differentiation.

During the incremental deformation, the longitudinal ends $x = \pm a$, moving with constant velocity U, are assumed to be frictionless (hence the shear traction-rates on these ends are zero) and the lateral faces $y = \pm b$ are supposed free of nominal traction-rate. That is,

$$\begin{aligned} v_1 = n_i U, \quad \dot{T}_2 &= n_i\dot{s}_{i2} = 0 \quad \text{on the faces} \quad x = \pm a, \\ \dot{T}_j &= n_i\dot{s}_{ij} = 0 \quad \cdots \quad y = \pm b, \end{aligned} \tag{3}$$

where n_i is unit outward normal to the boundary surface.

Using (2) and the incompressibility condition $v_{i,i} = 0$, the boundary conditions (3) reduce to:

$$\begin{aligned} v_1 &= \pm U \\ \dot{\sigma}_{12} &= 0 \quad \text{at} \quad x = \pm a, \quad \text{and} \\ \dot{\sigma}_{2j} - \sigma_{jk}v_{2,k} &= 0 \quad \text{at} \quad y = \pm b. \end{aligned} \tag{4}$$

In view of (2), the equations of continuing equilibrium $\dot{s}_{ij,i} = 0$ become

$$\dot{\sigma}_{ij,i} = 0. \tag{5}$$

Material Properties: The Jaumann derivative of the true stress tensor $\mathfrak{D}\sigma_{ij}/\mathfrak{D}t$, which vanishes under rigid body rotation, is used to represent the material behaviour. This is related to $\dot{\sigma}_{ij}$ by:

$$\mathfrak{D}\sigma_{ij}/\mathfrak{D}t = \dot{\sigma}_{ij} + \sigma_{ik}\omega_{kj} + \sigma_{jk}\omega_{ki} , \tag{6}$$

where $\omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ is the antisymmetric part of the velocity gradient tensor. The material response to a change in stress is taken in the following form which is due to Hill [1]:

$$\mathfrak{D}\sigma_{ij}/\mathfrak{D}t = K_{ijkl}(\epsilon_{kl} - \epsilon_{kl}^p) + p\delta_{ij} \tag{7}$$

with

$$\begin{aligned} \epsilon_{ij}^p &= h^{-1}m_{ij}(m_{kl}(\mathfrak{D}\sigma_{kl}/\mathfrak{D}t)) \quad \text{when} \quad m_{ij}(\mathfrak{D}\sigma_{ij}/\mathfrak{D}t) > 0 \\ &= 0 \qquad\qquad\qquad\qquad\qquad \cdots \qquad\qquad \leq 0. \end{aligned} \tag{8}$$

Here K_{ijkl} are the elastic moduli in the current state having the property

$K_{ijkl} = K_{jikl} = K_{ijlk} = K_{klij}$
$\epsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) =$ total strain rate,
$\epsilon_{ij}^p =$ plastic part of the strain rate,
$h =$ positive scalar measure of the current rate of work-hardening,
$m_{ij} =$ components of the unit outward normal to the local yield surface in six-dimensional stress-space,

and

$p =$ an unknown hydrostatic pressure rate.

For an isotropic elastic solid,

$$K_{ijkl} = \lambda\delta_{ij}\,\delta_{kl} + \mu(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}) \tag{9}$$

where λ and μ are the Lamé constants.

For a body in a state of plane strain, the internal distribution of the velocity field may be taken in the form

$$w = 0, \qquad u = u(x, y), \qquad v = v(x, y).$$

Hence

$$\epsilon_{33} = \epsilon_{13} = \epsilon_{23} = 0$$

and the incompressibility condition simplifies to

$$\epsilon_{11} + \epsilon_{22} = 0. \tag{10}$$

Taking plastic incompressibility and the current distribution of stress (1) into account, the components of m_{ij} for an isotropic solid are written as

$$m_{ij} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

By the use of (6), (8), (9), (10) and (11), equation (7), for the material loading everywhere, can be expressed in the form

$$\dot{\sigma}_{11}(1 + \delta/2) - \dot{\sigma}_{22}(\delta/2) = 2\mu\epsilon_{11} + p$$
$$-\dot{\sigma}_{11}(\delta/2) + \dot{\sigma}_{22}(1 + \delta/2) = 2\mu\epsilon_{22} + p$$
$$\dot{\sigma}_{33} = p = \tfrac{1}{2}(\dot{\sigma}_{11} + \dot{\sigma}_{22}) \tag{12}$$
$$\dot{\sigma}_{12} = 2\mu\epsilon_{12} - \sigma\omega_{12}$$
$$\dot{\sigma}_{13} = \dot{\sigma}_{23} = 0,$$

where $\delta = 2\mu/h$.

3. Homogeneous deformation. A solution of (5) which satisfies the boundary condition (3) and the incompressibility condition (10) may be found to be

$$u = Ux/a, \qquad v = -Uy/a,$$
$$\dot{\sigma}_{11} = 4\mu h/(2\mu + h)U/a, \qquad \dot{\sigma}_{33} = p = 2\mu h/(2\mu + h)U/a, \tag{13}$$
$$\dot{\sigma}_{22} = \dot{\sigma}_{12} = \dot{\sigma}_{13} = \dot{\sigma}_{23} = 0.$$

This solution preserves the rectangular shape of the specimen.

The value of the tangent modulus E_t at the current instant is

$$E_t = 4\mu h/(2\mu + h), \tag{14}$$

and the corresponding rate of loading on the faces $x = \pm a$ is

$$\dot{T}_1 = (E_t - \sigma)U/a. \tag{15}$$

Now we seek a solution of (5), which represents a nonhomeogeneous deformation, with the velocity boundary condition $v_1 = 0$ on the faces $x = a$ and the traction-rate boundary conditions (4ii) and (4iii). The nonhomogeneous deformation superposed on the homogeneous deformation will still satisfy the equations of continuing equilibrium (5) and the boundary condition (4).

4. Nonhomogeneous deformation without length change. For a body in a state of plane strain, (5) is identically satisfied for $j = 3$. The other two equations are satisfied if

$$\dot{\sigma}_{11} = \varphi_{yy}, \qquad \dot{\sigma}_{12} = -\varphi_{xy}, \qquad \dot{\sigma}_{22} = \varphi_{xx} \tag{16}$$

where $\varphi(x, y)$ is a scalar stress function and subscripts denote partial differentiation.

Eliminating u and v from (12i), (12ii) and (12iv) and using (16) leads to

$$(1 - \theta)\varphi_{yyyy} + 2(1 - \delta)/(1 + \delta)\varphi_{xxyy} + (1 + \theta)\varphi_{xxxx} = 0 \tag{17}$$

where $\theta = (\sigma/2\mu)$.

On the longitudinal ends $x = \pm a$, $\dot{\sigma}_{12} = 0$. Hence a solution for φ in (17) is sought in the form $\varphi = \Phi(y) \cos \nu x$, $\nu = (n\pi/a)$, n being an integer. Substituting this in (17) leads to

$$(1 - \theta)\Phi_{yyyy} - 2\nu^2((1 - \delta)/(1 + \delta))\Phi_{yy} + \nu^4(1 + \theta)\Phi = 0. \tag{17a}$$

Seeking a solution of (17a) in the form $\Phi = \alpha \exp(\nu\beta y)$, we find the characteristic equation giving β to be

$$(1 - \theta)\beta^4 - 2\frac{1 - \delta}{1 + \delta}\beta^2 + (1 + \theta) = 0. \tag{18}$$

The roots of this equation are

$$\beta_1 = -\beta_3 = \frac{1}{(1 - \theta)^{1/2}}(r_1 + r_2)^{1/2}, \qquad \beta_2 = -\beta_4 = \frac{1}{(1 - \theta)^{1/2}}(r_1 - r_2)^{1/2}, \tag{18a}$$

where

$$r_1 = (1 - \delta)/(1 + \delta), \qquad r_2 = [\theta^2 - 4\delta/(1 + \delta)^2]^{1/2}$$

and $\theta = \sigma/2\mu$ is positive or negative according as the applied load is tensile or compressive.

Two distinct cases may arise: (i) unequal roots corresponding to $r_2 \neq 0$, (ii) equal roots corresponding to $r_2 = 0$.

(i) Unequal roots: The general solution of (17) for φ may be written as

$$\varphi = \cos \nu x(\alpha_1 \cosh \nu\beta_1 y + \alpha_2 \cosh \nu\beta_2 y + \alpha_3 \sinh \nu\beta_1 y + \alpha_4 \sinh \nu\beta_2 y)$$

and the expressions for u and v follows as

$$\begin{aligned} u &= \nu(1 + \delta)/4\mu \sin \nu x[(1 + \beta_1^2)(\alpha_1 \cosh \nu\beta_1 y + \alpha_3 \sinh \nu\beta_2 y) \\ &\quad + (1 + \beta_2^2)(\alpha_2 \cosh \nu\beta_2 y + \alpha_4 \sinh \nu\beta_2 y)], \\ v &= -\nu(1 + \delta)/4\mu \cos \nu x[(\beta_1 + 1/\beta_1)(\alpha_1 \sinh \nu\beta_1 y + \alpha_3 \cosh \nu\beta_1 y) \\ &\quad + (\beta_2 + 1/\beta_2)(\alpha_2 \sinh \nu\beta_2 y + \alpha_4 \cosh \nu\beta_2 y)] \end{aligned} \tag{19}$$

from (12i), (12ii) and (10).

In passing, it may be noted that elastic solids are included in this case. For an elastic solid the work-hardening parameter h tends to infinity and hence $\delta = 0$ in the limit, and therefore

$$\beta_1 = -\beta_3 = 1, \qquad \beta_2 = -\beta_4 = (1 + \theta)/(1 - \theta))^{1/2}.$$

(ii) Equal roots: This case follows for a rigid-plastic solid for which the elastic modulus tends to infinity, and hence δ tends to infinity. For such a solid, $\beta_1 = \beta_3 = -\beta_2 = -\beta_4 = \sqrt{-1}$. Hence the solution of (17) may now be taken in the form

$$\varphi = \cos \nu x(\alpha_1 \cos \nu y + \alpha_2 y \sin \nu y + \alpha_3 \sin \nu y + \alpha_4 y \cos \nu y)$$

and the corresponding expressions for u and v are obtained as

$$u = (1/h) \sin \nu x(\alpha_2 \cos \nu y + \alpha_4 \sin \nu y), \qquad v = -(1/h) \cos \nu x(\alpha_2 \sin \nu y + \alpha_4 \cos \nu y).$$

Using (16) and (19), the stress-rate $\dot{\sigma}_{ij}$ corresponding to nonhomogeneous deformation can be obtained in terms of the constants α_i $(i = 1, \cdots, 4)$. The values of these constants are to be determined from the traction-rate boundary condition on $y = \pm b$. For vanishing traction-rate, this procedure generates an eigenvalue problem characterising a bifurcation of equilibrium. The associated modes of deformation may be (i) symmetric, (ii) antisymmetric, or (iii) localized at the surface. The three cases are considered separately.

5. Symmetric mode of deformation. The nature of the velocity field v_i in this case is such that

$$u(x, y) = u(x, -y) \qquad v(x, y) = -v(x, -y).$$

Only even functions of y need therefore be considered in the expressions for φ and u. Substituting for v and $\dot{\sigma}_{ij}$ and using (19), the boundary conditions (4*i*) yield:

$$\begin{gathered} \alpha_1[(1 + \beta_1^2)^2/\beta_1] \sinh \nu\beta_1 b + \alpha_2[(1 + \beta_2^2)^2/\beta_2] \sinh \nu\beta_2 b = 0 \\ \alpha_1 \cosh \nu\beta_1 b + \alpha_2 \cosh \nu\beta_2 b = 0. \end{gathered} \tag{20}$$

A necessary and sufficient condition for existence of nontrivial solutions for α_i is

$$\frac{\beta_2 \tanh \nu\beta_1 b}{\beta_1 \tanh \nu\beta_2 b} = \frac{(1 + \beta_2^2)^2}{(1 + \beta_1^2)^2}, \tag{21}$$

where $\nu = n\pi/a$, n being a positive integer.

It is easily seen that the equation (21), in general, contains both real and imaginary quantities. In order to reduce (21) to a more convenient form so that only real part is retained, the following artifice is employed. Let the roots be denoted by

$$\beta = k_1 \pm \sqrt{-1}\, k_2\,, \qquad k_1\,,\, k_2 \text{ being real.}$$

Hence, from (18a)

$$k_1^2 - k_2^2 = (1 - \delta)/(1 + \delta)\cdot 1/(1 - \theta)$$

and

$$k_1^2 + k_2^2 = +[(1 + \theta)/(1 - \theta)]^{1/2}.$$

The boundary condition now yields the following critical condition, which is equivalent to (21), for bifurcation of equilibrium:

$$\frac{k_1 \sin 2\nu k_2 b}{k_2 \sinh 2\nu k_1 b} = -\frac{(k_1^2 + k_2^2)[2 - \theta(1 + \delta)] + \theta(1 + \delta)}{(k_1^2 + k_2^2)[2 - \theta(1 + \delta)] - \theta(1 + \delta)}.$$

Writing Ω for the left-hand side, this equation can be rewritten in the form

$$f_1(\theta) = f_2(\theta) \tag{22}$$

where

$$f_1 = (1 - \Omega)/(1 + \Omega) \qquad f_2 = ((1 + \theta)/(1 - \theta))^{1/2}[1 - 2/\theta(1 + \delta)],$$

The function f_1 regarded as a function of θ generates a family of curves, each corresponding to a different value of ν. The point of intersection of the f_2 curve with this family gives the value of $\theta = \sigma/2\mu$ at bifurcation. The number of half-waves arising at the instant of instability is given by the value of ν corresponding to the curve f_1 on which the critical stress point is located. As an illustration, curves showing the variation of $f_1(\theta)$, $f_2(\theta)$ with θ are given in Fig. 1 for δ $(= 2\mu/h) = 10$ and for (a) $n = 1$, $b/a = 0.25$ and (b) $n = 1$, $b/a = 1$. It is seen that f_1 curve corresponding to $n = 1$, $b/a = 0.25$ has only one point of intersection A with the f_2 curve. The corresponding mode of deformation is of necking type (under tensile load). The f_1 curve for $n = 1$, $b/a = 1$ has two points of intersection with the f_2 curve, the point B corresponds to necking mode (under tensile load) and the point C corresponds to the bulging mode (under compressive load).

Two cases of particular interest are (a) when the ratio of width b to length a is very small, and (b) when the ratio b/a is very large, ν being assumed finite in each case.

(a) The ratio b/a tends to zero and consequently Ω tends to unity. Hence, $f_1 = 0$. Referring to Fig. 2, it is seen that in this case there is no solution for values of δ less than unity. This means that a bifurcation of the necking type cannot occur in a specimen for which the current rate of hardening is greater than the current value of 2μ.

It may be mentioned that the number of half-waves arising at the instant of instability is indeterminate.

(b) For an infinitely large value of b/a, Ω tends to zero and hence $f_1 = 1$. In this case, it may be seen from Fig. 2 that bifurcation may be expected both under tensile load (corresponding to a necking mode) or under compressive load (corresponding to a bulging mode). Here too, as in case (a), the number of half-waves occurring at the instant of instability remains indeterminate. It is worth stressing that this form of instability can also occur for a specimen with a finite b/a ratio if the number of half-waves arising at the instant of instability is large. This may be a reason why, in practice, a large number of half-waves is sometimes observed in a specimen subjected to either a sudden pull or severe bending.

A similar calculation for the case of equal roots, i.e. for a rigid-plastic solid, yields the following value of the critical stress at bifurcation:

$$\sigma = h(2\nu b + \sin 2\nu b)/\sin 2\nu b, \qquad \nu = (n\pi b/a).$$

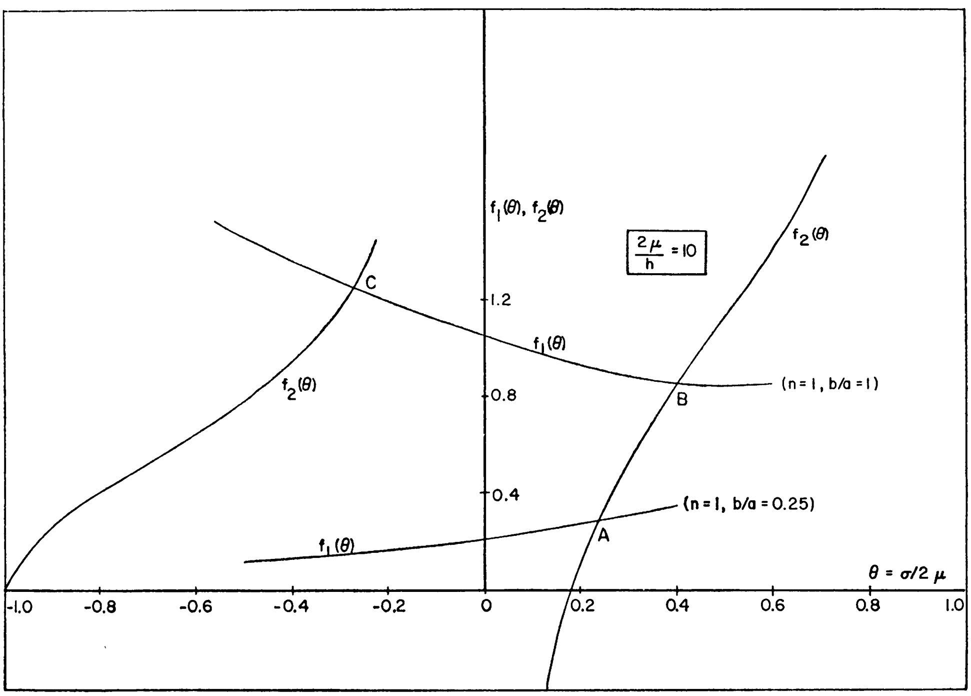

Fig. 1

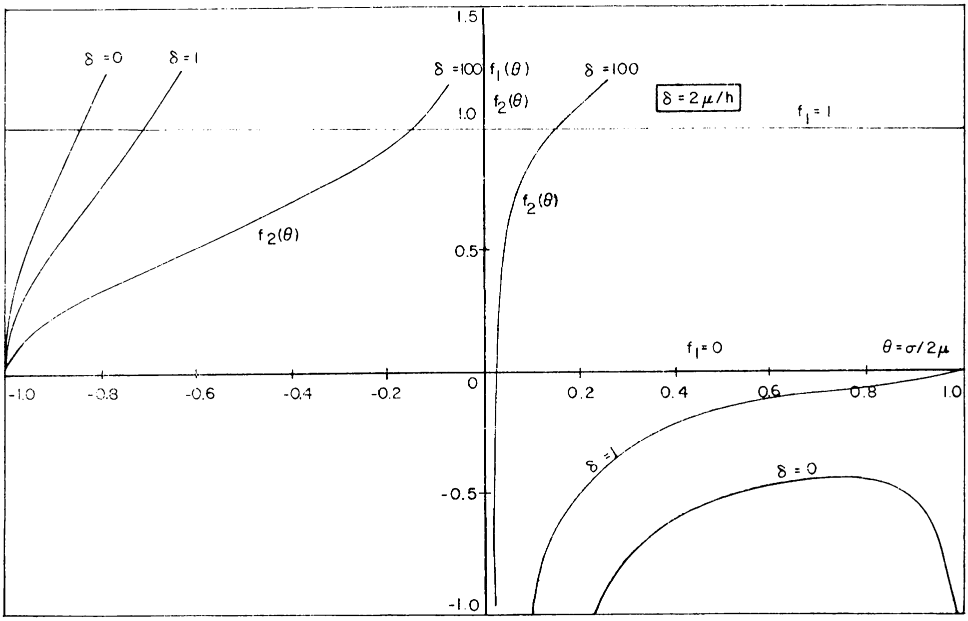

FIG. 2

This result, for $n = 1$, is the same as that obtained by Cowper and Onat [2].

The effect of shear stiffening on the critical stress for a specimen with small b/a ratio is obtained by retaining the first two terms in the power series expansion of Ω in terms of b/a. The value of the critical stress is then found to be:

$$\sigma = E_t(1 + \pi^2 b^2/3a^2), \qquad E_t = 4\mu h/(h + 2\mu).$$

6. Antisymmetric mode of deformation. The deformation in this case is characterized by

$$u(x, y) = -u(x, -y), \qquad v(x, y) = v(x, -y).$$

Hence φ is of the form

$$\varphi = \cos \nu x(\alpha_3 \sinh \nu\beta_1 y + \alpha_4 \sinh \nu\beta_2 y).$$

Using the expression for u and v from (19), the boundary condition (4i) yields:

$$\alpha_3 \cosh \nu\beta_1 b(1 + \beta_1^2)^2/\beta_1 + \alpha_4 \cosh \nu\beta_2 b(1 + \beta_2^2)^2/\beta_2 = 0$$

$$\alpha_3 \sinh \nu\beta_1 b + \alpha_4 \sinh \nu\beta_2 b = 0.$$

Hence the critical condition for bifurcation of equilibrium is

$$\frac{\beta_1 \tanh \nu\beta_1 b}{\beta_2 \tanh \nu\beta_2 b} = \frac{(1 + \beta_1^2)^2}{(1 + \beta_2^2)^2}, \qquad \nu = \frac{n\pi}{a}. \tag{23}$$

Equation (23) can be solved for σ in terms of the discrete parameter ν and the current value of $\delta = 2\mu/h$ to be found from the known stress-strain curve. It is interesting to note that for large values of b/a, (23) reduces to a form which is identical to the one already discussed in case (b) in Sec. 5. It is shown in the next section that this case essentially corresponds to instability localized at the surface.

For a specimen with a small value of b/a and finite ν, (23) furnishes an approximate value of critical stress as

$$\sigma = -\tfrac{1}{3}E_t(n\pi b/a)^2[1 + (n\pi b/a)^2(7\delta - 17)/(1 + \delta)]$$

when $E_t = 4\mu/(1 + \delta)$, $\delta = 2\mu/h$. In the first approximation, this agrees with the Shanley tangent modulus formula.

For a rigid-plastic solid, a similar procedure leads to the following value for σ:

$$\sigma = -h\,\frac{2\nu b - \sin 2\nu b}{\sin 2\nu b}, \qquad \nu = \frac{n\pi}{a}.$$

This again, for $n = 1$, is the same as the result obtained by Cowper and Onat [2].

7. Surface instabilities. The disturbance is assumed localized at the surface of the body and to decay rapidly as the depth from the surface increases. For convenience the $y = 0$ plane is now shifted to lie on the surface.

The basic differential equation entering in the instability investigation remains the same and so do the roots of the characteristic equation (18).

The stress function φ is now taken in the form

$$\varphi = \cos \nu x[\alpha_1 \exp(-\nu\beta_1 y) + \alpha_2 \exp(-\nu\beta_2 y)], \qquad \beta_1, \beta_2 > 0. \tag{24}$$

For the boundary $y = 0$ free from traction-rate, the boundary conditions (4i) reduce to

$$\dot{\sigma}_{21} - \sigma v_x = 0$$

$$\dot{\sigma}_{22} = 0 \quad \text{at} \quad y = 0.$$

Substituting from (24) and using (10), (12i) and (12ii), the boundary conditions yield

$$\alpha_1(1 + \beta_1^2)^2/\beta_1 + \alpha_2(1 + \beta_2^2)^2/\beta_2 = 0$$

$$\alpha_1 + \alpha_2 = 0.$$

A necessary and sufficient condition for nontrivial solutions for α_1, α_2 to exist is $\beta_2(1 + \beta_1^2)^2 - \beta_1(1 + \beta_2^2)^2 = 0$. Using the relation $\beta_1\beta_2 = ((1 + \theta)/(1 - \theta))^{1/2}$, this can be rewritten as

$$f_1 = 1, \qquad f_1 = f_2 \equiv ((1 + \theta)/(1 - \theta))^{1/2}[1 - 2/\theta(1 + \delta)]. \tag{25}$$

This has already been discussed in case (b) in Sec. 5. It may be noted that the instability localized at the surface may occur for a body with a large b/a ratio when subjected either to a compressive or a tensile stress. The number of half-waves remains indeterminate in each case. It may be pointed out further that surface instability accompanied by the formation of a large number of half-waves on the surface may also occur for a specimen with a finite b/a ratio when subjected to a sudden severe pull. For a body subjected to sudden severe bending, the formation of a large number of half-waves on the inside surface, that is, on the compression side, has actually been observed.

For a semi-infinite elastic solid, for which δ tends to zero, (25) yields $\theta^3 - 2\theta + 2 = 0$. This is the same as the equation obtained by Biot [1] and has only the single real root $\sigma = -1.68\mu$.

8. Loading criterion. The analysis has been based on the assumption that the material loads everywhere. The criterion for plastic loading is (Hill [3]):

$$m_{ij}(\mathfrak{D}\sigma_{ij}/\mathfrak{D}t) > 0.$$

Using (1), (10), (11) and (12), this criterion reduces to $\epsilon_{11} = u_x > 0$, u being the velocity along the length of the specimen, and is obtained by superimposing the nonhomogeneous deformation on the homogeneous deformation. Substituting from (13i), (19i) and using (18a) and (20), the loading criterion yields for the symmetric mode of deformation

$$U/a - (\nu^2/E_t)\alpha_1 \left|(\beta_1^2 - \beta_2^2) \cos \nu x \cosh \nu\beta_1 y\right| > 0.$$

Hence from (15), the rate of loading required for the material to load everywhere is:

$$\dot{T}_1 > ((E_t - \sigma)/E_t)\alpha_1(n\pi b/a)^2 \cosh \nu\beta_1 b$$

where σ is given by (21).

It may be seen that the rate of loading $\dot{T}_1$ required for no unloading to take place becomes increasingly small as the stress σ approaches the tangent modulus E_t .

For the buckling mode, proceeding in a similar manner, we find that the material loads everywhere if

$$(\dot{T}_1/\alpha_1) > ((E_t - \sigma)/E_t)(n\pi b/a)^2 \sinh \nu\beta_1 b.$$

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