THE LINEAR THEORY OF SECOND-GRADE ELASTIC MATERIALS*

BY

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Abstract. Second-grade elastic materials have an energy density which depends on both first and second deformation gradients. The behaviour of these materials is investigated for deformations sufficiently small that the equations of motion can be linearized. The elastic materials with couple stress studied by Mindlin and Tiersten [1] are a special case of the materials studied here, and our results reduce to theirs with an appropriate choice for our elastic constants.

1. Introduction. Some time ago Toupin [2], Mindlin and Tiersten [1], and Koiter [3] proposed a theory of elastic materials with couple stress based on the Cosserat equations of motion and on the assumption that the rate of working of the couple stress is the scalar product of that stress with the vorticity. This assumption allows constitutive relations to be derived for stress and couple stress in terms of the stored energy density. The linearised form of the constitutive relations was obtained by Mindlin and Tiersten who used them to solve a number of problems both of wave propagation and static elasticity. Later work (for example, by Muki and Sternberg [4]) has provided solutions to many further problems.

A curious feature of this theory, pointed out later by Toupin [5], is that while shear waves become dispersive, dilatational waves remain nondispersive, as in the classical theory of elasticity. Toupin [5] showed that the theory is a special case of the theory of second-grade materials—that is, materials for which the energy density depends on both the first and second deformation gradients—and expressed the view that within this wider class of materials the dispersion anomaly would not occur. It is the purpose of this paper to investigate the behaviour of second-grade materials for deformations sufficiently small that linearised constitutive relations are applicable.

In Sec. 2 we derive the equations of motion from an action principle, assuming the action density to depend on the first and second deformation gradients, material velocity and velocity gradient. Throughout the rest of the paper we assume this last variable to be absent, although, as discussed in the final section, its presence makes little change. In Sec. 3 some simplifications are made on the basis that the action density consists of separate kinetic and potential parts, using the principle of frame indifference and restricting ourselves to isotropic materials, and in Sec. 4 the system is linearised by taking only quadratic terms in the energy density. This leads to linear expressions for the stress and hyperstress tensors in terms of the displacement gradients, and to a generalisation of Navier's equations of motion. In the energy density we are forced to introduce five elastic constants beyond the usual Lamé constants, but only two combinations of these enter the Navier's equations.

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In the following sections we examine various problems of wave propagation. For the case of plane body waves there are two wave solutions for both the shear and dilatation cases, one of which is propagating and dispersive, the other of which is nonpropagating. The decay lengths for the nonpropagating waves are different in the two cases. For the vibration frequencies of a finite slab, we obtain the same frequency equation as Mindlin and Tiersten for the thickness-shear vibrations, but a new frequency equation for the longitudinal mode. The equation for the torsional vibration frequencies of a circular cylinder is also different from Mindlin and Tiersten's result with a new term involving a combination of elastic constants which vanishes in their special case.

Certain restrictions on the values of the elastic constants arise from the requirement that the energy density is positive definite. Some of these are obtained in Sec. 8, where it is shown in particular that the decay lengths for both nonpropagating wave modes are real.

2. Equations of motion. We consider a continuous body the particles of which are described by their position vectors $\mathbf{X}$ in some reference configuration. These have components (X_α) with respect to a given Cartesian coordinate frame. At time t the particle $\mathbf{X}$ has position $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ in space, and $\mathbf{x}$ has components (x_i) with respect to a second Cartesian frame. Particle velocity is $\dot{\mathbf{x}} = (\partial/\partial t)\mathbf{x}(\mathbf{X}, t)$.

The first two deformation gradients are defined as

$$x_{i,\alpha} = \frac{\partial}{\partial X_\alpha} x_i(\mathbf{X}, t), \qquad x_{i,\alpha\beta} = \frac{\partial^2}{\partial X_\beta\, \partial X_\alpha} x_i(\mathbf{X}, t), \tag{1}$$

and from these we obtain

$$e_{\alpha\beta} = \tfrac{1}{2}(x_{i,\alpha}x_{i,\beta} - \delta_{\alpha\beta}) \tag{2}$$

$$q_{\alpha\beta\gamma} = x_{i,\alpha}x_{i,\beta\gamma}\ . \tag{3}$$

The $e_{\alpha\beta}$ are components of the strain tensor, while $q_{\alpha\beta\gamma}$ are related to the strain gradients:

$$\begin{aligned} q_{\alpha\beta\gamma} &= e_{\alpha\beta,\gamma} + e_{\alpha\gamma,\beta} - e_{\beta\gamma,\alpha} \\ 2e_{\alpha\beta,\gamma} &= q_{\gamma\beta\alpha} + q_{\gamma\alpha\beta}\ . \end{aligned} \tag{4}$$

Second-grade elastic materials are usually defined as materials for which the stress is a function of both $x_{i,\alpha}$ and $x_{i,\alpha\beta}$. We wish to follow Toupin [5] and use an action principle to formulate the theory, so we extend consideration to those materials which have an action density depending on the following variables:

$$L = L(\dot{x}_i\ , \dot{x}_{i,\alpha}\ , x_{i,\alpha}\ , x_{i,\alpha\beta}\ , \mathbf{X}). \tag{5}$$

The action associated with a part P of the body and an interval I of time is

$$A(P, I) = \int_I \int_P L\, dV\, dt. \tag{6}$$

Then we take the following variational principle, for a small change δx_i in the motion:

$$\delta A + \int_I \int_P (F_i\ \delta x_i + C_{i\alpha}\ \delta x_{i,\alpha})\, dV\, dt + \int_I \int_{\partial P} (T_i\ \delta x_i + D_{i\alpha}\ \delta x_{i,\alpha})\, dS\, dt \\ - \left[\int_P P_i^*\ \delta x_i\, dV + \int_{\partial P} Q_i^*\ \delta x_i\, dS\right]_{t_1}^{t_2} = 0. \tag{7}$$

Here F_i is a body force and $C_{i\alpha}$ a dipolar force (for example a body couple), T_i is a surface traction and $D_{i\alpha}$ a surface dipolar traction, P_i^* the initial and final momentum density and Q_i^* a dipolar momentum density which must be prescribed on the surface ∂P. Some of these terms are mathematically redundant (for example the $C_{i\alpha}$ term can be combined with the F_i and T_i terms), but since they arise from physically separable effects we prefer to include them.

Making the definitions

$$\begin{aligned} {}^0P_i &= \frac{\partial L}{\partial \dot{x}_i}, \qquad Q_{i\alpha} = \frac{\partial L}{\partial \dot{x}_{i,\alpha}}, \qquad P_i = {}^0P_i - Q_{i\alpha,\alpha} \\ {}^0T_{i\alpha} &= -\frac{\partial L}{\partial x_{i,\alpha}}, \qquad M_{i\alpha\beta} = -\frac{\partial L}{\partial x_{i,\alpha\beta}}, \qquad T_{i\alpha} = {}^0T_{i\alpha} - M_{i\alpha\beta,\beta} \end{aligned} \tag{8}$$

we can write

$$\delta A = \int_I \int_P \{P_i \, \delta\dot{x}_i + (Q_{i\alpha} \, \delta\dot{x}_i)_{,\alpha} - T_{i\alpha} \, \delta x_{i,\alpha} - (M_{i\alpha\beta} \, \delta x_{i,\alpha})_{,\beta}\} \, dV \, dt.$$

Integrating various terms by parts and setting the coefficient of the variation δx_i equal to zero in the different integration regions, we obtain

$$\dot{P}_i = T_{i\alpha,\alpha} + F_i - C_{i\alpha,\alpha} \quad \text{in} \quad I \times P \tag{9}$$

$$\begin{aligned} P_i &= P_i^* \quad \text{on} \quad \partial I \times P \\ Q_{i\alpha}N_\alpha &= Q_i^* \quad \text{on} \quad \partial I \times \partial P \end{aligned} \tag{10}$$

(here $\mathbf{N}$ is the outward normal from P). We are left with the surface term

$$\int_I \int_{\partial P} \{(T_i - T_{i\alpha}N_\alpha + C_{i\alpha}N_\alpha - \dot{Q}_{i\alpha}N_\alpha) \, \delta x_i + (D_{i\alpha} - M_{i\alpha\beta}N_\beta) \, \delta x_{i,\alpha}\} \, dS \, dt = 0.$$

Here only the normal derivative $\delta x_{i,\alpha}N_\alpha$ is independent of δx_i itself, which leads to

$$D_{i\alpha}N_\alpha - M_{i\alpha\beta}N_\alpha N_\beta = 0 \qquad (\text{on } I \times \partial P), \tag{11}$$

and expressing the tangential derivative in terms of δx_i gives, after using Eq. (11),

$$T_i - T_{i\alpha}N_\alpha + C_{i\alpha}N_\alpha - \dot{Q}_{i\alpha}N_\alpha - D_\alpha(D_{i\alpha} - M_{i\alpha\beta}N_\beta) = 0 \text{ on } I \times \partial P. \tag{12}$$

Here we have introduced a tangential differentiation operator

$$D_\alpha = (\partial/\partial X_\alpha - N_\alpha N_\beta \, \partial/\partial X_\beta). \tag{13}$$

3. Second-grade materials. The conventional second-grade materials are obtained by taking the special case

$$L = \tfrac{1}{2}\rho\dot{x}_i\dot{x}_i - W(x_{i,\alpha}, x_{i,\alpha\beta}, \mathbf{X}). \tag{14}$$

For these, $P_i = \rho\dot{x}_i$, $Q_{i\alpha} = 0$, and the dipolar velocities do not enter the theory. If we add the requirement that L (and hence W) is invariant under rigid rotation of the x_i-coordinate frame ($x_i \to R_{ij}x_j$ where $\mathbf{R}$ is orthogonal) then it follows that W must be a function only of $(e_{\alpha\beta})$ and $(q_{\alpha\beta\gamma})$ (or alternatively $(e_{\alpha\beta,\gamma})$) and not of the complete deformation gradients: $W = W(e_{\alpha\beta}, e_{\beta\gamma,\delta}, \mathbf{X})$. Then

$$
\begin{aligned}
{}^0T_{i\lambda} &= \frac{\partial W}{\partial e_{\alpha\beta}} \cdot \frac{\partial e_{\alpha\beta}}{\partial x_{i,\lambda}} + \frac{\partial W}{\partial e_{\alpha\beta,\gamma}} \frac{\partial e_{\alpha\beta,\gamma}}{\partial x_{i,\lambda}} \\
&= x_{i,\alpha} \frac{1}{2}\left(\frac{\partial W}{\partial e_{\alpha\lambda}} + \frac{\partial W}{\partial e_{\lambda\alpha}}\right) + x_{i,\alpha\gamma} \cdot \frac{1}{2}\left(\frac{\partial W}{\partial e_{\alpha\lambda,\gamma}} + \frac{\partial W}{\partial e_{\lambda\alpha,\gamma}}\right),
\end{aligned}
$$

and

$$
M_{i\lambda\mu} = \frac{\partial W}{\partial e_{\alpha\beta,\gamma}} \cdot \frac{\partial e_{\alpha\beta,\gamma}}{\partial x_{i,\lambda\mu}} = x_{i,\alpha} \cdot \frac{1}{2}\left(\frac{\partial W}{\partial e_{\alpha\lambda,\mu}} + \frac{\partial W}{\partial e_{\lambda\alpha,\mu}}\right). \tag{15}
$$

Thus

$$
T_{i\lambda} = x_{i,\alpha}\left[\frac{\partial W}{\partial e_{(\alpha\lambda)}} - \left(\frac{\partial W}{\partial e_{(\alpha\lambda),\mu}}\right)_{,\mu}\right], \tag{16}
$$

using brackets to denote symmetrisation.

Further restrictions on W arise from the symmetry properties of the material. We shall assume the material is homogeneous in the sense that there exists a reference configuration for the whole body called a uniform configuration for which W is the same for all particles—i.e. taking a uniform configuration as reference, $W = W(e_{\alpha\beta}, e_{\alpha\beta,\gamma})$ is independent of $\mathbf{X}$. If a uniform configuration exists, then any other configuration obtained from it by a homogeneous deformation is also uniform. From now on we assume that the reference configuration (X_α) is uniform.

Let (X'_α) be coordinates with respect to a second Cartesian system in the same reference configuration, so that $X'_\alpha = R_{\alpha\beta}X_\beta$ where $\mathbf{R}$ is orthogonal. Using this system leads to a strain tensor $e'_{\alpha\beta}$ where

$$
\begin{aligned}
e'_{\alpha\beta} &= e_{\gamma\delta}R_{\gamma\alpha}R_{\delta\beta} \\
e'_{\alpha\beta,\gamma} &= e_{\delta\epsilon,\xi}R_{\epsilon\beta}R_{\xi\gamma}R_{\delta\alpha}\ .
\end{aligned}
$$

The material symmetry group is the set of all transformations $\mathbf{R}$ for which $W(e'_{\alpha\beta}, e'_{\alpha\beta,\gamma}) = W(e_{\alpha\beta}, e_{\alpha\beta,\gamma})$.

A material is isotropic if there exists a configuration, called an undistorted state, with respect to which the symmetry group is the whole orthogonal group. Again, if one undistorted state exists, then there are many undistorted states. For let (Y_α) be the Cartesian coordinates of the particle (X_α) in a state obtained from the reference state $\{(X_\alpha)\}$ by a homogeneous dilatation:

$$
Y_\alpha = kX_\alpha \qquad (\alpha = 1, 2, 3).
$$

Then $\partial x_i/\partial Y_\alpha = k^{-1}x_{i,\alpha}$ and the strain relative to the $\mathbf{Y}$-configuration,

$$
\begin{aligned}
f_{\alpha\beta} &= \tfrac{1}{2}((\partial x_i/\partial Y_\alpha)(\partial x_i/\partial Y_\beta) - \delta_{\alpha\beta}) \\
&= k^{-2}e_{\alpha\beta} + \tfrac{1}{2}(k^{-2} - 1)\ \delta_{\alpha\beta}\ .
\end{aligned}
$$

Thus the strain energy relative to the Y-configuration,

$$
W_Y(f_{\alpha\beta}, f_{\alpha\beta,\gamma}) = W_X(k^2f_{\alpha\beta} + \tfrac{1}{2}(k^2 - 1)\delta_{\alpha\beta}, k^3f_{\alpha\beta,\gamma})
$$

and W_Y clearly always has the same symmetry group as W_X. So if X is an undistorted state, then so is Y.

Physically we expect W_X to become large as both $k \to 0$ and $k \to \infty$ for fixed $f_{\alpha\beta}$, so that there will be a value of k for which

$$\left.\frac{\partial W_X}{\partial k}\right|_{f_{\alpha\beta}=0} = 0,$$

i.e.

$$\left.\frac{\partial W_X}{\partial e_{\alpha\alpha}}\right|_{e_{\alpha\beta}=(k^2-1)\delta_{\alpha\beta}/2} = 0$$

and hence

$$\left.\frac{\partial W_Y}{\partial f_{\alpha\alpha}}\right|_{f_{\alpha\beta}=0} = 0. \tag{17}$$

The state (Y) for which this relation holds is called the natural state.

4. Linearisation. We shall consider a homogeneous isotropic material, using the natural state as reference, for which the strains and strain gradients are small. Then making a Taylor expansion, we obtain

$$\begin{aligned} W(e_{\alpha\beta}, e_{\alpha\beta,\gamma}) = W_0 + A_{\alpha\beta}e_{\alpha\beta} + B_{\alpha\beta\gamma}e_{\alpha\beta,\gamma} + C_{\alpha\beta\gamma\delta}e_{\alpha\beta}e_{\gamma\delta} \\ + D_{\alpha\beta\gamma\delta\epsilon}e_{\alpha\beta}e_{\gamma\delta,\epsilon} + E_{\alpha\beta\gamma\delta\epsilon\xi}e_{\alpha\beta,\gamma}e_{\delta\epsilon,\xi} + \cdots . \end{aligned} \tag{18}$$

For an isotropic material, all the coefficient tensors must be isotropic and hence must be of the forms:

$$\begin{aligned} A_{\alpha\beta} &= A\delta_{\alpha\beta} \\ B_{\alpha\beta\gamma} &= B\epsilon_{\alpha\beta\gamma} \\ C_{\alpha\beta\gamma\delta} &= C_1\delta_{\alpha\beta}\delta_{\gamma\delta} + C_2\delta_{\alpha\gamma}\delta_{\beta\delta} + C_3\delta_{\alpha\delta}\delta_{\beta\gamma} \\ D_{\alpha\beta\gamma\delta\epsilon} &= D_1\delta_{\alpha\beta}\epsilon_{\gamma\delta\epsilon} + D_2\delta_{\alpha\gamma}\epsilon_{\beta\delta\epsilon} + D_3\delta_{\alpha\delta}\epsilon_{\beta\gamma\epsilon} + D_4\delta_{\alpha\epsilon}\delta_{\beta\gamma\delta} \\ &\quad + D_5\delta_{\beta\gamma}\epsilon_{\alpha\delta\epsilon} + D_6\delta_{\beta\delta}\epsilon_{\alpha\gamma\epsilon} + D_7\delta_{\beta\epsilon}\epsilon_{\alpha\gamma\delta} \\ &\quad + D_8\delta_{\gamma\delta}\epsilon_{\alpha\beta\epsilon} + D_9\delta_{\gamma\epsilon}\epsilon_{\alpha\beta\delta} + D_{10}\delta_{\delta\epsilon}\epsilon_{\alpha\beta\gamma} \\ E_{\alpha\beta\gamma\delta\epsilon\xi} &= E_1\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\epsilon\xi} + E_2\delta_{\alpha\beta}\delta_{\gamma\epsilon}\delta_{\delta\xi} + E_3\delta_{\alpha\beta}\delta_{\gamma\xi}\delta_{\delta\epsilon} \\ &\quad + E_4\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\epsilon\xi} + E_5\delta_{\alpha\gamma}\delta_{\beta\epsilon}\delta_{\delta\xi} + E_6\delta_{\alpha\gamma}\delta_{\beta\xi}\delta_{\delta\epsilon} \\ &\quad + E_7\delta_{\alpha\delta}\delta_{\beta\gamma}\delta_{\epsilon\xi} + E_8\delta_{\alpha\delta}\delta_{\beta\epsilon}\delta_{\gamma\xi} + E_9\delta_{\alpha\delta}\delta_{\beta\xi}\delta_{\gamma\epsilon} \\ &\quad + E_{10}\delta_{\alpha\epsilon}\delta_{\beta\gamma}\delta_{\delta\xi} + E_{11}\delta_{\alpha\epsilon}\delta_{\beta\delta}\delta_{\gamma\xi} + E_{12}\delta_{\alpha\epsilon}\delta_{\beta\xi}\delta_{\gamma\delta} \\ &\quad + E_{13}\delta_{\alpha\xi}\delta_{\beta\gamma}\delta_{\delta\epsilon} + E_{14}\delta_{\alpha\xi}\delta_{\beta\delta}\delta_{\gamma\epsilon} + E_{15}\delta_{\alpha\xi}\delta_{\beta\epsilon}\delta_{\gamma\delta} . \end{aligned}$$

(Note that in the sixth-order isotropic tensor, the product of two permuation tensors can be decomposed into terms of the types listed.) Since the strains are measured from the natural state,

$$\left.\frac{\partial W}{\partial e_{\alpha\alpha}}\right|_{e_{\alpha\beta}=0} = A = 0.$$

Many of the terms in the expansion vanish since $e_{\alpha\beta}$ and $e_{\alpha\beta,\gamma}$ are symmetric in α, β while $\epsilon_{\alpha\beta\gamma}$ is antisymmetric, while other sets of terms give identical contributions.

After simplifying we obtain

$$\begin{aligned} W(e_{\alpha\beta}\,, e_{\alpha\beta,\gamma}) = W_0 &+ \tfrac{1}{2}\lambda(e_{\alpha\alpha})^2 + \mu e_{\alpha\beta}e_{\alpha\beta} + D\epsilon_{\alpha\beta\gamma}e_{\alpha\kappa}e_{\kappa\beta,\gamma} \\ &+ F_1 e_{\alpha\alpha,\gamma}e_{\beta\beta,\gamma} + F_2 e_{\alpha\alpha,\beta}e_{\beta\gamma,\gamma} + F_3 e_{\alpha\beta,\beta}e_{\alpha\gamma,\gamma} \\ &+ F_4 e_{\alpha\beta,\gamma}e_{\alpha\beta,\gamma} + F_5 e_{\alpha\beta,\gamma}e_{\alpha\gamma,\beta}\,, \end{aligned} \tag{19}$$

where $\lambda = 2C_1$ and $\mu = C_2 + C_3$ are the usual Lamé constants,

$D = D_2 + D_3 + D_5 + D_6\,,\quad F_1 = E_3\,,\quad F_2 = E_1 + E_2 + E_6 + E_{13}\,,$

$F_3 = E_4 + E_5 + E_7 + E_{10}\,,\quad F_4 = E_8 + E_{11}$ and $F_5 = E_9 + E_{12} + E_{14} + E_{15}\,.$

We shall follow Mindlin and Tiersten and assume that the materials we deal with are centro-symmetric, so that W is invariant under the inversion $X_\alpha \to -X_\alpha$. Then since the D-term is a pseudoscalar with respect to (X_α), we must require $D = 0$. It follows that

$$\begin{aligned} X_{\kappa,i}M_{i\lambda\mu} = \frac{\partial W}{\partial e_{(\kappa\lambda),\mu}} &= \{2F_1\delta_{\kappa\lambda}e_{\beta\beta,\mu} + F_2\delta_{\kappa\lambda}e_{\mu\beta,\beta} + F_2\delta_{\lambda\mu}e_{\beta\beta,\kappa} \\ &+ 2F_3\delta_{\lambda\mu}e_{\kappa\beta,\beta} + 2F_4 e_{\kappa\lambda,\mu} + 2F_5 e_{\kappa\mu,\lambda}\}_{(\kappa\lambda)} \end{aligned} \tag{20}$$

and

$$\begin{aligned} X_{\kappa,i}T_{i\lambda} &= \frac{\partial W}{\partial e_{(\kappa\lambda)}} - \left(\frac{\partial W}{\partial e_{(\kappa\lambda),\mu}}\right)_{\mu} \\ &= \lambda e_{\alpha\alpha}\delta_{\kappa\lambda} + 2\mu e_{\kappa\lambda} - \{2F_1\delta_{\kappa\lambda}e_{\beta\beta,\mu\mu} + F_2\delta_{\kappa\lambda}e_{\mu\beta,\beta\mu} \\ &\quad + F_2 e_{\beta\beta,\kappa\lambda} + 2F_3 e_{\kappa\beta,\beta\lambda} + 2F_4 e_{\kappa\lambda,\mu\mu} + 2F_5 e_{\kappa\mu,\lambda\mu}\}_{(\kappa\lambda)}\,. \end{aligned} \tag{21}$$

If (x_i) and (X_α) are now taken with respect to the same coordinate frame, the displacement vector is $u_\alpha = x_\alpha - X_\alpha$, and $u_{\alpha,\beta} \equiv \partial u_\alpha/\partial X_\beta \ll 1$. Then $e_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha})$ to first order in the displacement gradients, and

$$\begin{aligned} M_{\kappa\lambda\mu} = \{&2F_1\delta_{\kappa\lambda}u_{\beta,\beta\mu} + \tfrac{1}{2}F_2\delta_{\kappa\lambda}(u_{\mu,\beta\beta} + u_{\beta,\beta\mu}) \\ &+ F_2\delta_{\lambda\mu}u_{\beta,\beta\kappa} + F_3\delta_{\lambda\mu}(u_{\kappa,\beta\beta} + u_{\beta,\beta\kappa}) \\ &+ F_4(u_{\kappa,\lambda\mu} + u_{\lambda,\kappa\mu}) + F_5(u_{\kappa,\mu\lambda} + u_{\mu,\kappa\lambda})\}_{(\kappa\lambda)}\,; \end{aligned} \tag{22}$$

$$\begin{aligned} T_{\kappa\lambda} = \lambda\Delta\delta_{\kappa\lambda} + \mu(u_{\kappa,\lambda} + u_{\lambda,\kappa}) - \{&2F_1 + F_2)\delta_{\kappa\lambda}\nabla^2\Delta + (F_2 + F_3 + F_5)\partial_\kappa\partial_\lambda\Delta \\ &+ (F_4 + \tfrac{1}{2}F_3 + \tfrac{1}{2}F_5)\nabla^2(u_{\kappa,\lambda} + u_{\lambda,\kappa})\} \end{aligned} \tag{23}$$

where $\Delta = u_{\alpha,\alpha}$ and $\partial_\kappa = \partial/\partial X_\kappa$. Finally, putting this into the momentum equation gives the generalisation of Navier's equation to second-grade materials:

$$\rho\ddot{u}_\alpha - F_\alpha + C_{\alpha\beta,\beta} = T_{\alpha\beta,\beta} = (\lambda + \mu)\,\partial_\alpha\Delta + \mu\nabla^2 u_\alpha - \eta_1\,\partial_\alpha\nabla^2\Delta - \eta_2\nabla^4 u_\alpha \tag{24}$$

where

$$\eta_1 = 2F_1 + 2F_2 + \tfrac{3}{2}F_3 + F_4 + \tfrac{3}{2}F_5 \quad\text{and}\quad \eta_2 = \tfrac{1}{2}F_3 + F_4 + \tfrac{1}{2}F_5. \tag{25}$$

The materials considered by Mindlin and Tiersten satisfy this same equation with $\eta_2 = -\eta_1$.

Finally it is now only a matter of substitution to express the boundary conditions in terms of (u_α).

5. Wave solutions. The displacement field $\mathbf{u}$ can always be decomposed as

$$\mathbf{u} = \nabla\varphi + \nabla \wedge \mathbf{H}.$$

Substituting this into Navier's equations, (24), with all body forces and couples absent, we obtain

$$\nabla\left[\frac{1}{c_1^2}\ddot{\phi} - \nabla^2\phi + l_1^2\nabla^4\phi\right] + \nabla \wedge \left[\frac{1}{c_2^2}\ddot{\mathbf{H}} - \nabla^2\mathbf{H} + l_2^2\nabla^4\mathbf{H}\right] = \mathbf{0}$$

where

$$c_1^2 = (\lambda + 2\mu)/\rho, \qquad c_2^2 = \mu/\rho, \qquad l_1^2 = (\eta_1 + \eta_2)/(\lambda + 2\mu), \qquad l_2^2 = \eta_2/\mu. \tag{26}$$

This is satisfied by

$$c_1^{-2}\ddot{\phi} - \nabla^2\phi + l_1^2\nabla^4\phi = 0, \qquad c_2^{-2}\ddot{\mathbf{H}} - \nabla^2\mathbf{H} + l_2^2\nabla^4\mathbf{H} = 0. \tag{27}$$

We now look for plane wave solutions. For the dilatational wave, we take $\varphi = \varphi_0 \exp i(k\mathbf{n}\cdot\mathbf{r} - wt)$ and obtain that

$$w^2 = c_1^2k^2(1 + l_1^2k^2). \tag{28}$$

For the rotation wave, $\mathbf{H} = \mathbf{H}_0 \exp i(k\mathbf{n}\cdot\mathbf{r} - wt)$ which leads to

$$w^2 = c_2^2k^2(1 + l_2^2k^2). \tag{29}$$

In contrast to Mindlin and Tiersten's case, both modes are dispersive for second-grade materials.

Usually we would want the solutions for a given value of w; solving for k^2 for each mode produces

$$k^2 = [-c_i^2 \pm (c_i^4 + 4c_i^2l_i^2w^2)^{1/2}]/2c_i^2l_i^2\ . \tag{30}$$

Always one value of k^2 is positive and one value negative, and we denote the positive value by k_i^2, the negative value by $-m_i^2$. For $l_i \ll c_i/w$, these are approximately

$$k_i \approx \frac{w}{c_i}\left[1 - \frac{2w^2l_i^2}{c_i^2}\right] \quad \text{and} \quad m_i \approx l_i^{-1}.$$

Thus there are two wave solutions in each mode for fixed frequency, one of which is propagating with slight dispersion, the other nonpropagating with decay length l_i.

6. Vibrations of a slab. Consider a slab with plane faces $y = \pm b$; these surfaces are traction-free and no body forces are acting. We shall look for solutions of the types

$$\text{(a)}\quad \mathbf{H}_x = \mathbf{H}_y = \mathbf{0}, \qquad \mathbf{H}_z = f(y)e^{iwt};$$

$$\text{(b)}\quad \phi = g(y)e^{iwt}$$

Then, from Eq. (27),

$$\left(\frac{d^2}{dy^2} + k_1^2\right)\left(\frac{d^2}{dy^2} - m_1^2\right)g(y) = 0 \tag{31}$$

$$\left(\frac{d^2}{dy^2} + k_2^2\right)\left(\frac{d^2}{dy^2} - m_2^2\right)f(y) = 0, \tag{32}$$

where k_i^2 and m_i^2 are given in Eq. (30).

For the boundary conditions (11) and (12) there is no x- or z-dependence of $M_{i\alpha\beta}$, so the tangential derivative of $M_{i\alpha\beta}$ in the last term of (12) vanishes. Thus, at $y = \pm b$,

$$M_{i\alpha\beta}N_\alpha N_\beta = M_{i22} = 0 \qquad T_{i\alpha}N_\alpha = T_{i2} = 0$$

In case (a), $u_x = f'(y)e^{i\omega t}$, $u_y = u_z = 0$. From Eqs. (22) and (23),

$$M_{\kappa 22} = \mu l_2^2 \frac{d^2}{dy^2} u_\kappa = \mu l_2^2 f'''(y) e^{i\omega t}\delta_{\kappa 1}$$

$$T_{\kappa 2} = \mu \frac{d}{dy} u_\kappa - \mu l_2^2 \frac{d^3}{dy^3} u_\kappa = \mu \delta_{\kappa 1} e^{i\omega t}[f''(y) - l_2^2 f^{(iv)}(y)].$$

The general solution of Eq. (32) is

$$f(y) = A \cos k_2 y + B \sin k_2 y + C \cosh m_2 y + D \sinh m_2 y.$$

Applying the four boundary conditions and eliminating A, B, C and D leads to two possible solutions:

$$\text{(i)} \quad A = C = 0 \qquad (f(y) \quad \text{is odd})$$

$$\tan (k_2 b) = -\left(\frac{k_2}{m_2}\right)^3 \tanh (m_2 b) \tag{34}$$

$$\text{(ii)} \quad B = D = 0 \qquad (f(y) \quad \text{is even})$$

$$\tan (k_2 b) = \left(\frac{m_2}{k_2}\right)^3 \tanh (m_2 b), \tag{35}$$

which is the same equation as for Mindlin and Tiersten's antisymmetric thickness shear wave.

In case (b), $u_x = u_z = 0$, $u_y = g'(y)e^{i\omega t}$. Then from Eqs. (22) and (23),

$$M_{\kappa 22} = (\lambda + 2\mu) l_1^2 g'''(y) e^{i\omega t} \delta_{\kappa 2},$$

$$T_{\kappa 2} = (\lambda + 2\mu)\delta_{\kappa 2} e^{i\omega t}[g''(y) - l_1^2 g^{(iv)}(y)].$$

The general solution of Eq. (31) is as above with $k_2 m_2 \to k_1 m_1$, and applying the boundary conditions we obtain a corresponding result. There are solutions with $g(y)$ an odd function of y, with eigenfrequencies determined by the equation

$$\tan (k_1 b) = -(k_1/m_1)^3 \tanh (m_1 b) \tag{36}$$

and even solutions with eigenfrequency equation

$$\tan (k_1 b) = (m_1/k_1)^3 \tanh (m_1 b). \tag{37}$$

7. Torsional vibrations of a circular cylinder. Consider the deformation obtained by taking $\varphi = 0$ and $H_r = H_\theta = 0$, $H_z = f(r)e^{i\omega t}$, where (r, θ, z) are cylindrical polar coordinates. The only nonzero displacement component is u_θ, whose physical component is

$$\hat{u}_\theta = -f'(r)e^{i\omega t}.$$

Substituting $\mathbf{H}$ into Eq. (27) gives

$$(\nabla^2 + k_2^2)(\nabla^2 - m_2^2) f(r) = 0 \tag{38}$$

For the torsion of a solid cylinder we need the solution to be regular at $r = 0$; hence

$$f(r) = AJ_0(k_2 r) + BI_0(m_2 r). \tag{39}$$

It is convenient to write formulas (22) and (23) in covariant form. Noting that for the present deformation $\Delta = 0$, and using a semicolon to denote covariant differentiation,

$$T_{\kappa\lambda} = \mu\{(u_{\kappa;\lambda} + u_{\lambda;\mu}) - l_2^2\nabla^2(u_{\kappa;\lambda} + u_{\lambda;\kappa})\}; \tag{40}$$

$$\begin{aligned} M_{\kappa\lambda\mu} = {} & \tfrac{1}{2}F_2 g_{\kappa\lambda}\nabla^2 u_\mu + \tfrac{1}{2}F_3(g_{\lambda\mu}\nabla^2 u_\kappa + g_{\kappa\mu}\nabla^2 u_\lambda) \\ & + (\tfrac{1}{2}F_5 + F_4)(u_{\kappa;\lambda;\mu} + u_{\lambda;\kappa;\mu}) + F_5 u_{\mu;\kappa;\lambda}\,. \end{aligned} \tag{41}$$

The only nonzero components of $\nabla^2 u_\alpha$ and $\nabla^2 u_{\alpha;\beta}$ are

$$\nabla^2 u_\theta = \left\{-g''(r) + \frac{1}{r}\,g'(r)\right\},$$

$$\nabla^2 u_{\theta;r} = \left\{-g'''(r) + \frac{2}{r}\,g''(r) - \frac{2}{r^2}\,g'(r)\right\},$$

$$\nabla^2 u_{r;\theta} = \left\{\frac{1}{r}\,g''(r) - \frac{1}{r^2}\,g'(r)\right\},$$

where $g(r) = rf'(r) = -u_\theta e^{-i\omega t}$. Furthermore,

$$u_{r;r;r} = u_{\theta;r;\theta} = u_{\theta;\theta;r} = u_{r;\theta;\theta} = 0,$$

$$u_{r;r;\theta} = u_{r;\theta;r} = \left\{\frac{1}{r}\,g'(r) - \frac{2}{r^2}\,g(r)\right\},$$

$$u_{\theta;r;r} = \left\{-g''(r) + \frac{2}{r}\,g'(r) - \frac{2}{r^2}\,g(r)\right\},$$

$$u_{\theta;\theta;\theta} = \{-rg'(r) + 2g(r)\}.$$

Consequently the only nonzero stress components are

$$T_{r\theta} = T_{\theta r} = \mu\left\{-g'(r) + \frac{2}{r}\,g(r) + l_2^2\left[-g'''(r) + \frac{3}{r}\,g''(r) - \frac{3}{r^2}\,g'(r)\right]\right\}, \tag{42}$$

$$M_{rr\theta} = \tfrac{1}{2}F_2\left(-g'' + \frac{1}{r}\,g'\right) + 2F_4\left(\frac{1}{r}\,g' - \frac{2}{r^2}\,g\right) + F_5\left(-g'' + \frac{3}{r}\,g' - \frac{4}{r^2}\,g\right), \tag{43}$$

$$\begin{aligned} M_{r\theta r} = M_{\theta rr} = {} & \tfrac{1}{2}F_3\left(-g'' + \frac{1}{r}\,g'\right) + F_4\left(-g'' + \frac{3}{r}\,g' - \frac{4}{r^2}\,g\right) \\ & + \tfrac{1}{2}F_5\left(-g'' + \frac{5}{r}\,g' - \frac{8}{r^2}\,g\right), \end{aligned} \tag{44}$$

$$M_{\theta\theta\theta} = (\tfrac{1}{2}F_2 + F_3)(-r^2 g'' + rg') + 2(F_4 + F_5)(-rg' + 2g). \tag{45}$$

The condition that the outer surface ($r = a$) of the cylinder be free of tractions is obtained by substituting these results into Eqs. (11) and (12). The first of these gives:

$$M_{irr} = 0 \quad \text{at} \quad r = a.$$

$$T_{i\alpha}N^\alpha - g^{\alpha\gamma}(M_{i\gamma\beta}N^\beta)_{,\alpha} + N^\gamma N^\delta(M_{i\gamma\beta}N^\beta)_{,\delta} = 0.$$

For $i = r$ and $i = \theta$ respectively we obtain from this that on $r = a$

$$T_{rr} + \frac{2}{r} M_{\theta rr} = 0, \qquad T_{r\theta} - 2rM_{r\theta r} = 0.$$

The first of these is identically satisfied since $M_{\theta rr} = 0$ is one of the hyperstress conditions, while the second reduces to $T_{r\theta} = 0$. Using the above expressions for stress and replacing $g(r)$ by $rf'(r)$ gives, on $r = a$:

$$f'' - \frac{1}{r} f' - l_2^2\left[f^{iv} - \frac{3}{r^2} f'' + \frac{3}{r^3} f'\right] = 0 \tag{46}$$

$$\tfrac{1}{2}F_3\left[-f''' - \frac{1}{r} f'' + \frac{1}{r^2} f'\right] + F_4\left[-f''' + \frac{1}{r} f'' - \frac{1}{r^2} f'\right] + \tfrac{1}{2}F_5\left[-f''' + \frac{3}{r} f'' - \frac{3}{r^2} f'\right] = 0. \tag{47}$$

If we now substitute the solution (39) into these conditions and eliminate A/B we get an equation for the eigenfrequencies. This simplifies to

$$(F_3 + F_4)\{y^3 I_1(y)J_2(x) + x^3 I_2(y)J_1(x)\} + (F_4 + F_5)\{y^3 I_3(y)J_2(x) - x^3 I_2(y)J_3(x)\} = 0 \tag{48}$$

where $x = k_2 a$ and $y = m_2 a$.

This frequency equation involves the elastic constants in combinations other than l_2^2, and hence differs from the analogous result of Mindlin and Tiersten. It can be shown to reduce to their result when $F_4 + F_5 = 0$.

8. Stability of equilibrium. For the natural state to be stable W must be a positive definite function of the strains and strain gradients. For the strain-dependence this leads to the usual requirements $\mu > 0$ and $3\lambda + 2\mu > 0$ while the strain gradient criterion will lead to a series of inequalities for $F_1, \cdots, F_5$.

In terms of the displacement gradients,

$$W = (F_1 + \tfrac{1}{2}F_2 + \tfrac{1}{4}F_3)u_{\alpha,\alpha\gamma}u_{\beta,\beta\gamma} + (\tfrac{1}{2}F_2 + \tfrac{1}{2}F_3)u_{\alpha,\alpha\beta}u_{\beta,\gamma\gamma} + \tfrac{1}{4}F_3 u_{\alpha,\beta\beta}u_{\alpha,\gamma\gamma} + (\tfrac{1}{2}F_4 + \tfrac{1}{4}F_5)u_{\alpha,\beta\gamma}u_{\alpha,\beta\gamma} + (\tfrac{1}{2}F_4 + \tfrac{3}{4}F_5)u_{\alpha,\beta\gamma}u_{\beta,\gamma\alpha}. \tag{49}$$

Thus

$$\frac{\partial^2 W}{\partial u_{\kappa,\lambda\mu}\,\partial u_{\rho,\sigma\tau}} = (2F_1 + F_2 + \tfrac{1}{2}F_3)\delta_{\kappa\lambda}\delta_{\rho\sigma}\delta_{\mu\tau} + (\tfrac{1}{2}F_2 + \tfrac{1}{2}F_3)(\delta_{\kappa\lambda}\delta_{\mu\rho}\delta_{\sigma\tau} + \delta_{\lambda\mu}\delta_{\rho\sigma}\delta_{\kappa\tau}) + \tfrac{1}{2}F_3\delta_{\lambda\mu}\delta_{\kappa\rho}\delta_{\sigma\tau} + (F_4 + \tfrac{1}{2}F_5)\delta_{\kappa\rho}\delta_{\lambda\sigma}\delta_{\mu\tau} + (\tfrac{1}{2}F_4 + \tfrac{3}{4}F_5)(\delta_{\lambda\rho}\delta_{\kappa\sigma}\delta_{\mu\tau} + \delta_{\mu\rho}\delta_{\kappa\sigma}\delta_{\lambda\tau})$$

and in particular

$$\frac{\partial^2 W}{\partial u_{\kappa,\lambda\mu}^2} = (2F_1 + F_2 + \tfrac{1}{2}F_3 + \tfrac{1}{2}F_4 + \tfrac{3}{4}F_5)\delta_{\kappa\lambda} + \tfrac{1}{2}F_3\delta_{\lambda\mu} + (F_4 + \tfrac{1}{2}F_5) + (F_2 + F_3 + \tfrac{1}{2}F_4 + \tfrac{3}{4}F_5)\delta_{\kappa\lambda}\delta_{\lambda\mu}.$$

It is certainly necessary that all these latter quantities should, when added to the corresponding quantity with $\lambda \leftrightarrow \mu$, be nonnegative, which leads to the conditions:

$$\begin{aligned} &F_4 + \tfrac{1}{2}F_5 \geq 0 \\ &2F_1 + F_2 + \tfrac{1}{2}F_3 + \tfrac{5}{2}F_4 + \tfrac{7}{4}F_5 \geq 0 \\ &\tfrac{1}{2}F_3 + F_4 + \tfrac{1}{2}F_5 \geq 0 \\ &F_1 + F_2 + F_3 + F_4 + F_5 \geq 0. \end{aligned} \tag{50}$$

The last two of these are just the requirements that $l_1^2 \geq 0$ and $l_2^2 \geq 0$.

The displacement gradients are independent quantities except that $u_{\alpha,\beta\gamma} = u_{\alpha,\gamma\beta}$, so that we require the 18×18 matrix whose elements are

$$\frac{\partial^2 W}{\partial u_{\kappa,(\lambda\mu)}\, \partial u_{\rho,(\sigma\tau)}} (2 - \delta_{\lambda\mu})(2 - \delta_{\sigma\tau}) = \mathfrak{M}_{\kappa\lambda\mu,\rho\sigma\tau}$$

to be positive definite. Consideration of the 2×2 principal minors of this matrix leads to the further conditions that

$$\begin{aligned} &F_3 + F_4 + \tfrac{1}{2}F_5 \geq 0 \\ &2G_3G_4 - (\tfrac{1}{2}F_2 + F_3)^2 \geq 0 \\ &2G_2G_4 - (2F_1 + \tfrac{3}{2}F_2 + F_3)^2 \geq 0 \\ &2G_2G_3 - (\tfrac{1}{2}F_2 + \tfrac{1}{2}F_3 + F_4 + \tfrac{3}{2}F_5)^2 \geq 0 \\ &2G_2G_3 - (\tfrac{1}{2}F_2 + F_3)^2 \geq 0 \\ &G_2G_3 - (\tfrac{1}{2}F_2 + \tfrac{1}{2}F_3 + \tfrac{1}{2}F_4 + \tfrac{3}{4}F_5)^2 \geq 0 \\ &2G_2G_4 - (2F_1 + \tfrac{3}{2}F_2 + F_3 + \tfrac{1}{2}F_4 + \tfrac{3}{4}F_5)^2 \geq 0 \\ &2G_2G_4 - (2F_1 + \tfrac{3}{2}F_2 + F_3 + F_4 + \tfrac{3}{2}F_5)^2 \geq 0 \\ &4F_1 + 2F_2 + F_3 + \tfrac{5}{2}F_4 + \tfrac{7}{4}F_5 \geq 0 \\ &F_4 + \tfrac{1}{2}F_5 \geq |\tfrac{1}{2}F_4 + \tfrac{3}{4}F_5| \end{aligned}$$

where $G_1, \cdots, G_4$ are the four expressions in Eq. (50).

9. Concluding remarks. The material of Mindlin and Tiersten is a special case of the present class of materials. These authors consider, in the small-strain case, a strain-energy density of the form

$$W = \tfrac{1}{2}\lambda(e_{\alpha\alpha})^2 + \mu e_{\alpha\beta}e_{\alpha\beta} + 2\eta\kappa_{\alpha\beta}\kappa_{\alpha\beta} + 2\eta'\kappa_{\alpha\beta}\kappa_{\beta\alpha}$$

where $\kappa_{\alpha\beta} = \epsilon_{\beta\gamma\delta}e_{\alpha\gamma,\delta}$. Comparing with Eq. (19) we get

$$F_1 = F_3 = -2\eta', \quad F_2 = 4\eta', \quad F_4 = -F_5 = 2(\eta + \eta'). \tag{51}$$

In particular this leads to $l_1^2 = 0$, $l_2^2 = \eta/\mu$, giving no dispersion of the dilatational waves.

A straightforward extension of the materials we have considered is obtained by allowing the dipolar velocities $\dot{x}_{i,\alpha}$ to enter the action density. The simplest case is to assume that this quantity consists of separate kinetic and potential parts. For small strains and strain-rates we would consider only quadratic dependences:

$$L = \tfrac{1}{2}\rho\dot{x}_i\dot{x}_i + \tfrac{1}{2}\varphi\dot{x}_{i,\alpha}\dot{x}_{i,\alpha} - W(x_{i,\alpha}x_{i,\alpha\beta}).$$

Then $Q_{i\alpha} = \varphi \dot{x}_{i,\alpha}$ and $P_i = \rho \dot{x}_i - \varphi \dot{x}_{i,\alpha\alpha}$, and the equations of motion (24), in the small-strain approximation, become

$$\rho \ddot{u}_\alpha - \varphi \ddot{u}_{\alpha,\beta\beta} - F_\alpha + C_{\alpha\beta,\beta} = (\lambda + \mu)\,\partial_\alpha \Delta + \mu \nabla^2 u_\alpha - \eta_1\,\partial_\alpha \nabla^2 \Delta - \eta_2 \nabla^4 u_\alpha\,.$$

In the boundary conditions, (12), $\varphi \ddot{u}_{i,\alpha}$ is added to $T_{i\alpha}$.

The dispersion equation for plane waves becomes

$$w^2(1 + \varphi k^2/\rho) = c_i^2 k^2 (1 + l_i^2 k^2)$$

which has the same qualitative features as the earlier equation. If we now let k_i^2, $-m_i^2$ be the two solutions of this equation for fixed w, then for the vibration frequencies of a finite slab we obtain the same equations as in Sec. 6 (Eqs. (34)–(37)). However, Eq. (48) for the torsional frequencies of a circular cylinder is changed by the new term in the boundary conditions and becomes

$$(F_3 + F_4)\left\{y^3 I_1(y) J_2(x) + x^3 I_2(y) J_1(x) + \frac{\phi}{\rho a^2}(y^2 - x^2) xy I_1(y) J_1(x)\right\}$$
$$+ (F_4 + F_5)\left\{y^3 I_3(y) J_2(x) - x^3 I_2(y) J_3(x) + \frac{\phi}{\rho a^2}\left[xy^3 I_3(y) J_1(x) + x^3 y I_1(y) J_3(x)\right]\right\} = 0.$$

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