—NOTES—

ON INCIDENCE MATRICES*

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Given an $m \times n$ incidence matrix $A(m, n)$, it is desired to "squeeze" as many of its 1's as possible into its upper left-hand corner by a sequence of row and column permutations. Such problems arise in designing switches for computers. To establish criteria for the optimal matrix of the entire class of optimal row and column permutations, we assign a weight w_{ij} to each position in the original matrix, where w_{ij} is the weight assigned to the ith row, jth column position. The weight of a matrix $A(m, n) = (a_{ij})$ is taken to be

$$\sum_{1 \le i \le m, 1 \le j \le n} a_{ij} w_{ij} .$$

We take w_{ij} to be strictly increasing in both i and j so that a 1 in the (i, j)th position is preferred to a 1 in the $(i, j + 1)$th, and so on. The problem, then, reduces to finding the matrix, derived from the original by arbitrary row permutations and column permutations, such that its weight is minimal among all such (finitely many) permutation-derived matrices. For basic terms used in this note, we refer to Ryser [1].

We first consider the case in which $w_{ij} = f(i) + g(j)$, where f and g are strictly increasing in both i and j, respectively. Under these conditions, we have the following characterization.

PROPOSITION 1. *A necessary and sufficient condition that an incidence matrix $A(m, n)$ be optimal (i.e., minimal-weight) is that it be monotone. That is, if $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ are the row sum and column sum vectors, then $r_1 \ge r_2 \ge \cdots \ge r_m$ and $s_1 \ge s_2 \ge \cdots \ge s_n$.*

Proof. Note that if A is the matrix and $\bar{W}(A)$ is its weight, then

$$\begin{aligned} \bar{W}(A) &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} w_{ij} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} [f(i) + g(j)] \\ &= \sum_{i=1}^{m} f(i) r_i + \sum_{j=1}^{n} g(j) s_j \end{aligned}$$

where we note that $\sum_{i=1}^{m} a_{ij} =$ number of 1's in column $j = s_j$ and, similarly, that $\sum_{j=1}^{n} a_{ij} = r_i$. In a matrix A, with $p > q$ and $r_q < r_p$, consider the matrix A' derived

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from A by permuting rows p and q. Now a row permutation clearly does not affect the number of 1's on a given column so that, if $R' = (r'_1, \cdots, r'_m)$ and $S' = (s'_1, \cdots s'_n)$ are the associated row and column vectors of A', then $S' = S$ and $r_i = r'_i$, $i \neq p, q$; $r_p = r'_q$, and $r_q = r'_p$. But

$$p > q \quad \text{implies} \quad f(p) > f(q)$$

and

$$r_p > r_q \quad \text{implies} \quad r_p f(p) + r_q f(q) > r_p f(q) + r_q f(p).$$

This follows from the fact that

$$(r_p - r_q)(f(p) - f(q)) > 0 \quad \text{implies} \quad r_p f(p) + r_q f(q) - (r_p f(q) + r_q f(p)) > 0.$$

However, we then have, since $r_p = r'_q$, $r_q = r'_p$, that

$$r_p f(p) + r_q f(q) > r'_p f(p) + r'_q f(q).$$

Since all other row and column sums are unaltered, we have

$$\begin{aligned} \bar{W}(A) &= \sum_i r_i f(i) + \sum_j s_j g(j) \\ &> \sum_i r'_i f(i) + \sum_j s'_j g(j) = \bar{W}(A'). \end{aligned}$$

Hence A is not an optimal matrix. To prove sufficiency, we need only note that $\bar{W}(A)$ depends on A only through its row and column sum vectors since the number of 1's in the given lines is unaltered by any sequence of permutations. Thus we see that the row and column sum vectors of the monotone matrices derivable from the given matrix are identical. Hence the weights for all monotone matrices are the same. From the necessity of monotonicity the conclusion follows.

This solves the question of optimality for the special weighting $f(i) + g(j)$. It seems reasonable next to consider weights $w_{ij} = f(i)g(j)$ where f and g were as above. The question is not so simple, as the following examples show.

In the case of a sum weighting there existed a sequence of permutations which decreased the weights of the intermediate matrices monotonically.[1] This is not the case for a product weighting.

EXAMPLE 1. Consider the weight $w_{ij} = ij$ and the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then

$$W(A) = 2 + 4 + 6 + 4 + 5 = 21.$$

It is clear that one column permutation or one row permutation each increases the weight in the resultant matrix. However,

$$A^* = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

is derivable from A and $W(A^*) = 2 + 4 + 3 + 4 + 5 = 18 < 21$.

[1] Just permute pairs of rows (columns) which are not monotone in the row (column) sum vector. This follows the proof of necessity.

At this point we should include a procedure for minimization by permutations of rows (columns) only against a given order of columns (rows): Compute the value of each row were it in the first position in the new matrix. Permute the rows so that the row with the greatest initial weight is first, and proceed in order, decreasing the rows in order of diminishing initial weight. Note that if the initial values of rows $1, \cdots, m$ were $I_1, I_2, \cdots, I_m$, then the contribution of the ith row in the kth position of the new matrix is

$$\begin{aligned} \frac{f(k)}{f(1)} I_i &= \frac{f(k)}{f(1)} \sum_{j=1}^{n} a_{ij} f(1) f(j) \\ &= \sum_{j=1}^{n} a_{ij} f(k) f(j). \end{aligned}$$

But $f(k)/f(1)$ is increasing in k. The same argument is used in showing the necessity of monotonicity in the $f + g$ weighting with I_i taking the role of r_i.

EXAMPLE 2. Monotonicity is unnecessary in either row or column in the $3 \times n$ case, as is shown by the following matrix A, with $w_{ij} = ij$,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here A is minimal for this particular sequence of rows and

$$W(A) = 4 + 8 + 12 + 16 + 20 + 12 + 14 + 16 + 18 = 120.$$

On the other hand,

$$A^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

which is minimal against a monotone row and yields

$$W(A^*) = 3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 = 135.$$

In the square matrix $A(4, 4)$ the necessity of monotonicity is also contradicted in the case $w_{ij} = ij$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Here

$$W(A) = 3 + 6 + 12 + 12 = 33.$$

This is contrasted with

$$A^* = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where

$$W(A^*) = 4 + 6 + 9 + 12 = 31$$

and it can be shown to be minimal.

Since monotonicity seems to be a "nice" necessary condition (i.e., we have an easy characterization) we now develop a characterization of the functions f, g for which fg is such a weighting.

Proposition 2. *Let $w_{ij} = f(i)g(j)$ and let h represent either f or g (i.e., both must satisfy the following criterion). Then a necessary and sufficient condition that a necessary condition for optimization is monotonicity of row and column sum vector is*

$$\sum_1^n h(i) > \sum_0^{n-2} h(m - i) \qquad \text{for all} \quad m \geq n \geq 2$$

with m, n chosen so that m is in the domain of definition of h.

Proof. Assume that for a given weighting of columns g, there exist m and n for which $\sum_1^n g(i) < \sum_0^{n-2} g(m - i)$. If $m - (2n - 1) > 2$ we may increase n by 1, for the increase will merely add $g(n + 1)$ to the left-hand sum and $g(m - n + 1)$ to the right-hand sum and $m - (2n - 1) > 2 - m - (2(n + 1) - 1) > 0$. Thus the two sums do not overlap and the inequality is preserved. We might then as well assume that $m - (2n - 1) \leq 2$. Hence if we constructed two rows, one of n 1's in the first n of $m(m \leq 2n)$ positions and a second of $(n - 1)$ 1's in the last $n - 1$ positions, they would not overlap and there would exist at most one 0 in common at the same column positions of both rows. Now assume an f row-weighting. The $g(j)f(i)$ combination had as a necessary condition for all matrices, monotone row and column sum vectors. We then consider one of the two following matrices:

$$\begin{array}{c} \begin{matrix} \overbrace{}^{n} & & \overbrace{}^{n-1} \end{matrix} \\ \begin{bmatrix} 1 & 1 \cdots 1 & 1 & 0 \cdots 0 \\ 1 & 1 \cdots 1 & 1 & 0 \cdots 0 \\ 1 & 1 \cdots 1 & 0 & 0 \cdots 0 \\ 0 & 0 \cdots 0 & 0 & 1 \cdots 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} \begin{matrix} \overbrace{}^{n} & \overbrace{}^{n-1} \end{matrix} \\ \begin{bmatrix} 1 & 1 \cdots 1 & 0 \cdots 0 \\ 1 & 1 \cdots 1 & 0 \cdots 0 \\ 0 & \cdots 0 & 1 \cdots 1 \end{bmatrix} \end{array}$$

In either case, there is but one monotone matrix obtainable, and by the "necessary" conditions they must be minimal. However, the criteria for minimization by row permutation only gives a lesser weight by:

$$\begin{array}{c} \overbrace{\hphantom{1\;\;1\cdots 1}}^{n}\;\;\hphantom{1}\;\;\overbrace{\hphantom{0\cdots 0}}^{n-1} \\ \begin{bmatrix} 1 & 1\cdots 1 & 1 & 0\cdots 0 \\ 1 & 1\cdots 1 & 1 & 0\cdots 0 \\ 0 & 0\cdots 0 & 0 & 1\cdots 1 \\ 1 & 1\cdots 1 & 0 & 0\cdots 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \overbrace{\hphantom{0\cdots 0}}^{n}\;\;\overbrace{\hphantom{1\cdots 1}}^{n-1} \\ \begin{bmatrix} 0\cdots 0 & 1\cdots 1 \\ 1\cdots 1 & 0\cdots 0 \\ 1\cdots 1 & 0\cdots 0 \end{bmatrix} \end{array}$$

Hence the assertion.

Conversely, the proof will make use of the optimization criteria developed between Examples 2 and 1. Note that if row k contains r_k 1's and row l has r_l 1's, then for row k to precede row l in the optimal matrix, with $r_k > r_l$, the sum of g-weights of row k (denoted G_k) must exceed the sum of g-weights in row l (G_l). Denote by G_k' and G_l' the sum of g-weights associated with 1's in rows k and l, respectively, with the 1's placed in the first r_k positions for G_k' and the last r_l positions for computation of G_l'. By monotonicity of g, $G_k' \leq G_k$ and $G_k' \geq G_l$. Hence $G_k' > G_l' \Rightarrow G_k > G_l$. Also, since the row for G_k' contains more 1's than that for G_l', in order that the monotonicity of the optimal matrix hold we have $G_k' > G_l'$. Hence the necessity of monotonicity of the rows in the optimal matrix is equivalent to the statement: "In minimizing by permutation of rows only, if $k > l$ and k 1's appear in the first k places of a row and l 1's appear in the last l places, then the row of k 1's appears before that of l 1's." From the criteria for minimization by column permutation this becomes

$$\sum_{j=1}^{n} f(1)g(j) > \sum_{j=0}^{n-2} f(1)g(m-j)$$

where we may take m to be the index at which the last 1 of the shorter row appears. But the above is equivalent to

$$\sum_{1}^{n} g(j) > \sum_{0}^{n-2} g(m-j) \qquad \text{for all} \quad m \geq n \geq 2.$$

Hence the assertion. A similar proof can be made for f. Note that the above proposition, in the case $n = 2$, implies that

$$f(1) + f(2) > f(m) \quad \text{for all} \quad m$$

and

$$g(1) + g(2) > g(m) \quad \text{for all} \quad m.$$

This implies that f and g must be bounded (increasing functions). We now further characterize these weights:

PROPOSITION 3. *Let $f(i)$ (or $g(j)$) be a weight function component giving rise to the necessity condition of the preceding proposition. Then $f(i)$ must be of the form $k - h(i)$ where $h(i)$ is strictly decreasing, $k > 0$, and an upper bound for h and for $p = \lim_{i\to\infty} h(i)$, is given by*

$$k - p \geq \sum_{1}^{\infty} (h(i) - p).$$

Further, any f of this form can be used in a "nice" product weighting.

Proof of sufficiency. If f is "nice", the preceding proposition gives $\sum_1^n f(i) > \sum_0^{n-2} f(m - i)$ for all m in the domain of definition of f. [We assume here that f can be extended, if necessary, to let the criterion for all n, and show later that such is always possible.] The previous remark gives $f(1) + f(2) = k > f(m)$, since f is increasing, $k - f(i)$ is decreasing, and $f = k - (k - f)$. Define $h(i) = k - f(i)$. Now $k - f(i) > k - f(1) = f(2)$ so $h(i)$ is decreasing and bounded below by $f(2) > 0$; hence there exists p such that $f(i) \to p > 0$ and $f(i) > p$ for each $i = 1, 2, \cdots$.

Now from Proposition 2, we have

$$\sum_1^n k - h(i) > \sum_0^{n-2} k - h(m - i),$$

or

$$k - \sum_1^n h(i) > -\sum_0^{n-2} h(m - i),$$

or

$$k > \sum_2^n [h(i) - h(m - n + i)] + h(1),$$

$$k > h(i) + \lim_{m\to\infty} \sum_2^n [h(i) - h(m - n + 1)],$$

$$k > h(1) + \sum_2^n (h(i) - p),$$

$$k - p > \sum_1^n (h(i) - p).$$

But $h(i) - p > 0$, so that the above sum has a limit and

$$k - p \geq \sum_1^\infty (h(i) - p).$$

Proof of necessity. Now if $k - p \geq \sum_1^\infty (h(i) - p)$ and $f(i) = k - h(i)$, where $h(i)$ is decreasing and $h(i) \to p$, then $h(i) > p$ and $k - p > \sum_1^n (h(i) - p)$ for any n. But $p < h(j)$ for all j, specifically for $j = m - n + 1$, so that

$$k - p \geq \sum_1^\infty (h(i) - p) > \sum_1^n (h(i) - p) > \sum_2^n (h(i) - h(m - n + i) + h(1) - p)$$

$$\Leftrightarrow k > h(1) + \sum_2^n h(i) - h(m - n + 1),$$

$$nk - \sum_1^n h(i) > (n - 1)k - \sum_0^{n-2} h(m - i),$$

$$\sum_1^n f(i) > \sum_0^{n-2} f(m - i).$$

This is the criterion of the second proposition. Hence the assertion.

PROPOSITION 4. *If $f(i)$ $(f(j))$ is defined and satisfied the criteria of Proposition* 1 *for* $i = 1, 2, \cdots, k$, *it can be extended on* $k+1, k+2, \cdots$.

Proof. Define $f(i) = f(k)/2^{i-k}$ for $i > k$. For $m \leq k$ the assertion of Proposition 1 is true by assumption. If $m > k$ and $n < k$, we note that the $n-1$ 1's in the $k, k-1, \cdots, k-n+2$ positions, "outweight" those in the $m, m-1, m-2, \cdots, m-k+2$ positions. If $m > k$ and $n > k$, we have by assumption that the 1's in positions $2, \cdots, k$ at least equal the weight of 1's in the first k positions of the lesser set, and the first 1 outweighs the kth 1 which outweighs any number of 1's which go after it. Hence the assertion.

In particular, Proposition 3 permits any decreasing sequence whose series converges as a possible weight component. For example, a necessary condition for minimization with weight function $w_{ij} = (1-e^{-i})(1-e^{-j})$ are monotone row and column sum vectors.

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References

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