EIGENFUNCTIONS CORRESPONDING TO THE BAND PASS KERNEL WITH LARGE CENTER FREQUENCY*

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Abstract. It is known that the eigenfunctions of the finite integral equation with band limited difference kernel are the prolate spheroidal wave functions. When the band pass difference kernel was considered, no such well-known functions were found, except for the limiting case of zero band width. In this report we show that for large values of the center frequency the eigenfunctions corresponding to the band pass kernel are related to the prolate spheroidal functions.

1. Introduction. Slepian [1] considered the following integral equation with band limited kernel,

$$\lambda u(t) = \int_{-1}^{1} \rho(t - s)u(s)\, ds \equiv Ku(t) \tag{1.1}$$

where

$$\rho(t) = (\pi t)^{-1} \sin ct \equiv \rho_c(t) \tag{1.2}$$

and showed that the self-adjoint differential operator

$$L(t) = \left\{\frac{d}{dt}\left[(1 - t^2)\frac{d}{dt}\right] + (\chi - c^2t^2)\right\} \tag{1.3}$$

commutes with the integral operator K.

The solutions of the differential equation $Lu = 0$ for discrete values of χ are the prolate spheroidal wave functions $S_{0n}(c, t)$ and are in turn the eigenfunctions of the integral equation (1.1). Morrison [2] considered the above integral equation with band pass kernel

$$\rho_{a,c}(t) = (ct)^{-1} \sin ct \cos at; \quad a > c > 0. \tag{1.4}$$

and found a fourth order operator L_1 which under some prescribed conditions will commute with the integral operator defined by Eqs. (1.1) and (1.4). Morrison then outlined a method for solving the differential equation $L_1u = 0$ and gave eigenfunctions for the limiting case of

$$\rho(t) = \lim_{c\to 0} \rho_{a,c}(t) = \cos at; \qquad a > 0.$$

In this note we consider the integral equation with band pass kernel for large a. Our method is based on expanding the band pass kernel

$$\rho_{a,c}(t - s) = [c(t - s)]^{-1} \sin c(t - s) \cos a(t - s); \quad a > c > 0. \tag{1.5}$$

in a bilinear form of $S_{0n}(c, t) \cos at$ and $S_{0n}(c, t) \sin at$, to show that for large a, $S_{0n}(c, t)$ $\sin at$ and $S_{0n}(c, t) \cos at$ are the eigenfunctions of the integral equation (1.1) with the band pass kernel defined by Eq. (1.5).

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This method is in parallel to the method Stone et al. [3] used for solutions of the integral equation with the kernel of their second order filter.

2. Eigenfunctions for integral equation with band pass kernel of large center frequency. From Morse and Feshbach [4] we find the expansion in prolate spheroidal coordinates of the Green function e^{ikR}/R. If we equate the imaginary parts in this expansion, do the simple calculations for R and regard the two different notations in [4] and [1], we get

$$\{c(t-s)\}^{-1}\sin c(t-s)\cos a(t-s)$$
$$= \sum_n \frac{\pi}{c}\lambda_n[S_{0n}(c,t)\cos at S_{0n}(c,s)\cos as + S_{0n}(c,t)\sin at S_{0n}(c,s)\sin as] \tag{2.1}$$

where the sum is from $n = 0$ to ∞ and $S_{0n}(c, t)$ are the orthonormal prolate spheroidal wave functions satisfying

$$\lambda_n(c)S_{0n}(c,t) = \int_{-1}^{1}\rho_c(t-s)S_{0n}(c,s)\,ds. \tag{2.2}$$

We remark here that the above bilinear form (2.1) can be obtained with the aid of Mercer's theorem.

Consider (1.1) with the kernel (1.4) as represented by the series (2.1).

(A) Into this equation substitute $u(s) = S_{0l}\cos as$ to get

$$\lambda u(t) = \frac{\pi}{c}\sum\lambda_n\Bigg[S_{0n}(c,t)\cos at\int_{-1}^{1}S_{0n}(c,s)S_{0l}(c,s)\cos^2 as\,ds$$
$$+ S_{0n}(c,t)\sin at\int_{-1}^{1}S_{0n}(c,s)S_{0l}(c,s)\frac{\sin 2as}{2}\,ds\Bigg]. \tag{2.3}$$

Now write $\cos^2 as = \frac{1}{2}(\cos 2as + 1)$. Riemann's Theorem [5] shows that the integrals involving pairs of prolate functions vanish as $a \to \infty$. After using the orthogonality property of $S_{0n}(c, s)$ and letting $\lambda = \pi\lambda_l(c)/2c$ we finally obtain

$$u(t) = S_{0l}(c,t)\cos at; \quad a \text{ large}. \tag{2.4}$$

(B) If we substitute $u(s) = S_{0l}$ sin as in Eq. (1.1) with the kernel (1.4) as represented by (2.1) in the same way for (A) we obtain $\lambda = \pi\lambda_l(c)/2c$ and

$$u(t) = S_{0l}(c,t)\sin at; \quad a \text{ large}. \tag{2.5}$$

3. Comparison of results. Morrison's results for the limiting case of $c \to 0$ and $\lambda = 0$ are

$$u = \left(\frac{\cos 2a}{2a} - \frac{\sin 2a}{4a^2}\right)\left(1 + \frac{\sin 2a}{2a}\right)^{-1}\cos at + t\sin at \quad \text{(even)} \tag{3.1}$$

$$u = \left(\frac{\cos 2a}{2a} - \frac{\sin 2a}{4a^2}\right)\left(1 - \frac{\sin 2a}{2a}\right)^{-1}\sin at + t\cos at \quad \text{(odd)}. \tag{3.2}$$

For large enough a these will become

$$\begin{aligned} u &= t\sin at \quad \text{(even) } (a \text{ large}) \\ u &= t\cos at \quad \text{(odd) } (a \text{ large}). \end{aligned} \tag{3.3}$$

For this special case of $c \to 0$ the $S_{0n}(c, t)$ in Eqs. (2.4) and (2.5) reduces to $P_n(t)$, the Legendre polynomials, and for small c we find from Slepian [6] that

$$\lambda_1(c) = \frac{32}{\pi} c^3 \exp\left\{-\frac{3c^2[1 + o(c^2)]}{25}\right\}. \tag{3.4}$$

So for $n = 1$ our results are also $\lambda = \pi\lambda_1/2c = 0$ and

$$\begin{aligned} u &= P_1(t) \sin at = t \sin at \qquad \text{(even) } (a \text{ large}) \\ u &= P_1(t) \cos at = t \cos at \qquad \text{(odd) } (a \text{ large}). \end{aligned} \tag{3.5}$$

For the case of nonzero eigenvalue of the integral equation (1.1) with the kernel (1.4) and with $c = 0$, Morrison's results are $\lambda = 1 + \sin 2a/2a$ and $\lambda = 1 - \sin 2a/2a$ with eigenfunctions $\cos at$ and $\sin at$ respectively. For large enough values of a this will reduce to the degenerate eigenvalue $\lambda = 1$ which may be related to our $\lambda = \lim_{c\to 0} (\pi/c)(\lambda_0/2) = 1$ and the corresponding eigenfunctions with $n = 0$, i.e., $S_{00}(0, t) \cos at = \cos at$ and $S_{00}(0, t) \sin at = \sin at$.

4. Comments. It is clear that for practical purposes, the present results depend on how large a, the center frequency, should be. For this purpose we have attempted to calculate the integral in Eq. (1.1) with the kernel (1.4) for the available tabulated values of $S_{0n}(c, t)$ and $\lambda_n(c)$ with $n = 0, 1, 2$ and 3 and $c = 1, 2, 4$. The preliminary results tend to show that for $a \sim 10c$ to $20c$ and $c > \pi n/2$ the difference between the two sides of Eq. (1.1) is not more than 1 to 0.1 percent of the exact left side. This is not unreasonable when we consider that most authors assume such range for a and that the conditions $c \gg \pi n/2$ defines the essentially band limited functions.

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References

1. D. Slepian and H. Pollak, *Prolate spheroidal wave functions, Fourier analysis and uncertainty.-I,* Bell System Tech. J. **40**, 43–64 (1961)
2. J. A. Morrison, *Eigenfunctions corresponding to band pass kernel in the case of degeneracy,* Quart. Appl. Math. **21**, 13–19 (1963)
3. W. M. Stone and R. L. Brock, *On the probability of detection with a post detection filter,* Boeing D-16921, Seattle (1955)
4. P. M. Morse, and H. Feshbach, *Methods of theoretical physics,* Vol. 2, McGraw-Hill, New York, p. 1505, 1953
5. P. Franklin, *A treatise on advanced calculus,* John Wiley, New York, p. 480, 1940
6. D. Slepian, *Asymptotic expansions for prolate spheroidal wave functions,* J. Math. Phys. (2) **44**, 99–140 (1965)