ON CERTAIN CLOSED-FORM SOLUTIONS TO PROBLEMS OF WAVE PROPAGATION IN A STRAIN-HARDENING ROD*

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1. Introduction. As has been stressed in a recent paper [1], exact, closed-form solutions to problems of plastic wave propagation are valuable even if they are based on highly idealized or even outright unrealistic constitutive equations, because they furnish welcome means of checking numerical results furnished by computer codes that use discrete models of plastic continua. In the earlier paper, closed-form solutions were developed for the propagation of longitudinal waves in a semi-infinite rod of a rigid, linearly workhardening, locking material. This assumed mechanical behavior qualitatively resembles the behavior of earth materials in uniaxial strain. In the present paper, an alternative mechanical behavior is considered that more closely approaches the behavior of earth materials in uniaxial strain but still permits the development of closed-form solutions for some problems of plastic wave propagation. The assumed stress-strain curve (Fig. 1) consists of a straight segment OA through the origin followed by a curve ADFC that is convex towards the strain axis. The fact that the secant modulus increases along such a curve with increasing strain is in accordance with the observed behavior of many earth materials in uniaxial strain. In the earlier model, the locking feature provided a crude representation of this stiffening effect. On account of the possibility of permanant compaction of earth materials, the slope of the initial straight loading line tends to be considerably smaller than that of typical unloading lines. With the view to obtaining closed-form solutions, we shall idealize this type of behavior by stipulating that, in our model, unloading takes place under constant strain (e.g. DE in Fig. 1). The dashed lines in Fig. 1 schematically indicate the behavior of earth materials in uniaxial strain.

A semi-infinite rod consisting of a material of this type is supposed to be subjected to a compressive stress $p(t)$ at the end $x = 0$ that is suddenly raised at $t = 0$ from zero to a value p_0 in excess of the yield stress σ_y (see Fig. 1), and then monotonically reduced to zero during the time interval $0 < t \leq \tau$. A general discussion of the problem is presented in Sec. 2, and in Sec. 3 closed-form solutions are given when the stress-strain law in loading has the form

$$\sigma = K_0\epsilon \qquad \text{for} \quad 0 \leq \epsilon \leq \epsilon_y ,$$

$$= \sigma_y + (\sigma^* - \sigma_y)\left(\frac{\epsilon - \epsilon_y}{\epsilon^* - \epsilon_y}\right)^n \qquad \text{for} \quad \epsilon \geq \epsilon_y ,$$

where K_0, ϵ_y, ϵ^*, σ^*, and $n \geq 1$ are constants, and $\sigma_y = K_0\epsilon_y$ (see Fig. 2). No change of strain is supposed to occur in unloading (rigid unloading). Zvolinski and Rykov [2] have

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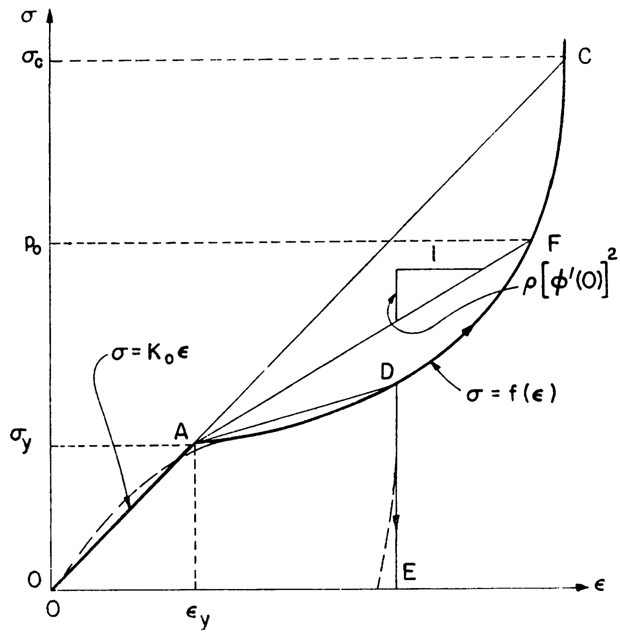

FIG. 1. Stress-strain diagram.

treated the special case where ϵ_y and hence σ_y vanish, but without obtaining closed form solutions except for the case of an endstress that is suddenly applied and thereafter kept constant or is kept constant for a finite duration of time and then suddenly decreased to zero. In treating the plastic deformation of a steel cylinder striking a target, Lee and Tupper [3] pointed out that an elastic, perfectly plastic relation between strain and *true* stress leads to a relation between strain and *nominal* stress represented by a curve that is convex towards the strain axis (and hence resembles the curves in Fig. 2).

2. General discussion of problem solution. To begin, we let t and x represent time and the Lagrangian coordinate along the rod, and denote by $\sigma(x, t)$, $u(x, t)$ and $v(x, t)$ the nominal stress, longitudinal displacement, and the particle velocity at time t and station x. Utilizing this notation and assuming that plane sections remain plane, the equation of motion of the rod can be written

$$\partial\sigma/\partial x + \rho\, \partial v/\partial t = 0. \tag{1}$$

Since $\epsilon = -\partial u/\partial x$ and $v = \partial u/\partial t$, the following compatibility equation must also be satisfied:

$$\partial\epsilon/\partial t + \partial v/\partial x = 0. \tag{2}$$

The constitutive relation of the rod material, which was qualitatively described in Sec. 1, furnishes the following additional equations (see Fig. 1):

(i) Loading ($d\sigma/dt > 0$):

$$\begin{aligned} \sigma &= K_0\epsilon \qquad \text{for} \quad 0 \le \epsilon \le \epsilon_y\,, \\ &= f(\epsilon) \qquad \text{for} \quad \epsilon \ge \epsilon_y\,. \end{aligned} \tag{3a}$$

It will be assumed that $f(\epsilon)$ in (3a) is of class C^2 in ϵ and satisfies the following conditions:

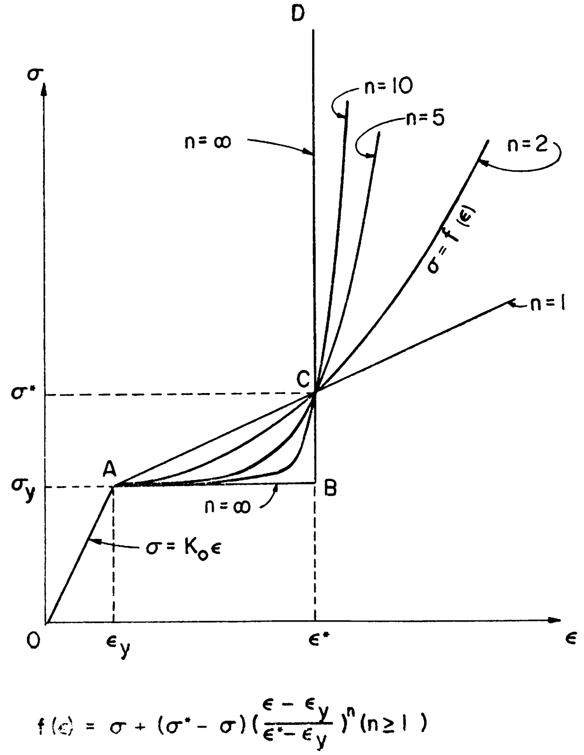

FIG. 2. Power stress-strain law.

$$f''(\epsilon) \geq 0, \quad f'(\epsilon_y^+) < K_0\,, \quad f'(\epsilon) \equiv K(\epsilon) > 0 \tag{3b}$$

where primes denote differentiation of a function with respect to its argument. The sign requirement on $f'(\epsilon)$ may be relaxed to include the case $f'(\epsilon) = 0$ at certain points of the interval. With a minor modification, the results to be derived in this section can easily be extended to the case for which $f'(\epsilon)$ and $f''(\epsilon)$ are discontinuous at a finite number of points (cf. Sec. 3).

(ii) Unloading ($d\sigma/dt \leq 0$):

$$\epsilon(x, t) = \epsilon_m(x) \quad \text{for} \quad \sigma(x, t) \leq \sigma_m(x) \tag{3c}$$

for all $\epsilon > 0$. Here $\epsilon_m(x)$ and $\sigma_m(x)$ denote the maximum strain and stress experienced at station x during the loading process.

For purposes of this study, the applied end stress $\sigma(0, t) \equiv p(t)$ is defined as

$$\begin{aligned} p(t) &= 0 \qquad \text{for} \quad t < 0, \\ p(0^+) &= p_0(>\sigma_y), \\ p'(t) &< 0 \qquad \text{for} \quad 0 < t < \tau, \\ p(t) &= 0 \qquad \text{for} \quad t \geq \tau. \end{aligned} \tag{4}$$

We shall assume that $p(t)$ is of class C^1 with respect to t in $0 < t < \tau$ (this continuity requirement may be relaxed to include a finite number of jumps in $p'(t)$, again with a minor modification of the analysis).

Eqs. (1)–(3), along with the boundary conditions (4) and proper initial conditions (which we assume to be zero), define the functions σ, ϵ, and v throughout the x, t-plane except at lines or curves of discontinuity, which are classified as strong (shock) or weak discontinuities according to whether these variables themselves or their derivatives are discontinuous. In the case of a strong discontinuity further relations are necessary to effect a solution. The latter take the form of jump conditions, obtained by the requirement of continuous displacements and the conservative of linear momentum across the shock; these are

$$\Delta\sigma = \rho\phi'(t)\Delta v \qquad \Delta v = \phi'(t)\Delta\epsilon. \tag{5}$$

The quantities $\Delta\sigma$, $\Delta\epsilon$, Δv in (5) denote jumps in σ, ϵ, v respectively across the shock and $x = \phi(t)$ defines the shock trajectory in the x, t-plane ($\phi'(t)$ is thus the velocity of propagation of the shock with respect to the undeformed configuration of the rod). At a weak discontinuity σ, ϵ, v are continuous and their values may be obtained by appropriate matching of solutions valid in adjacent regions.

Let σ_c denote the stress at the intersection point C (if any) of the curve $\sigma = f(\epsilon)$ (segment ABDG of Fig. 1) with the extension of the segment OA of Fig. 1. Three cases may arise in connection with the magnitude of the initial applied impact stress p_0 if $\sigma_c \neq \infty$: (i) $\sigma_y < p_0 < \sigma_c$, (ii) $p_0 \geq \sigma_c$ and (iii) $p_0 \leq \sigma_y$. Only the first case, which includes the third case as a special case, will be considered here in detail. The second case can be obtained from the first by a minor modification of the analysis, as will be shown at the end of this section.

Consider now the solution of the posed problem. Referring to Fig. 3, let us divide the x, t-plane into regions bounded by weak or strong discontinuities. We shall employ the symbols σ_n, ϵ_n and v_n to denote stress, strain and velocity in the nth region. Then, the solution of our problem for $\sigma_y < p_0 < \sigma_c$ is as follows:

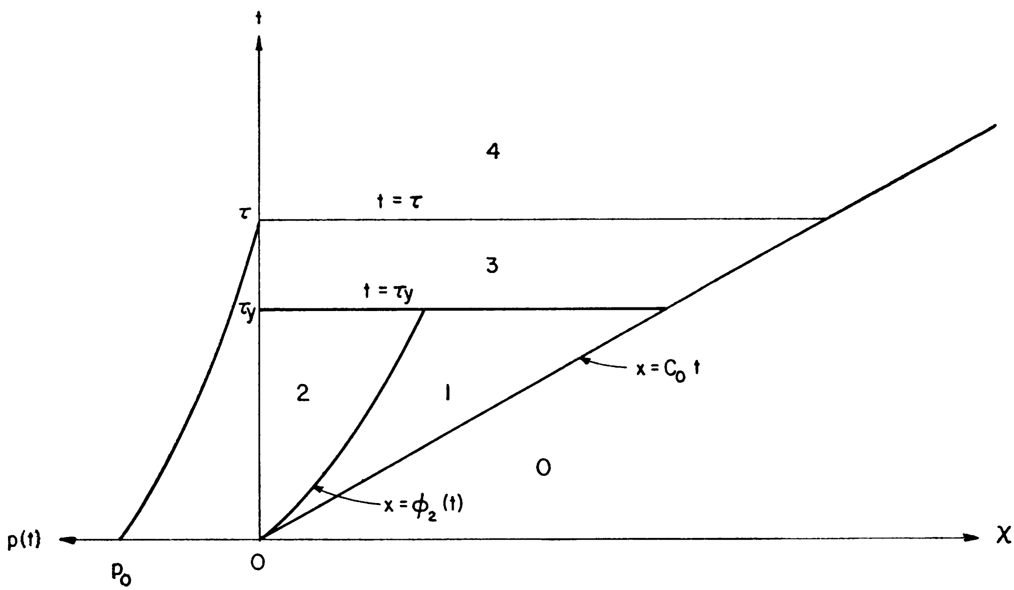

FIG. 3. x, t-plane ($\sigma_y < p_0 < \sigma_c$).

Region 0. The rod is undisturbed, and

$$\sigma_0 = \epsilon_0 = v_0 = 0. \tag{6}$$

These expressions are valid for $x > 0$ and $0 \leq t < x/c_0$, where $c_0 = (K_0/\rho)^{1/2}$.

Region 1. This is a region of constant state and is separated from region 0 by the strong discontinuity line $x = c_0 t$, across which the first of the stress-strain relations (3a) applies. Since the initial end-stress p_0 is applied instantly and has a magnitude greater then σ_y, we have

$$\sigma_1 = \sigma_y, \quad \epsilon_1 = \epsilon_y = \sigma_y/(\rho c_0^2), \quad v_1 = v_y = \sigma_y/(\rho c_0), \tag{7}$$

where the shock relations (5) with $\phi(t) = c_0 t$ have been employed. The stress-strain state in (7) corresponds to point A in Fig. 1.

Region 2. This region is separated from region 1 by the shock $x = \phi_2(t)$. In view of (3b), when the initial end-stress is increased beyond the yield stress σ_y, each increment of stress is propagated along the rod with a monotonically increasing velocity $c(\epsilon) = [f'(\epsilon)/\rho]^{1/2} = c[g(\sigma)]$, where $g(\sigma)$ is defined by

$$\epsilon = g(\sigma) = f^{-1}(\sigma). \tag{8}$$

Here f^{-1} is the inverse mapping of the function f, i.e. $\epsilon = f^{-1}(\sigma)$. Such a condition leads to the development of a shock wave eminating from the origin of the x, t-plane. If superimposed bars are used to indicate values at the shock, the shock relations (5) yield the following expression for the shock trajectory $x = \phi_2(t)$:

$$\rho[\phi_2'(t)]^2 = (\bar{\sigma}_2(t) - \bar{\sigma}_1(t))/(\bar{\epsilon}_2(t) - \bar{\epsilon}_1(t)). \tag{9}$$

In view of the right member of (9), the quantity $\rho[\phi_2'(t)]^2$ is precisely the slope of the chord with end points $(\bar{\sigma}_1, \bar{\epsilon}_1)$ and $(\bar{\sigma}_2, \bar{\epsilon}_2)$ (e.g., the line A D in Fig. 1) on the loading portion of the stress-strain curve provided that $\bar{\sigma}_2 > \bar{\sigma}_1$. If the stress just behind the shock is less than that ahead of the shock, the shock velocity becomes infinite in view of (3c) and (9). If we define region 2 to be a region for which $\bar{\sigma}_2(t) \geq \sigma_y$, then across the shock the second of stress-strain relation (3a) (or (8)) applies and

$$\bar{\sigma}_2 = f(\bar{\epsilon}_2) \quad \text{or} \quad \bar{\epsilon}_2 = g(\bar{\sigma}_2) = f^{-1}(\bar{\sigma}_2). \tag{10}$$

By definition of τ_y, unloading must take place in regions 2–4 (since the stress level at the impact face is not maintained). Hence, from (3c) we obtain

$$\epsilon_2(x, t) = \epsilon_m(x) = \bar{\epsilon}_2(t)|_{t=\Psi_2(x)} \quad \text{for} \quad \sigma_2(x, t) \leq \bar{\sigma}_2(t)|_{t=\Psi_2(x)}, \tag{11}$$

where $\Psi_2(x)$ is the inverse of $\phi_2(t)$, i.e.,

$$t = \Psi_2(x) = \phi_2^{-1}(x). \tag{12}$$

It is evident that $\bar{\sigma}_2(t)$ is a monotonically decreasing function of t. With the aid of the stress-strain diagram and by virtue of (7) and (9), it can be observed that the shock velocity $\phi_2'(t)$ is also a monotonically decreasing function of t. The latter satisfies the initial condition

$$\phi_2'(0) = \{(p_0 - \sigma_y)/\rho[g(p_0) - \epsilon_y]\}^{1/2}. \tag{13}$$

Substituting now (11) in (2), we find that

$$v_2(t) = \bar{v}_2(t); \tag{14}$$

thus, (1) may be integrated with respect to x to furnish

$$\sigma_2(x, t) - p(t) + \rho x v_2'(t) = 0. \tag{15}$$

By virtue of the boundary conditions

$$\sigma_2(0, t) = p(t) \quad \text{and} \quad \sigma_2[\phi_2(t), t] = \bar{\sigma}_2(t), \tag{16}$$

the following two equations result from (15):

$$\sigma_2(x, t) = p(t) - x[p(t) - \bar{\sigma}_2(t)]/\phi_2(t) \tag{17}$$

$$p(t) - \bar{\sigma}_2(t) = \rho\phi_2(t)\frac{dv_2}{dt}. \tag{18}$$

In view of (7), the shock relations (5) become

$$\bar{\sigma}_2 - \sigma_y = \rho\phi_2'(t)(v_2 - v_y), \tag{19a}$$

$$v_2 - v_y = \phi_2'(t)(\bar{\epsilon}_2(t) - \epsilon_y). \tag{19b}$$

Equation (19a) indicates that

$$v_2(0) = v_y + (p_0 - \sigma_y)/[\rho\phi_2'(0)]. \tag{20}$$

Adding (18) and (19a) together and integrating the resulting equation with respect to t, we obtain

$$v_2(t) = v_y + G(t)/[\rho\phi_2(t)]; \qquad G(t) \equiv \int_0^t (p(\xi) - \sigma_y)\, d\xi \tag{21}$$

where we have used the initial condition (20) and $\phi_2(0) = 0$. Upon substitution of (21) into (19), the following expressions for stress and strain at the shock are obtained:

$$\bar{\sigma}_2(t) = \sigma_y + \phi_2'(t)G(t)/\phi_2(t) \tag{22a}$$

$$\bar{\epsilon}_2(t) = \epsilon_y + G(t)/[\rho\phi_2(t)\phi_2'(t)]. \tag{22b}$$

Substituting these equations in (10), (11) and (17), we arrive at the following expressions for $\epsilon_2(x)$, $\sigma_2(x, t)$ and $\phi_2(t)$ respectively:

$$\epsilon_2(x) = \epsilon_y + G(\Psi_2(x))/[\rho x\phi_2'(\Psi_2(x))], \tag{23a}$$

$$\sigma_2(x, t) = p(t) - x[p(t) - \sigma_y - \phi'(t)G(t)/\phi_2(t)]/\phi_2(t), \tag{23b}$$

$$\sigma_y + \phi_2'(t)G(t)/\phi_2(t) = f[\epsilon_y + G(t)/(\rho\phi_2(t)\phi_2'(t))], \tag{24a}$$

or

$$g[\sigma_y + \phi_2'(t)G(t)/\phi_2(t)] = \epsilon_y + G(t)/(\rho\phi_2(t)\phi_2'(t)). \tag{24b}$$

Equation (24), together with the initial condition

$$\phi_2(0) = 0, \tag{24c}$$

constitutes an initial-value problem for the determination of the shock trajectory $x = \phi_2(t)$. This equation is a nonlinear differential equation of the first order. For certain specific stress-strain relations, closed-form solutions are obtainable, as will be shown in the next section. Once $\phi_2(t)$ is known, $\sigma_2(x, t)$, $\epsilon_2(x)$ and $v_2(t)$ can be obtained from (21)–(23).

In the above expressions, the unknown quantities σ_2, ϵ_2 and v_2 were expressed in terms of the unknown function $\phi_2(t)$, which is governed by the initial-value problem (24). An alternative way is to let $\bar{\epsilon}_2(t)$ (or $\bar{\sigma}_2(t)$) play the role of $\phi_2(t)$; then, from (10), (19) and (22), we have

$$\bar{\epsilon}_2'(t) = h(\bar{\epsilon}_2(t), t), \qquad (0 < t < \tau_v) \tag{25a}$$

$$\bar{\epsilon}_2'(0) = g(p_0), \tag{25b}$$

where

$$h(\bar{\epsilon}_2(t), t) \equiv \frac{2[f(\bar{\epsilon}_2) - \sigma_v][p(t) - f(\bar{\epsilon}_2)]}{G(t)[f'(\bar{\epsilon}_2) + (f(\bar{\epsilon}_2) - \sigma_v)/(\bar{\epsilon}_2 - \epsilon_v)]}, \tag{25c}$$

$$(f'(\bar{\epsilon}_2) \equiv df/d\bar{\epsilon}_2).$$

Integrating (25a) with respect to t and using (25b), we have

$$\bar{\epsilon}_2(t) = g(p_0) + \int_0^t h(\bar{\epsilon}_2(s), s)\, ds \qquad (0 \leq t \leq \tau_v), \tag{26}$$

which is a nonlinear Volterra integral equation. Therefore, the theory of integral equations can be employed to prove the existence and uniqueness of the solution of (26) on an appropriate time interval $0 \leq t \leq T \leq \tau_v$; in particular, we may use the Picard method of successive approximations for an approximate solution [4].

When the stresses just ahead and behind the shock $x = \phi_2(t)$ reach the same value ($\bar{\sigma}_2 = \sigma_v$ in this case), say at $t = \tau_v$, we have from (22a)

$$G(\tau_v) = \int_0^{\tau_v} p(\xi)\, d\xi - \sigma_v \tau_v = 0, \tag{26a}$$

and (21) and (22b)

$$\bar{\epsilon}_2(\tau_v) = \epsilon_v, \qquad v_2(\tau_v) = v_v. \tag{26b}$$

Thus, the stress-strain state at $t = \tau_v$ at the shock front corresponds to point A in Fig. 1. Note that the time τ_v may be greater or smaller than τ depending on the form of $p(t)$. For simplicity, however, only the case $\tau_v \leq \tau$ will be considered here. (The solution method for the case $\tau_v > \tau$ is essentially the same.)

Region 3. This region is separated from regions 1 and 2 by the strong discontinuity line $t = \tau_v$ (see discussion following Eq. (9)). Unloading now takes place in the portion $0 \leq x < c_0 t$ of the rod. Substituting $\phi(t) = c_0 t$ in the equation of rigid-body translation (18) and the shock relations (5), we obtain

$$p(t) - \bar{\sigma}_3(t) = \rho c_0 t \frac{dv_3}{dt} \tag{27a}$$

$$\bar{\sigma}_3(t) = \rho c_0 v_3(t) = \rho c_0^2 \bar{\epsilon}_3(t). \tag{27b}$$

Eliminating $\bar{\sigma}_3$ from (27a) and the first equality of (27b), and solving the resulting equation by utilizing the initial condition

$$v_3(\tau_v) = v_v = \sigma_v/\rho c_0, \tag{28}$$

we obtain

$$v_3(t) = \left[\int_{\tau_v}^t p(\xi)\, d\xi + \sigma_v \tau_v\right] \Big/ (\rho c_0 t). \tag{29}$$

Substituting (29) in (27b) we find

$$\bar{\sigma}_3(t) = \left[\int_{\tau_y}^{\tau} p(\xi)\,d\xi + \tau_y\sigma_y\right]\Big/ t$$
$$\bar{\epsilon}_3(t) = \left[\int_{\tau_y}^{\tau} p(\xi)\,d\xi + \tau_y\sigma_y\right]\Big/ (\rho c_0^2 t). \tag{30}$$

The quantities $\sigma_3(x, t)$ and $\epsilon_3(x)$ are obtained from (11), (17) and (30) as follows:

$$\sigma_3(x, t) = p(t) - \frac{x}{c_0 t}\left\{p(t) - \left[\int_{\tau_y}^{\tau} p(\xi)\,d\xi + \tau_y\sigma_y\right]\Big/ t\right\}, \tag{31}$$

$$\begin{aligned} \epsilon_3(x) &= \epsilon_2(x) \quad \text{for} \quad 0 \le x \le \phi_2(\tau_y) \qquad \text{(cf. (23a))} \\ &= \epsilon_y \quad \text{for} \quad \phi_2(\tau_y) \le x \le c_0 t \\ &= \left[\int_{\tau_y}^{t=x/c_0} p(\xi)\,d\xi + \sigma_y\tau_y\right]\Big/ (\rho c_0 x) \qquad \text{for} \quad c_0\tau_y \le x \le c_0\tau. \end{aligned} \tag{32}$$

It is easily shown from (7), (23b), (26a) and (31) that $t = \tau_y$ is a line of strong discontinuity (the stress is discontinuous while the strain and velocity are continuous across $t = \tau_y$).

Region 4. This region is separated from regions 3 by the line of weak discontinuity $t = \tau$. Since $p(t)$ is everywhere zero in this region and the shock equation $x = c_0 t$ is the same as that of region 3, the solution is readily obtained from the expressions (29)–(31) by setting $t = \tau$ for the upper limit of integration. For example,

$$v_4(t) = \left[\int_{\tau_y}^{\tau} p(\xi)\,d\xi + \sigma_y\tau_y\right]\Big/ (\rho c_0 t). \tag{33}$$

Eq. (33) indicates that unloading continues indefinitely.

All expressions obtained thus far are valid for the case $\sigma_y < p_0 < \sigma_c$. With the following minor modifications, they are also valid for the cases $p_0 \ge \sigma_c$ and $p_0 \le \sigma_y$.

Case of $p_0 \ge \sigma_c$. The x, t-plane for this case is illustrated in Fig. 4. A new region, which we denote by "A", now appears because $\phi_A'(t) \ge c_0$ (see the discussion immediately following Eq. (9)) in $0 < t \le \tau_c$, where τ_c is the critical time defined by $\bar{\sigma}_A(\tau_c) = \sigma_c$. The expressions for $v_A(t)$, $\sigma_A(x, t)$ and $\epsilon_A(x)$ and the differential equation for $\phi_A(t)$ are obtained from (20)–(24) by setting $\sigma_y = v_y = \epsilon_y = 0$. For example, from (21) and (24), we have

$$v_A = \int_0^t p(\xi)\,d\xi/(\rho\phi_A(t)) \tag{34a}$$

and

$$\phi_A'(t) \int_0^t p(\xi)\,d\xi/\phi_A(t) = f\left\{\int_0^t p(\xi)\,d\xi/[\rho\phi_A(t)\phi_A'(t)]\right\} \tag{34b}$$

$$\phi_A(0) = 0. \tag{34c}$$

For $t > \tau_c$, the stress immediately behind the shock has a value less than $p_0(\bar{\sigma}_2(t) < p_0)$ and thus $\phi_2'(t) < c_0$. Therefore, the shock $x = \phi_A(\tau_c) + c_0(t - \tau_c)$, which is tangent to the shock trajectory $x = \phi_A(\tau_c)$ at $t = \tau_c$ and $x = \phi_A(\tau_c)$, propagates ahead of the

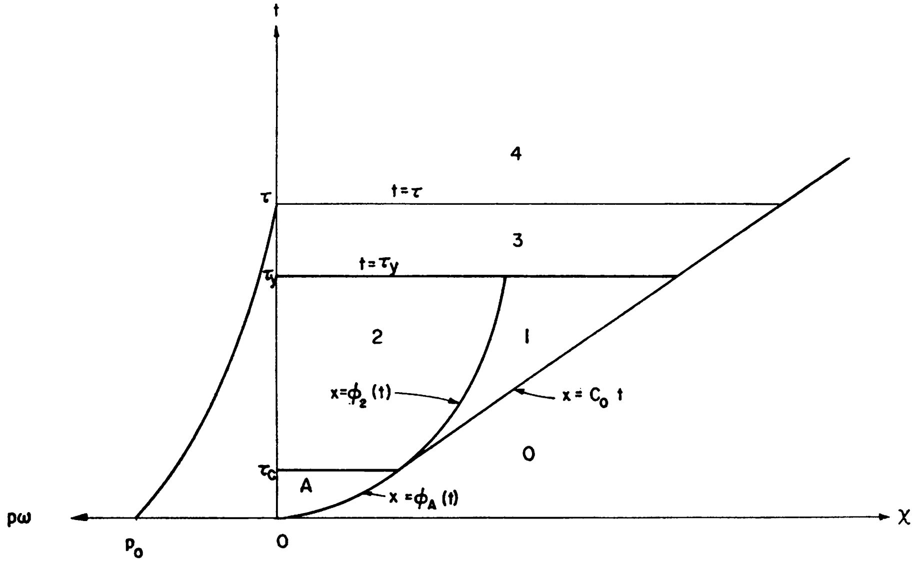

FIG. 4. x, t-plane ($p_0 \geq \sigma_c$).

shock $x = \phi_2(t)$. Hence, except for initial conditions, the problem for $t > \tau_c$ is again reduced to that of case (i)($\sigma_y < p_0 < \sigma_c$). Employing the initial conditions[1]

$$v_2(\tau_c) = v_A(\tau_c) = \int_0^{\tau_c} p(\xi)\, d\xi/(\rho\phi_A(\tau_c)), \tag{35a}$$

$$\phi_2(\tau_c) = \phi_A(\tau_c) \tag{35b}$$

in place of (20) and (24c) respectively, the expressions (or differential equations) for $\sigma_2(x, t)$, $v_2(t)$, $\phi_2(t)$ in $0 \leq x \leq \phi_2(\tau_y)^-$ and[2] $\epsilon_2(x)$ in $\phi_A(\tau_c) \leq x \leq \phi_2(\tau_y)$ are readily obtained from (21)–(24) by replacing the function $G(t)$ with $G(t) + a$, where $a \equiv \sigma_y(\tau_c - \phi(\tau_c)/(c_0) \leq 0$. Similarly, the solution in regions 3 and 4 are obtained from previous cases by shifting the origin of the shock $x = c_0t$ from (0, 0) to $(\phi_A(\tau_c), \tau_c)$, etc.

Case of $p_0 \leq \sigma_y$. This is a special case of the case (i)($\sigma_y < p_0 < \sigma_c$) in which regions 1 and 2 (Fig. 3) and τ_y vanish. Thus the solutions (in regions 3, 4) are obtained from (29), (32) and (33) by setting $\tau_y = 0$.

Other special cases. A combination of the vanishing of one or both of σ_y and ϵ_y yields three special cases. Here σ_c becomes infinite and c_0 is either 0 or infinity, which result in certain simplifications of our solutions.

In the remainder of this paper, unless otherwise noted, the initial impact stress p_0 is assumed to be less than σ_c.

3. Closed-form solutions for the power stress-strain law. Consider now a class of materials whose stress-strain relation is defined by (3a)–(3c)(allowing $f'(\epsilon) = 0$ at $\epsilon = \epsilon_y^+$) in which the function $f(\epsilon)$ takes the form

$$f(\epsilon) = \sigma_y + k_n(\epsilon - \epsilon_y)^n (n \geq 1,\ k_n = \text{const.} > 0,\ \sigma \geq \sigma_y). \tag{36}$$

[1] The condition (35a) follows from (5) for $\phi'(t) = \infty$.

[2] $\epsilon_2(x) = \epsilon_A(x)$ in $0 \leq x \leq \phi_2(\tau_c)$.

For such a stress-strain law, (24) becomes

$$[\phi_2(t)]^{n-1}[\phi_2'(t)]^{n+1} = (k_n/\rho^n)[G(t)]^{n-1}; \qquad \phi(0) = 0, \tag{37}$$

which can easily be integrated to yield

$$\phi_2(t) = \left[\frac{2n}{n+1}\left(\frac{k_n}{\rho^n}\right)^{1/(n+1)} F_n(t)\right]^{(n+1)/2n} \tag{38a}$$

where

$$F_n(t) \equiv \int_0^t [G(\xi)]^{(n-1)/(n+1)}\, d\xi.$$

If we express the shock equation in the form $t = \Psi_2(x)$, then (38a) gives

$$F_n[\Psi_2(x)] = \frac{n+1}{2n}\left(\frac{\rho^n}{k_n}\right)^{1/(n+1)} x^{2n/(n+1)}. \tag{38b}$$

Equation (38a), together with (21)—(23), furnishes

$$\begin{aligned} v_2(t) &= v_y + G(t)/[(\rho^n k_n)^{1/2n}((2n/(n+1))F_n(t))^{(n+1)/2n}], \\ \bar{\sigma}_2(t) &= \sigma_y + (n+1)[G(t)]^{2n/(n+1)}/[2nF_n(t)], \\ \sigma_2(x, t) &= p(t) - x[p(t) - \bar{\sigma}_2(t)]/\phi_2(t), \\ \epsilon_2(x) &= \epsilon_y + \{G[\Psi_2(x)]/[(\rho k_n)^{1/2}x]\}^{2/(n+1)}, \end{aligned} \tag{39}$$

where $\Psi_2(x)$ is implicitly expressed by (38b).

Before proceeding further, let us examine the character of the stress just behind the shock. Integrating by parts, we can rewrite the second of (39) in the following form:

$$\begin{aligned} \bar{\sigma}_2(t) &= \sigma_y - \int_0^t [G(\xi)]^{(n-1)/(n+1)}[p(\xi) - \sigma_y]\, d\xi/F_n(t) \\ &= p(t) - \int_0^t F_n(\xi)p'(\xi)\, d\xi/F_n(t), \end{aligned} \tag{40}$$

from which

$$\bar{\sigma}_2'(t) = [G(t)]^{(n-1)/(n+1)}\left[\int_0^t F_n(\xi)p'(\xi)\, d\xi\right] \Big/ [F_n(t)]^2. \tag{41}$$

In view of (4) and the definition of $G(t)$ given in (21), it is evident that

$$G(t) > 0, \qquad F_n(t) > 0 \quad \text{for} \quad 0 < t < \tau_y \tag{42}$$

which, upon combining with (40)–(41), implies

$$\bar{\sigma}_2(t) > p(t) \quad \text{and} \quad \bar{\sigma}_2'(t) < 0, \tag{43}$$

as was expected. Similarly, it can be shown that $\phi_2(t)$ is a monotonically increasing function of t while $\phi_2'(t)$, $v_2(t)$ and $\epsilon_2(t)$ are monotonically decreasing functions of t in $0 < t < \tau_y$. If the condition $p'(t) < 0$ in (4) is replaced by $p'(t) \leq 0$, then (40) and (41) reduce to

$$\bar{\sigma}_2(t) \geq p(t) \quad \text{and} \quad \bar{\sigma}'(t) \leq 0. \tag{44}$$

Thus, in view of the unloading stress-strain relation (3c), the results obtained in the present and previous sections remain unchanged. Therefore, having obtained the closed-form solutions in region 2, we thus have a complete closed-form solution in the entire x, t-plane since the solutions in all other regions are the same as those in Sec. 2.

Writing

$$k_n = (\sigma^* - \sigma_y)/(\epsilon^* - \epsilon_y)^n (\sigma_y < \sigma^* = \text{const.}, \epsilon_y < \epsilon^* = \text{const.}) \tag{45}$$

the function (38) reduces to

$$f(\epsilon) = \sigma_y + (\sigma^* - \sigma_y)(\epsilon - \epsilon_y)^n/(\epsilon^* - \epsilon_y)^n (\epsilon \geq \epsilon_y). \tag{46}$$

The $\sigma = f(\epsilon)$ curves corresponding to (46) for various n are illustrated in Fig. 3. When n approaches infinity, (46) becomes

$$\begin{aligned} \lim_{n\to\infty} f(\epsilon) &= \sigma_y \quad \text{for} \quad \epsilon < \epsilon^* \\ \lim_{n\to\infty} \epsilon &= \epsilon^* \quad \text{for} \quad \epsilon > \epsilon^* \end{aligned} \tag{47}$$

which correspond to the segments AB, BD in Fig. 2. Substituting (45) into (37)–(39) we obtain, as n approaches infinity,

$$\begin{aligned} \phi_2(t) &= [2F_\infty(t)/\rho(\epsilon^* - \epsilon_y)]^{1/2}\left(F_\infty(t) \equiv \int_0^t G(\xi)\,d\xi\right), \\ v_2(t) &= v_y + G(t)[(\epsilon^* - \epsilon_y)/(2\rho F_\infty(t))]^{1/2}, \\ \sigma_2(x, t) &= p(t) - x[p(t) - \sigma_y - (G(t))^2/(2F_\infty(t))]/\phi_2(t), \\ \epsilon_2 &= \epsilon^*, \end{aligned} \tag{48}$$

which are the solution in region 2 for the loading stress-strain curve OABD in Fig. 2. Note that this solution remains valid in an interval $0 \leq t < T$ even if $p(t)$ is not a monotonically decreasing function of t, as long as the conditions

$$p_0 > \sigma_y \quad \text{and} \quad \bar{\sigma}_2(t) \geq \sigma_y \quad \text{in} \quad 0 < t < T \tag{49}$$

are satisfied. The last condition requires that

$$G(t) \geq 0 \quad \text{for} \quad 0 < t < T. \tag{50}$$

The closed-form solutions given above can be used to construct a new solution for the stress-strain curve composed of a finite number of segments of the class of curves (36) (with different n assigned to each segment).

Remarks concerning the case $p_0 \geq \sigma_c$. Substituting (36) in (34b), we obtain

$$\left[\phi_A'(t) \int_0^t p(\xi)\,d\xi\right] \Big/ \phi_A(t) = \sigma_y + k_n\left[\frac{1}{\rho\phi_A'(t)\phi_A(t)} \int_0^t p(\xi)\,d\xi - \epsilon_y\right]^n, \tag{51a}$$

$$\phi_A'(0) = 0. \tag{51b}$$

This equation cannot be integrated in a closed-form unless $n \to \infty$ (locking material).[3] The solution for $t > \tau_c$ (regions 2–4), however, can be obtained in a closed-form as before.

4. Concluding remarks. One-dimensional wave propagation in a semi-infinite strain-

[3] Note that σ_y and ϵ_y are both nonvanishing in order that the inequality $\sigma_c > p_0$ be meaningful.

hardening rod, subjected to compressible end-stress, was considered in this paper. It was shown that an analytical solution is possible in the entire x, t-plane provided the constitutive equations of the rod are as indicated in Fig. 1 and the end-stress is applied suddenly and thereafter monotonically decreases to zero. In particular, closed-form solutions are obtained if the strain-hardening portion of the constitutive equation obeys a power law. It was also indicated that these closed-form solutions can be combined in various ways to construct new closed-form solutions.

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References

1. R. C. Shieh, G. A. Hegemier and W. Prager, *Closed-form solutions to problems of wave propagation in rigid, work-hardening, locking rods,* Defense Atomic Support Agency Technical Report No. 1, April 1968
2. L. Zvolinski and G. Rykov, *Reflection of plastic wave from an obstacle,* in *Stress waves in anelastic solids,* ed. H. Kolsky and W. Prager, Springer, Berlin, 1964, pp. 207–221
3. E. H. Lee and S. J. Tupper, *Analysis of plastic deformation in a steel cylinder striking a rigid target,* J. Appl. Mech. **21**, 63–70 (1954)
4. T. Saaty, *Modern nonlinear equations,* McGraw-Hill, New York, 1967, pp. 276–300