CANONICAL APPROACH TO BIHARMONIC VARIATIONAL PROBLEMS*

BY

A. M. ARTHURS

University of York, England

Abstract. A canonical approach to biharmonic variational problems is presented. It provides a new form of the principle of stationary energy and a new derivation of the principle of minimum potential energy.

1. Introduction. In a recent series of papers [1]–[4], variational principles associated with the canonical equations

\[ T \Phi = \frac{\partial W}{\partial U}, \quad T^* U = \frac{\partial W}{\partial \Phi} \]

have been studied. The work of Noble [1] dealt with the one-dimensional case \( T = \frac{d}{dx}, \ T^* = -\frac{d}{dx} \), while the operators \( T = \text{grad}, \ T^* = -\text{div} \) have been discussed in papers on diffusion and related topics [2]–[4].

The purpose of this note is to present some results for the operators \( T = \nabla^2, \ T^* = \nabla^2 \). The theory is used to derive a canonical form of the principle of stationary energy for biharmonic problems and to provide a new derivation of the principle of minimum potential energy.

2. Theory. We consider a physical problem which is described by the pair of canonical Euler equations

\[ T \Phi = \frac{\partial}{\partial U} W(r, \Phi, U), \quad \text{in} \ R, \]

\[ T^* U = \frac{\partial}{\partial \Phi} W(r, \Phi, U) \]

in which \( T \) and \( T^* \) are linear operators, \( r \) is a position vector and \( \Phi \) and \( U \) are functions of \( r \). The region \( R \) is a part of the \( xy \)-plane which has a piecewise smooth boundary \( B \). The boundary conditions are taken to be

\[ \Phi = \varphi_0 \quad \text{on} \ B \]

\[ \frac{\partial \Phi}{\partial n} = f(U) \]

where \( n \) is the outward pointing normal to the boundary, and \( \varphi_0 \) and \( f \) are given functions. The operator \( T^* \) in (2) is the adjoint of \( T \) in the sense that

\[ (U, T \Phi) = \langle T^* U, \Phi \rangle, \]

where the inner products are defined by

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\((U, T\Phi) = \int_R UT\Phi \, dx \, dy - \int_B U \frac{\partial \Phi}{\partial n} \, ds + \int_B F(U) \, ds, \) (6)
\((T*U, \Phi) = \int_R (T*U)\Phi \, dx \, dy - \int_B \frac{\partial U}{\partial n} \, \Phi \, ds + \int_B F(U) \, ds, \) (7)

with
\[ F(U) = \int^U f(U') \, dU'. \] (8)

Definitions (6) and (7) are appropriate to the case \( T = \nabla^2, \) \( T* = \nabla^2. \) If we introduce the functional
\[ I(\Phi, U) = \int_R W(r, \Phi, U) \, dx \, dy - (U, T\Phi), \] (9)
then the following results are obtained:

**Stationary property.** \( I(\Phi, U) \) is stationary at \((\varphi, u)\) if Eqs. (1)–(4) hold simultaneously at \((\varphi, u)\).

**Extremum principle.** Choose a trial function \( \Phi \) which is equal to \( \varphi_0 \) on \( B \), and determine \( U(\Phi) \) so that Eqs. (1) and (4) are satisfied identically. Then, if (2) holds at \((\varphi, u)\) we have from (9)
\[ G(\Phi) = I(\Phi, U(\Phi)) = I(\varphi, u) + \delta^2 I(\Phi) + O(\Phi - \varphi)^3, \] (10)

where
\[ \delta^2 I(\Phi) = \frac{1}{2} \int_R \left\{ (\Phi - \varphi)^2 \left[ \frac{\partial^2 W}{\partial \Phi^2} \right]_{\varphi, u} - (U(\Phi) - u)^2 \left[ \frac{\partial^2 W}{\partial U^2} \right]_{\varphi, u} \right\} \, dx \, dy + \frac{1}{2} \int_B (U(\Phi) - u)^2 \left[ \frac{df}{dU} \right]_{\varphi, u} \, ds. \] (11)

If terms of third and higher orders can be neglected (or if they vanish), it follows that
\[ G(\Phi) \leq I(\varphi, u) \quad \text{if} \quad \delta^2 I \leq 0, \] (12)
or
\[ G(\Phi) \geq I(\varphi, u) \quad \text{if} \quad \delta^2 I \geq 0. \] (13)

Thus we have an upper or a lower bound for \( I(\varphi, u) \) depending on the sign of \( \delta^2 I. \) The pair of functions \((\varphi, u)\) furnishes the exact solution of the problem in Eqs. (1)–(4).

**3. The biharmonic equation.** The equation which governs the small deflection bending of a thin elastic plate is [5], [6]
\[ \nabla^4 \Phi = q/D \quad \text{in} \ R, \] (14)

where \( \Phi \) is the deflection normal to the surface, \( q(x, y) \) is the distribution of normal loading and \( D \) is the flexural rigidity. The boundary conditions [5], [6] for a plate which is clamped on part \( B_1 \) and simply supported on \( B - B_1 \) are
\[ \Phi = \frac{\partial \Phi}{\partial n} = 0 \quad \text{on} \ B_1, \] (15)
and
\[
\Phi = \frac{\partial \Phi}{\partial n} - \frac{1}{\kappa}(1 - \nu) \nabla^2 \Phi = 0 \quad \text{on } B - B_1. \tag{16}
\]
Here \(\kappa\) is the local curvature of \(B\) and \(\nu\) is Poisson's ratio. In addition we assume that these deflection conditions satisfy Eq. (14) explicitly. We now write (14) as the pair of equations

\[
\begin{align*}
\nabla^2 \Phi &= U \quad \text{in } R, \tag{17} \\
\nabla^2 U &= q/D \tag{18}
\end{align*}
\]

This way of writing (14) is equivalent to that used by Morley [5], but the canonical form of (17) and (18) does not seem to have been emphasized previously. The boundary value problem in (15)–(18) is a special case of that discussed in Sec. 2 and corresponds to

\[
\begin{align*}
W(r, \Phi, U) &= \frac{1}{2}U^2 + q\Phi/D, \tag{19} \\
T &= \nabla^2, \quad T^* = \nabla^2, \tag{20} \\
\varphi_0 &= 0 \quad \text{on } B, \tag{21} \\
f(U) &= 0 \quad \text{on } B_1, \tag{22} \\
= U/\kappa(1 - \nu) \quad \text{on } B - B_1.
\end{align*}
\]

Putting these in (9) we obtain

\[
I(\Phi, U) = \int_R \left\{ \frac{1}{2} U^2 + \frac{q}{D} \Phi - U\nabla^2 \Phi \right\} \, dx \, dy + \int_{B-B_1} \left\{ U \frac{\partial \Phi}{\partial n} - \frac{U^2}{2\kappa(1 - \nu)} \right\} \, ds. \tag{23}
\]

From Sec. 2 we see that \(I(\Phi, U)\) in (23) is stationary at \((\varphi, u)\) where \(\varphi, u\) are the exact solutions of (15)–(18). This is a canonical form of the principle of stationary energy. When \(\Phi = \varphi\) and \(U = u\) we find that (23) gives

\[
I(\varphi, u) = \frac{1}{2} \int_R u^2 \, dx \, dy - \frac{1}{2(1 - \nu)} \int_{B-B_1} \frac{1}{\kappa} u^2 \, ds. \tag{24}
\]

The functional \(G(\Phi)\) given by (10) is found to be

\[
G(\Phi) = \int_R \left\{ -\frac{1}{2} \left(\nabla^2 \Phi\right)^2 + \frac{q}{D} \Phi \right\} \, dx \, dy + \frac{1}{2} \int_{B-B_1} \kappa(1 - \nu) \left(\frac{\partial \Phi}{\partial n}\right)^2 \, ds, \quad \Phi = 0 \quad \text{on } B. \tag{25}
\]

In addition Eq. (11) becomes

\[
\delta^2 I(\Phi) = -\frac{1}{2} \int_R (U(\Phi) - u)^2 \, dx \, dy + \frac{1}{2(1 - \nu)} \int_{B-B_1} \frac{1}{\kappa} (U(\Phi) - u)^2 \, ds. \tag{26}
\]

If

\[
\kappa(1 - \nu) < 0, \tag{27}
\]

then

\[
\delta^2 I(\Phi) \leq 0, \tag{28}
\]

and hence, by (12), we have the variational principle

\[
G(\Phi) \leq I(\varphi, u). \tag{29}
\]

Eq. (29) is equivalent to the principle of minimum potential energy [5], [6].
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References