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A MONOTONE PROPERTY OF THE SOLUTION OF A STOCHASTIC BOUNDARY VALUE PROBLEM*

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1. Introduction. In addition to their mathematical interest, differential equations have long been a valuable tool for physicists and engineers. Mathematically, a typical problem in differential equations is composed of an operator L, a forcing function f and auxiliary conditions. In recent years it has been found useful to study the solution of a differential equation when one of the three components of the problem is random. The following remarks may suggest why this is beneficial. Essentially all of the known quantities in the operator L are determined through physical measurements; therefore, a certain amount of random error is unavoidably introduced by imperfect measuring instruments. Sometimes particular quantities in the problem are assumed to be homogeneous, but since this is rarely the case physically, a random error is again incurred. Even in cases where the nonhomogeneities are completely specified, the complete formulation of the problem may be too complicated to solve mathematically; however, by considering an "average value" for this nonhomogeneity we may be able to obtain an "average" solution which is related to the solution of the desired problem. Therefore, for several reasons a study of stochastic differential equations may aid in understanding some physical problems better.

In ordinary differential equations an extensive literature exists for stochastic initial value problems, as can be verified, for example, by considering Middleton [13] and his bibliography. On the other hand, a very limited amount of work has been done in stochastic boundary value problems. Even though many results from random initial value problems are applicable to random boundary value problems, the latter often raise questions which are unanswered by a study of the former. Boyce [5], [6], [7], Goodwin [9] and Haines [10], [11] have studied many properties of random eigenvalues and random eigenfunctions of a boundary value problem by a variety of techniques. Bharucha-Reid [4] and others have proved the existence and uniqueness of solutions to stochastic boundary value problems by using arguments from functional analysis and measure theory.

Let us consider the Sturm-Liouville problem

$$\begin{aligned} Ly(x) &= -(p(x)y'(x))' + q(x)y(x) = f(x), \\ y(0) &- ay'(0) = 0, \\ y(1) &+ by'(1) = 0. \end{aligned} \tag{1}$$

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If any of the quantities $p(x)$, $q(x)$, $f(x)$, a and b, are random, then the solution $y(x)$ is also a random process and can only be described in a statistical sense. In this paper we will consider only the case when $f(x)$ is a random process. Because this forcing function $f(x)$ is a random function, we can give only its statistical properties (such as its mean and covariance) in an attempt to describe it. Of course, when the process is known to be normal (Gaussian), the mean and the covariance completely determine the process. Frequently, one assumes a process is normal because of convenience and because such an assumption is not entirely unreasonable physically. Even if no assumption is made about a normal distribution for the process in question, discussions are frequently limited to the first two moments. We shall be primarily concerned with studying the variation of the covariance of the solution of problem (1) when the coefficient $q(x)$ changes. The covariance of the random function f is defined at points x_1 and x_2 in $I = [0, 1]$ as

$$\langle f(x_1)f(x_2)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t_1 t_2 \, dF_{x_1,x_2}(t_1, t_2)$$

where $\langle\ \rangle$ denotes ensemble average or expectation and where

$$F_{x_1,x_2}(t_1, t_2) = \text{probability } (f(x_1) < t_1, \quad f(x_2) < t_2).$$

Thus, $y(x)$, the solution of problem (1), has covariance given by

$$\langle y(x_1)y(x_2)\rangle = \int_0^1\int_0^1 G(x_1, z_1)G(x_2, z_2)\,\langle f(z_1)f(z_2)\rangle\, dz_1\, dz_2 \tag{2}$$

where $G(x, z)$ is the Green's function of problem (1). We assume throughout this paper that the covariance of $f(x)$ is measurable and is bounded almost everywhere in I^2; i.e., $\langle f(x_1)f(x_2)\rangle \in L^\infty(I)^2$.

We shall show that under suitable restrictions the covariance of the solution of problem (1) is monotonically nonincreasing as a function of the coefficient $q(x)$; i.e., as $q(x)$ increases from $q_1(x)$ to $q_2(x)$ with $q_1(x) \leq q_2(x)$ for all $x \in I$, $\langle y(x_1)y(x_2)\rangle$ for $q(x) = q_2(x)$ is less than or equal to $\langle y(x_1)y(x_2)\rangle$ for $q(x) = q_1(x)$. This proof relies heavily on the Neumann expansion theorem for linear operators and on oscillation theorems.

We shall examine not only coefficients $q(x)$ which are nonnegative but also a restricted class of negative $q(x)$'s. We are first forced to determine restrictions on $q(x) \leq 0$ which render the Green's function positive. Although papers by Aronszajn and Smith [1], Pak [14] and others have dealt with this question from an existence viewpoint, we provide quantitative bounds on $q(x)$. Since the Green's function is always nonnegative if $q(x) \geq 0$, these restrictions will not be needed when $q(x) \geq 0$. Some of the lemmas appearing in Sec. 2 are special cases of more general results obtained in oscillation theorems and maximum principles as given in Swanson [16] and in Protter and Weinberger [15]. In order to make this paper more nearly self-contained, we have given proofs of these results rather than citations.

2. The positive Green's function. When $q(x)$ is negative, we may feel that $q(x)$ will have to be limited in a manner that will exclude all eigenvalues of the homogeneous problem corresponding to (1). But other than this limitation it is hard to anticipate physically what restrictions must be placed on $q(x)$ in order to be assured that the covariance of the solution is monotonic in $q(x)$, much less in what direction the monotone property exists. To help guide us toward the right direction, we shall first consider a simple example and then try to generalize our result.

Let $u(x)$ and $v(x)$ be the solutions of

$$-u''(x) = f(x), \qquad u(0) = 0 = u(1) \tag{3}$$

and

$$-v''(x) - v(x) = f(x), \qquad v(0) = 0 = v(1) \tag{4}$$

respectively. Then

$$\langle v(x_1)v(x_2)\rangle - \langle u(x_1)u(x_2)\rangle = \int_0^1 \int_0^1 (G(x_1, z_1)G(x_2, z_2) - H(x_1, z_1)H(x_2, z_2)) * \langle f(z_1)f(z_2)\rangle \, dz_1 \, dz_2 \tag{5}$$

where $G(x, z)$ is the Green's function associated with problem (4) and $H(x, z)$ is the Green's function associated with problem (3). Thus we see that

$$\langle v(x_1)v(x_2)\rangle - \langle u(x_1)u(x_2)\rangle \geq 0$$

if $G(x, z) \geq H(x, z) \geq 0$ and if we consider only forcing functions with nonnegative covariances. Now

$$G(x, z) = \begin{cases} \sin(x) \sin(1 - z)/\sin(1), & \text{if } 0 \leq x \leq z, \\ \sin(z) \sin(1 - x)/\sin(1), & \text{if } z \leq x \leq 1 \end{cases} \tag{6}$$

and

$$H(x, z) = \begin{cases} x(1 - z), & \text{if } 0 \leq x \leq z, \\ z(1 - x), & \text{if } z \leq x \leq 1. \end{cases} \tag{7}$$

But for $0 \leq x \leq z$

$$\frac{\sin(x)}{x} \frac{\sin(1 - z)}{1 - z}$$

is a monotonically decreasing function of x for fixed z since $0 \leq x \leq 1$. Hence,

$$\frac{\sin(x)}{x} \frac{\sin(1 - z)}{1 - z} \geq \frac{\sin(z)}{z} \frac{\sin(1 - z)}{1 - z} \geq \sin(1);$$

therefore,

$$G(x, z) = \sin(x) \sin(1 - z)/\sin(1) \geq x(1 - z) = H(x, z).$$

Similarly, $G(x, z) \geq H(x, z)$ for $z \leq x \leq 1$. Thus we see that when problems (3) and (4) are in the form of problem (1), an increase in $q(x)$ decreases the covariance of the solution.

As in this example, we consider problem (1) when $q(x)$ is nonpositive in this section and in Sec. 3, and we define $r(x) = -q(x)$. In trying to describe $\langle y(x_1)y(x_2)\rangle$, we look initially at the Green's function of the problem. Under the restrictions

$$\begin{aligned} &p(x) > 0 \quad \text{in} \quad I, \qquad p(x) \in C^{(1)}(I), \\ &r(x) \geq 0 \quad \text{in } I \text{ and } r(x) \not\equiv 0 \quad \text{if } a = b = \infty, \qquad r(x) \in C(I), \end{aligned} \tag{8}$$

we wish to establish the positivity of the Green's function of problem (1) or at least determine what conditions can be imposed on $r(x)$ to guarantee a positive Green's function. We proceed to this goal via a sequence of lemmas. Throughout this paper we shall use the following notation:

$$\begin{aligned} M &= \max_{x\in I} p(x), & K &= \min_{x\in I} p(x) \\ R &= \max_{x\in I} r(x), & Q &= \max_{x\in I} q(x). \end{aligned} \tag{9}$$

Because all of the following lemmas contain similar results either for the endpoint $x = 0$ or for the endpoint $x = 1$, we shall give a proof at only one endpoint and merely state the result for the other endpoint.

LEMMA 1. *If $R < K$, $0 \leq r_0 \leq 1$, restrictions (8) are satisfied and $u(x)$ is the solution of the problem*

$$-(p(x)y'(x))' - r_0 r(x)u(x) = 0 \tag{10}$$

$$u(0) = 0, \qquad u'(0) = 1, \tag{11}$$

then $u'(x) > 0$ in I. If (11) is replaced by

$$u(1) = 0, \qquad u'(1) = 1,$$

then $u'(x) > 0$ in I.

Proof. Since $R/K < 1 < \pi^2$, $u(x)$ cannot oscillate in I according to Ince [12, p. 227]; hence, $u(x) > 0$ in I. Now $(p(x)u'(x))' = -r_0 r(x)u(x) \leq 0$. Integrating from 0 to x, we have

$$\int_0^x (p(x)u'(x))'\,dx = p(x)u'(x) - p(0)u'(0) \leq 0$$

or $u'(x) \leq p(0)/p(x) \leq p(0)/K$. Integrating again from 0 to x we get

$$\int_0^x u'(x)\,dx = u(x) - u(0) \leq p(0)/K \leq p(0)/R.$$

Therefore, $\max_{x\in I} u(x) < p(0)/R$. But

$$(p(x)u'(x))' = -r_0 r(x)u(x) \geq -R(\max_{x\in I} u(x)).$$

Thus,

$$u'(x) \geq (p(0) - R(\max u(x))x)/p(x) > 0.$$

LEMMA 2. *If $R < K$, restrictions (8) are fulfilled and $u(x)$ satisfies*

$$Lu(x) = -(p(x)u'(x))' - r(x)u(x) = 0 \tag{12}$$

$$u(1) = c \neq 0, \qquad u'(1) = 0, \tag{13}$$

then $u(x)$ always has the same sign as c throughout I. Furthermore, $u'(x)$ always has the same sign as c in I. If (13) is replaced by

$$u(0) = c \neq 0, \qquad u'(0) = 0,$$

then $u(x)$ always has the same sign as c, but $u'(x)$ has the opposite sign to c throughout I.

Proof. Suppose $u(z)$ is not always of the same sign as c. Then $u(\alpha) = 0$ for some $\alpha \in [0, 1)$. Hence, $u(z)$ is a nontrivial solution to

$$Lu(z) = 0, \qquad u(\alpha) = 0 = u'(1),$$

which is equivalent to

$$\begin{gathered} -(p(x)u'(x))' - (1-\alpha)^2 r(x)u(x) = 0 = L_\alpha u(x) \\ u(0) = 0, \qquad u'(1) = 0 \end{gathered} \tag{14}$$

under the transformation $x = (z - \alpha)/(1 - \alpha)$. However, if $u(x)$ satisfies problem (14), then so does $ku(x)$ for any constant k. In particular, with $k = 1/u'(0)$, $u(x)/u'(0)$ does. Note that $u'(0)$ is not zero since then

$$L_\alpha u(x) = 0, \qquad u(0) = 0 = u'(0),$$

which implies that $u(x)$ is identically zero. But this contradicts $u(1) = c \neq 0$. Now let $v(x) = u(x)/u'(0)$. Then

$$L_\alpha v(x) = 0, \qquad v(0) = 0, \qquad v'(0) = 1.$$

By Lemma 1, $v'(x) > 0$ in I, in particular $v'(1) > 0$. But this contradicts $v'(1) = u'(1)/u'(0) = 0$. Thus $u(z)$ is never zero in I and always has the same sign as c.

If we integrate $(p(x)u'(x))' = -r(x)u(x)$ from x to 1, we have

$$\int_x^1 (p(x)u'(x))'\,dx = p(1)u'(1) - p(x)u'(x) = -u'(x)p(x) = \int_x^1 (-r(x)u(x))\,dx.$$

Thus,

$$u'(x) = p^{-1}(x) \int_x^1 r(x)u(x)\,dx > 0, \quad \text{if} \quad c > 0,$$

$$< 0, \quad \text{if} \quad c < 0.$$

LEMMA 3. *Let $u(x)$ satisfy*

$$Lu(x) = 0, \tag{15}$$

$$u(0) = 0, \qquad u'(0) = 1. \tag{16}$$

If $R < K$ and restrictions (8) *are satisfied, then*

$$u(x) \geq (p(0)/M)(1 - R/K)x$$

in I. If (16) *is replaced by*

$$u(1) = 0, \qquad u'(1) = 1,$$

then

$$u(x) \leq (-p(1)/M)(1 - R/K)(1 - x).$$

Proof. Since $u(x)$ is nonoscillatory in I, $(p(x)u'(x))' = -r(x)u(x) \leq 0$. Integration from 0 to x yields

$$\int_0^x (p(x)u'(x))'\,dx = p(x)u'(x) - p(0)u'(0) \leq 0.$$

Thus, $u'(x) \leq p(0)/p(x) \leq p(0)/K$. Again integrating from 0 to x gives $\max_{x \in I} u(x) \leq p(0)/K$. But

$$(p(x)u'(x))' = -r(x)u(x) \geq -Rp(0)/K.$$

Integrating this inequality between 0 and x yields

$$\int_0^x (p(x)u'(x))'\,dx = p(x)u'(x) - p(0)u'(0) \geq -Rp(0)/K.$$

Thus,

$$u'(x) \geq p(0)(1 - R/K)/p(x) \geq p(0)(1 - R/K)/M.$$

Again if we integrate from 0 to x, we have

$$u(x) - u(0) = u(x) \geq p(0)(1 - R/K)x/M.$$

Hence, $u(x) \geq p(0)(1 - R/K)x/M$.

COROLLARY 1. *If $R < K$ and restrictions* (8) *are satisfied, then the solution of*

$$Lu(x) = 0 \tag{17}$$

$$u(0) = a, \qquad u'(0) = 1 \tag{18}$$

is bounded below by $p(0)(1 - R/K)x/M$ if $a \geq 0$. If (18) *is replaced by*

$$u(1) = -b, \qquad u'(1) = 1,$$

then $u(x)$ is bounded above by $-p(1)(1 - R/K)(1 - x)/M$ if $b \geq 0$.

Proof. Let us think of the solution $u(x)$ of problem (17)–(18) as $v(x) + aw(x)$ where $v(x)$ and $w(x)$ satisfy

$$Lv(x) = 0, \qquad v(0) = 0, \qquad v'(0) = 1 \tag{19}$$

and

$$Lw(x) = 0, \qquad w(0) = 1, \qquad w'(0) = 0 \tag{20}$$

respectively. Then $v(x) \geq 0$ by Ince's [12] oscillation theorem, and $w(x) \geq 0$ by Lemma 2. Thus,

$$u(x) = v(x) + aw(x) \geq v(x) \geq p(0)(1 - R/K)x/M$$

by Lemma 3. Hence, $u(x) \geq p(0)(1 - R/K)x/M$.

In our formula for Green's function

$$G(x, z) = \begin{cases} y_1(x)y_2(z)/-p(x)W(y_1, y_2)(x), & \text{if } 0 \leq x \leq z, \\ y_1(z)y_2(x)/-p(x)W(y_1, y_2)(x), & \text{if } z \leq x \leq 1, \end{cases}$$

we require that $y_1(x)$ must satisfy

$$Ly(x) = 0, \qquad y(0) - ay'(0) = 0 \tag{21}$$

or equivalently

$$Ly(x) = 0, \qquad y(0) = a, \qquad y'(0) = 1 \tag{22}$$

and that $y_2(x)$ must satisfy

$$Ly(x) = 0, \qquad y(1) + by'(1) = 0 \tag{23}$$

or equivalently

$$Ly(x) = 0, \qquad y(1) = -b, \qquad y'(1) = 1. \tag{24}$$

We can think of $y_2(x)$ as the sum $v(x) + bw(x)$ where

$$Lv(x) = 0, \qquad v(1) = 0, \qquad v'(1) = 1 \tag{25}$$

and

$$Lw(x) = 0, \qquad w(1) = -1, \qquad w'(1) = 0. \tag{26}$$

Ince [12] shows that $R/K < \pi^2$ implies $v(x) \leq 0$ in I. Lemma 2 shows that $R/K < 1$ implies $w(x) \leq 0$ in I. Therefore, if $R/K < 1$, $y_2(x) \leq 0$ in I. Similarly, we have $y_1(x) \geq 0$ in I as shown in Corollary 1. Hence, $G(x, z)$ always has the same sign as $W(y_1, y_2)(x)$ since $p(x) > 0$ in I. But we know that $p(x)W(y_1, y_2)(x)$ is a constant. Let us "evaluate" $p(x)W(y_1, y_2)(x)$ at $x = 1$. Thus,

$$W(1) = y_1(1) + by_1'(1). \tag{27}$$

We want to retain a positive Green's function; therefore, we now find what conditions on $r(x)$ will guarantee this.

Now with $y_1(x) = y(x)$

$$(p(x)y'(x))' = -r(x)y(x) \leq 0. \tag{28}$$

Upon integrating (28) from 0 to x, we have $p(x)y'(x) - p(0)y'(0) \leq 0$. Thus,

$$y'(x) \leq p(0)/p(x) \leq p(0)/K. \tag{29}$$

Again we integrate from 0 to x and get

$$y(x) \leq a + p(0)/K. \tag{30}$$

Hence,

$$\max_{x \in I} y_1(x) \leq a + p(0)/K. \tag{31}$$

But $(p(x)y'(x))' = -r(x)y(x) \geq -R(a + p(0)/K)$, which gives $p(x)y'(x) - p(0)y'(0) \geq -R(a + p(0)/K)$ when integrated from 0 to x. Thus,

$$y_1'(x) \geq (p(0) - R(a + p(0)/K))/p(x). \tag{32}$$

But Corollary 1 shows that $y_1(x) \geq p(0)(1 - R/K)x/M$. Therefore, in order to obtain $y_1(1) + by_1'(1) > 0$, we can require

$$p(0)(1 - R/K)/M + b(p(0) - R(a + p(0)/K))/p(1) > 0$$

or

$$R < R^{(1)} = \frac{Kp(0)(p(1) + bM)}{p(0)p(1) + abMK + bMp(0)} \leq K. \tag{33}$$

But we can find a similar result for

$$W(0) = ay_2'(0) - y_2(0) \tag{34}$$

since we could also have "evaluated" $W(y_1, y_2)(x)$ at 0. If $W(0) > 0$, then our Green's function is still nonnegative. In a manner completely analogous to that which produced result (33), we can obtain

$$-b - p(1)/K \leq \min_{x \in I} y_2(x) \tag{35}$$

and

$$(p(1) - R(b + p(1)/K))/p(x) \leq y_2'(x). \tag{36}$$

Thus, requiring

$$R < R^{(2)} = \frac{Kp(1)(p(0) + aM)}{p(0)p(1) + abMK + aMp(1)} \leq K \tag{37}$$

forces $W(0)$ to be positive.

Consequently, if R is less than either $R^{(1)}$ or $R^{(2)}$ (not the minimum), then $G(x, z)$ will be nonnegative for all x and z in I. We summarize our results with this theorem.

THEOREM 1. *If conditions* (8) *are satisfied, the Green's function of problem* (1) *with* $r(x) \geq 0$ *is nonnegative for all* x *and* z *in* I *provided* $R < \max(R^{(1)}, R^{(2)})$ *where* $R^{(1)}$ *is given in* (33) *and* $R^{(2)}$ *is given in* (37).

We present the next example to show that if $R > \max(R^{(1)}, R^{(2)})$, then the Green's function is not necessarily positive throughout the unit square. Let

$$-y''(x) - 16y(x) = f(x), \qquad y(0) = 0 = y(1).$$

Then

$$G(x, z) = \begin{cases} \sin(4x)\sin(4(1-z))/(4\sin(4)), & \text{if } 0 \leq x \leq z, \\ \sin(4z)\sin(4(1-x))/(4\sin(4)), & \text{if } z \leq x \leq 1. \end{cases}$$

Thus,

$$G(.5, .5) = \sin(2)\sin(2)/(4\sin(4)) < 0.$$

But, of course, $R = 16 > 1 = K = R^{(1)} = R^{(2)}$.

3. The monotonicity of $<y(x_1)y(x_2)>$ **with** $q(x) \leq 0$. Let

$$L_1y(x) = -(p(x)y'(x))' - r_1(x)y(x) = f(x),$$
$$y(0) - ay'(0) = 0 = y(1) + by'(1) \tag{38}$$

and

$$L_2u(x) = -(p(x)u'(x))' - r_2(x)u(x) = f(x),$$
$$u(0) - au'(0) = 0 = u(1) + bu'(1). \tag{39}$$

We want to investigate the relationship of the covariance of the solution $y(x)$ of problem (38) and the covariance of the solution $u(x)$ of problem (39) where $f(x)$ is a random function with nonnegative covariance and $r_1(x) \geq r_2(x) \geq 0$ for all x in I.

For the inverse operators L_1^{-1} and L_2^{-1}, we have

$$y(x) = L_1^{-1}f(x) \tag{40}$$

and

$$u(x) = L_2^{-1}f(x) \tag{41}$$

so that

$$\langle y(x_1)y(x_2)\rangle = (L_1^{-1}(x_1))(L_1^{-1}(x_2))\langle f(x_1)f(x_2)\rangle$$

and

$$\langle u(x_1)u(x_2)\rangle = (L_2^{-1}(x_1))(L_2^{-1}(x_2))\langle f(x_1)f(x_2)\rangle$$

since L_1^{-1} and L_2^{-1} are linear. But

$$\begin{aligned} L_1^{-1} &= (L_1)^{-1} = (L_1 - r_2(x) + r_2(x))^{-1} \\ &= (L_2 - (r_1(x) - r_2(x)))^{-1} \\ &= L_2^{-1}(I - (r_1(x) - r_2(x))L_2^{-1})^{-1}. \end{aligned} \tag{42}$$

Since the operator $(r_1(x) - r_2(x))L_2^{-1}$ maps $L^\infty(I)$ into $L^\infty(I)$ and is complete under the sup norm ($||v(x)|| = \sup_{x\in I} |v(x)|$, for $v(x) \in L^\infty(I)$), the Neuman expansion gives us

$$L_1^{-1} = L_2^{-1}\left(\sum_{n=0}^{\infty} (r_1(x) - r_2(x))^n (L_2^{-1})^n\right)$$

if

$$||(r_1(x) - r_2(x))L_2^{-1}|| < 1. \tag{43}$$

Furthermore,

$$\begin{aligned} \langle y(x_1)y(x_2)\rangle &= (L_1^{-1}(x_1))(L_1^{-1}(x_2))\langle f(x_1)f(x_2)\rangle \\ &= (L_2^{-1}(x_1))(L_2^{-1}(x_2))\left(\sum_{m=0}^{\infty} (r_1(x_1) - r_2(x_1))^m (L_2^{-1}(x_1))^m\right) \\ &\quad * \left(\sum_{n=0}^{\infty} (r_1(x_2) - r_2(x_2))^n (L_2^{-1}(x_2))^n\right)\langle f(x_1)f(x_2)\rangle \\ &= L_2^{-1}(x_1)L_2^{-1}(x_2)\langle f(x_1)f(x_2)\rangle \\ &\quad + \sum_{m+n>0}^{\infty} (r_1(x_1) - r_2(x_1))^m (r_1(x_2) - r_2(x_2))^n \\ &\quad * (L_2^{-1}(x_1))^{m+1}(L_2^{-1}(x_2))^{n+1}\langle f(x_1)f(x_2)\rangle. \end{aligned} \tag{44}$$

Thus,

$$\begin{aligned} \langle y(x_1)y(x_2)\rangle = \langle u(x_1)u(x_2)\rangle &+ \sum_{m+n>0}^{\infty} (r_1(x_1) - r_2(x_1))^m (r_1(x_2) - r_2(x_2))^n \\ &* (L_2^{-1}(x_1))^{m+1}(L_2^{-1}(x_2))^{n+1}\langle f(x_1)f(x_2)\rangle. \end{aligned} \tag{45}$$

Therefore,

$$\langle y(x_1)y(x_2)\rangle - \langle u(x_1)u(x_2)\rangle \geq 0$$

if $r_1(x) \geq r_2(x)$ for all x in I and if

$$
\begin{aligned}
(L_2^{-1}(x_1))^{m+1}(L_2^{-1}(x_2))^{n+1}\langle f(x_1)f(x_2)\rangle &= \int_0^1 \int_0^1 \cdots \int_0^1 G(x_1\,,\,x_{1,1})G(x_{1,1}\,,\,x_{1,2}) \cdots \\
&\quad * G(x_{1,m}\,,\,x_{1,m+1})G(x_2\,,\,x_{2,1})G(x_{2,1}\,,\,x_{2,2}) \cdots \\
&\quad * G(x_{2,n}\,,\,x_{2,n+1})\langle f(x_{1,m+1})f(x_{2,n+1})\rangle \\
&\quad * dx_{1,1} \cdots dx_{1,m+1}\, dx_{2,1} \cdots dx_{2,n+1} \\
&\geq 0
\end{aligned} \tag{46}
$$

where $G(x, z)$ is the Green's function corresponding to problem (39). But $G(x, z)$ is nonnegative under the restrictions of Theorem 1. Consequently, we have proved the following theorem.

THEOREM 2. *Under restrictions* (8), *the covariance of the solution of problem* (1), *acted upon by a random function f with nonnegative covariance, is monotone nondecreasing as $r(x)$ increases from $r_2(x)$ to $r_1(x)$ provided*

$$
\begin{aligned}
&(1) \quad r_1(x) \geq r_2(x) \geq 0, \\
&(2) \quad \|r_i(x)\| < \max (R^{(1)}, R^{(2)}), \quad \text{for} \quad i = 1, 2, \\
&(3) \quad \|(r_1(x) - r_2(x))L_2^{-1}(x)\| < 1 \quad \text{for all} \quad x \in I.
\end{aligned} \tag{47}
$$

We want to see how large our change in $r(x)$ may be and still retain the monotone property of the covariance of the solution. We know that if we take our initial $r(x)$ as $r_0(x)$ and change to $r_n(x)$ with $r_n(x) \geq r_0(x)$, then by condition (47.2)

$$
R_n = \|r_n(x)\| < \max (R^{(1)}, R^{(2)}). \tag{48}
$$

Condition (47.3) imposes the only restriction on $\|r_n(x) - r_0(x)\|$. Therefore, we want to change from $r_0(x)$ to $r_n(x)$ through a sequence of functions; viz., $r_0(x)$ to $r_1(x)$, $r_1(x)$ to $r_2(x)$, $\cdots$, $r_{n-1}(x)$ to $r_n(x)$ so that

$$
r_i(x) \leq r_{i+1}(x), \quad \text{for } i = 0, 1, \cdots , n - 1
$$

and

$$
\|r_i(x) - r_{i-1}(x)\| < \min (B^{(1)}(r_n), B^{(2)}(r_n)) \tag{49}
$$

where $B^{(1)}(r_n)$ and $B^{(2)}(r_n)$ are the reciprocals of the upper bounds of $G(x, z; r_n)$. Indeed,

$$
\begin{aligned}
|G(x, z; r_n)| &\leq \frac{(y_1(0)/K + p(0)y_1'(0)K)(-y_2(1)/K + p(1)y_2'(1)/K)}{p(0)(1 - R_n/K)/M + b(p(0) - R_n(a + p(0)/K))/p(1)} \\
&= 1/B^{(1)}(r_n)
\end{aligned}
$$

and

$$
\begin{aligned}
|G(x, z, r_n)| &\leq \frac{(y_1(0)/K + p(0)y_1'(0)K)(-y_2(1)/K + p(1)y_2'(1)/K)}{p(1)(1 - R_n/K)/M + a(p(1) - R_n(b + p(1)/K))/p(0)} \\
&= 1/B^{(2)}(r_n).
\end{aligned}
$$

If condition (49) is satisfied and if L_i represents the operator L of problem (1) with $-q(x) = r_i(x)$ for $i = 0, 1, \cdots , n$, then

$$
\|(r_i(x) - r_{i-1}(x))L_i^{-1}(x)\|
$$

will be bounded above either by

$$\frac{Kp(0)p(1) + bKMp(0) - R_n(p(0)p(1) + abMK + bMp(0))}{Kp(0)p(1) + bKMp(0) - R_i(p(0)p(1) + abMK + bMp(0))} \leq 1,$$

or by

$$\frac{Kp(0)p(1) + aKMp(1) - R_n(p(0)p(1) + abMK + aMp(1))}{Kp(0)p(1) + aKMp(1) - R_i(p(0)p(1) + abMK + aMp(1))} \leq 1,$$

since $R_i = ||r_i(x)|| \leq R_n$. Therefore, even though $||r_0(x) - r_n(x)||$ may be too large for Theorem 2 to be applicable, we can still obtain the result of this theorem by finding a sequence $\{r_i(x)\}_{i=1}^{n}$ with the properties

$$\begin{aligned} &(1) \quad ||r_i(x) - r_{i-1}(x)|| < B^{(j)}(r_n), \quad \text{for } j = 1, 2 \text{ and } i = 1, 2, \cdots, n, \\ &(2) \quad r_{i-1}(x) \leq r_i(x), \quad \text{for } i = 1, 2, \cdots, n, \\ &(3) \quad R_i < K, \quad \text{for } i = 0, 1, \cdots, n. \end{aligned} \tag{50}$$

Indeed,

$$r_i(x) = r_{i-1}(x) + (r_n(x) - r_{i-1}(x))B(r_n)/R_n \tag{51}$$

will be such a sequence if $B(r_n) = \min (B^{(1)}(r_n), B^{(2)}(r_n))$. Thus, repeated application of Theorem 2 to this sequence yields this theorem without restriction (47.3).

4. The monotonicity of $\langle y(x_1)y(x_2) \rangle$ with $q(x) \geq 0$. We shall extent our monotonicity result to include all continuous functions $q(x) \geq 0$ in I. Since an increase in $r(x)$ corresponds to a decrease in $q(x)$, we anticipate that the relation will be as stated in the following theorem.

THEOREM 3. *Under restrictions* (8) *with $q(x) \geq 0$ in I, the covariance of the solution of problem* (1), *acted upon by random function $f(x)$ with nonnegative covariance, is monotonically nonincreasing as $q(x)$ increases from $q_1(x)$ to $q_2(x)$ provided that*

$$||(q_1(x) - q_2(x))L_2^{-1}(x)|| < 1 \tag{52}$$

for all x in I.

Proof. We know that the Green's function for problem (1) is always nonnegative when $q(x) \geq 0$; therefore, we do not have to impose extra restrictions on $q(x)$ to guarantee a nonnegative Green's function. Mimicking our proof of Theorem 2, we have

$$\begin{aligned} \langle y(x_1)y(x_2)\rangle &= \langle (L_1^{-1}f(x_1))(L_1^{-1}f(x_2))\rangle \\ &= (L_1^{-1} + q_2(x_1) - q_2(x_1))^{-1} \\ &\quad * (L_1^{-1} + q_2(x_2) - q_2(x_2))^{-1}\langle f(x_1)f(x_2)\rangle \\ &= L_2^{-1}(x_1)(I - (q_2(x_1) - q_1(x_1))L_2^{-1}(x_1))^{-1} \\ &\quad * L_2^{-1}(x_2)(I - (q_2(x_2) - q_1(x_2))L_2^{-1}(x_2))^{-1} \\ &\quad * \langle f(x_1)f(x_2)\rangle \\ &= \langle u(x_1)u(x_2)\rangle + \sum_{m+n>0}^{\infty} (q_2(x_1) - q_1(x_1))^m(q_2(x_2) \\ &\quad - q_1(x_2))^n(L_2^{-1}(x_1))^{m+1}(L_2^{-1}(x_2))^{n+1}\langle f(x_1)f(x_2)\rangle \end{aligned} \tag{53}$$

where $q_2(x)$ corresponds to the solution $u(x)$ of $L_2u(x) = f(x)$ and $q_1(x)$ corresponds to the solution $y(x)$ of $L_1y(x) = f(x)$. Therefore, if $q_2(x) \geq q_1(x) \geq 0$ for all x in I, then $\langle y(x_1)y(x_2)\rangle \geq \langle u(x_1)u(x_2)\rangle$ provided $\|(q_2(x) - q_1(x))L_2^{-1}(x)\| < 1$ for all x in I. This proves our theorem.

Again we want to see how small $\|q_2(x) - q_1(x)\|$ must be for this theorem to apply. We write the Green's function of problem (1) in the familiar form

$$G(x, z) = \begin{cases} y_1(x)y_2(z)/(-p(x)W(y_1, y_2)(x)), & \text{if } 0 \leq x \leq z, \\ y_1(z)y_2(x)/(-p(x)W(y_1, y_2)(x)), & \text{if } z \leq x \leq 1 \end{cases}$$

where now $y_1(x)$ satisfies

$$Ly(x) = 0, \qquad y(0) = a, \qquad y'(0) = 1, \quad \text{if } 0 \leq a \leq 1, \tag{54}$$

or $y_1(x)$ satisfies

$$Ly(x) = 0, \qquad y(0) = 1, \qquad y'(0) = 1/a, \qquad \text{if } 1 \leq a \leq \infty, \tag{55}$$

and where $y_2(x)$ satisfies

$$Ly(x) = 0, \qquad y(1) = -b, \qquad y'(1) = 1, \quad \text{if } 0 \leq b \leq 1, \tag{56}$$

or $y_2(x)$ satisfies

$$Ly(x) = 0, \qquad y(1) = -1, \qquad y'(1) = 1/b, \quad \text{if } 1 \leq b \leq \infty. \tag{57}$$

In the following lemma we shall establish that $y_1(x)$ is always positive and that $y_2(x)$ is always negative.

LEMMA 4. *The solution of*

$$Ly(x) = 0 \tag{58}$$

$$y(1) = -c < 0, \qquad y'(1) = d > 0 \tag{59}$$

is negative in I under restrictions (8). *Correspondingly, if* (59) *is replaced by*

$$y(0) = c > 0, \qquad y'(0) = d > 0$$

then $y(x) > 0$ in I.

Proof. Suppose $y(z)$ is not always negative. Since $y(1) < 0$, there must be some α in $[0, 1)$ such that $y(\alpha) = 0$. Under the transformation $x = (z - \alpha)/(1 - \alpha)$, we have a nontrivial solution to

$$-(p(x)y'(x))' + (1 - \alpha)^2 q(x)y(x) = 0,$$

$$y(0) = 0, \qquad y(1) = -c.$$

But $y_x(1) = y_z(1)\, dz/dx = (1 - \alpha)\, d$ so that

$$(1 - \alpha)^2 = \frac{\int_0^1 (p(x)y'(x))'y(x)\, dx}{\int_0^1 q(x)y^2(x)\, dx}$$

$$= \frac{p(1)(1 - \alpha)\, d(-c) - \int_0^1 p(x)(y'(x))^2\, dx}{\int_0^1 q(x)y^2(x)\, dx} < 0;$$

thus, a contradiction if $0 \leq \alpha < 1$. Hence, $y(x) < 0$ in I.

Again we would like to see whether restriction (52) can be removed from Theorem 3 by means of repeated application of this theorem to a sequence $\{q_i(x)\}_{i=0}^n$ where $q_0(x)$ is $q(x)$ initially and $q_n(x)$ is $q(x)$ at the end. Since

$$||(q_i(x) - q_{i-1}(x))L_i^{-1}(x)|| \leq ||q_i(x) - q_{i-1}(x)|| \; ||G(x, z; q_i)||$$

where L_i is the operator L of equation (1) with $q(x) = q_i(x)$, we see that we need an upper bound on $G(x, z; q_i)$ in order to determine $||q_i(x) - q_{i-1}(x)||$ so that restriction (52) is fulfilled. Our objective then is to obtain an explicit upper bound for $G(x, z)$ in terms of $Q = \max_{x \in I} q(x)$.

We denote this upper bound by $D(Q)$. Let $Q_i = ||q_i(x)||$ and consider $D(Q)$ for $Q_0 \leq Q \leq Q_n$. Let $Q_{\max}$ be that $Q \in [Q_0, Q_n]$ such that $D(Q)$ is a maximum. Then by choosing

$$||q_i(x) - q_{i-1}(x)|| < 1/D(Q_{\max}),$$

we have

$$||(q_i(x) - q_{i-1}(x))L_i^{-1}(x)|| \leq D(Q_i) \; ||q_i(x) - q_{i-1}(x)|| < 1.$$

From our expression for Green's function we see that

$$|G(x, z)| \leq y_1(1)(-y_2(0))/(\min p(x)W(y_1, y_2)(x)).$$

This follows because $y_1(x) > 0$ in I and because

$$(p(x)y_1'(x))' = q(x)y_1(x) \geq 0$$

implies $p(x)y_1'(x) \geq p(0)y_1'(0) > 0$ or $y_1'(x) > 0$ in I. Similarly, $y_2(x) < 0$ in I and $y_2'(x) > 0$ in I.

We proceed to bound $y_1(1)$ above. Let $S = M \min_{x \in I} p'(x)/p(x)$. S may be any finite real number. Consider the solution of the problem

$$u''(x) + (S/M)u'(x) - (Q/K)u(x) = 0$$
$$u(0) = a, \qquad u'(0) = 1,$$

which will be

$$u(x) = T^{-1} \exp(-Sx/2M)((1 + aS/2M) \sinh(Tx) + aT \cosh(Tx))$$

where $T = (S^2/4M^2 + Q/K)^{1/2}$. Furthermore, both $u(x)$ and $u'(x)$ are nonnegative in I. But

$$\begin{aligned} u''(x) + (p'(x)/p(x))u'(x) - (q(x)/p(x))u(x) & \\ &= u''(x) + (S/M)u'(x) - (Q/K)u(x) \\ &\quad + (p'(x)/p(x) - S/M)u'(x) + (Q/K - q(x)/p(x))u(x) \\ &\geq 0. \end{aligned}$$

Thus, applying Theorem 13, page 26, from Protter and Weinberger [15], $u(x) \geq y_1(x)$ for all $x \in I$ if $y_1(x)$ satisfies problem (54); i.e., if $0 \leq a \leq 1$. Hence,

$$y_1(1) \leq T^{-1} \exp(-S/2M)((1 + aS/2M) \sinh(T) + aT \cosh(T))$$

if $0 \leq a \leq 1$. Similarly, if $1 \leq a \leq \infty$, we find that $y_1(x)$ is bounded above by

$$T^{-1} \exp(-Sx/2M)((1/a + S/2M) \sinh(Tx) + T \cosh(Tx))$$
$$= a^{-1}T^{-1} \exp(-Sx/2M)((1 + aS/2M) \sinh(Tx) + aT \cosh(Tx)).$$

Thus, for all a

$$y_1(1) \leq T^{-1} \exp(-S/2M)((1 + aS/2M) \sinh(T) + aT \cosh(T)).$$

Of course, if $1 \leq a \leq \infty$ we are able to obtain a smaller upper bound as indicated above.

In an analogous manner we find that

$$y_2(0) \geq -T^{-1} \exp(S/2M)((1 - bS/2M) \sinh(T) + bT \cosh(T))$$

for $0 \leq b \leq 1$ and

$$y_2(0) \geq -T^{-1} \exp(S/2M)((1/b - S/2M) \sinh(T) + T \cosh(T))$$

for $1 \leq b \leq \infty$. Thus,

$$|G(x, z)| \leq K^{-1}T^{-2}((1 + aS/2M) \sinh(T) + aT \cosh(T)) * \frac{((1 - bS/2M) \sinh(T) + bT \cosh(T))}{\min W(y_1, y_2)(x)}.$$

We assume that a and b are not both ∞, say $a \neq \infty$. Then

$$\begin{aligned} p(x)W(y_1, y_2)(x) &= p(1)W(y_1, y_2)(1) \\ &= p(1) \begin{vmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1) \end{vmatrix} \\ &\geq K \begin{cases} y_1(1), & \text{if } 0 \leq b \leq 1, \\ y_1'(1), & \text{if } 1 \leq b \leq \infty. \end{cases} \end{aligned}$$

But $(p(x)y_1'(x))' = q(x)y_1(x) \geq 0$, which yields after integrating from 0 to x

$$\begin{aligned} y_1'(x) \geq p(0)y_1'(0)/p(x) \geq p(0)y_1'(0)/M &= p(0)/M, \quad \text{if } 0 \leq a \leq 1, \\ &= p(0)/aM, \quad \text{if } 1 \leq a \leq \infty. \end{aligned}$$

After a second such integration, we obtain

$$y_1(x) \geq y_1(0) + p(0)y_1'(0)x/M = \begin{cases} a + p(0)x/M, & \text{if } 0 \leq a \leq 1, \\ 1 + p(0)x/aM, & \text{if } 1 \leq a \leq \infty. \end{cases}$$

Thus, $y_1(1) \geq p(0)/M$ for all a. Hence, for all a and b

$$\begin{aligned} |G(x, z)| \leq K^{-1}T^{-2}&((1 + aS/2M) \sinh(T) + aT \cosh(T)) \\ &* ((1 - bS/2M) \sinh(T) + bT \cosh(T)) \\ &* M/p(0). \end{aligned}$$

If $a = b = \infty$, we can obtain a positive lower bound on $W(y_1, y_2)$ since $\int_0^1 q(x)\,dx > 0$. Consequently, we obtain the following theorem.

THEOREM 4. *Under restrictions* (8) *with* $q(x) \geq 0$ *in* I, *the covariance of the solution of problem* (1), *acted upon by random function* f *with nonnegative covariance, is monotonically nonincreasing as* $q(x)$ *increases from* $q_1(x)$ *to* $q_2(x)$.

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