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POTENTIAL AND RAYLEIGH-SCATTERING THEORY FOR A SPHERICAL CAP*

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Abstract. Harmonic functions are constructed for spherical-harmonic prescriptions of either a potential or its normal derivative on a spherical cap. The dipole-moment tensor and the Rayleigh-scattering properties of a spherical bowl, including the limiting case of a Helmholtz resonator, are determined. The results are uniformly valid with respect to the polar angle of the cap and resolve certain discrepancies in the existing literature.

1. Introduction. We consider harmonic functions of the form

$$\psi(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \psi_n^m(r, \theta)(C_n^m \cos m\phi + S_n^m \sin m\phi), \tag{1.1}$$

where (r, θ, ϕ) are spherical polar coordinates, and either ψ_n^m (Dirichlet problem) or $\partial\psi_n^m/\partial r$ (Neumann problem) must reduce to the Legendre function P_n^m (cos θ) on a spherical cap ($\equiv$ *bowl*), $r = 1$ and $0 \leq \theta < \theta_1$ (see Fig. 1). We refer all lengths to the dimensional radius of the sphere, say a.

The solution of the Dirichlet problem for $m = n = 0$ was given originally by Kelvin [1], who determined the charge distribution on a conducting bowl through the spherical inversion of a disk. Ferrers [2] subsequently obtained the general axisymmetric solution of the Dirichlet problem through an expansion in zonal harmonics; Gallop [3] obtained similar results through inversion. Basset [4] obtained the solution for a conducting bowl in a transverse field (Dirichlet problem for $m = n = 1$) through inversion. Basset also claimed to obtain the solution of the hydrodynamic problem of a spherical bowl in an otherwise uniform flow (the Neumann problem for $n = 1$) through radial differentiation of the solution to the Dirichlet problem, although he did not give explicit results. In fact, this procedure yields physically unacceptable singularities at the rim of the bowl (Rayleigh [5] noticed the flaw in the analogous procedure for the diffraction problem for an aperture in an infinite screen). Collins [6] obtained general solutions of both the Dirichlet and Neumann problems for a spherical cap and the correct solution for the hydrodynamic problem.[1]

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[1] Alternative solutions of both the Dirichlet and Neumann potential problems may be obtained by separation of variables in toroidal coordinates (Hobson [12, Secs. 267, 268]); see, e.g., Neumann's [13] solution of Kelvin's problem. This procedure is, in principle, more direct than the expansions in spherical harmonics adopted here; however, it is less flexible in practice in consequence of the recondite character of toroidal functions *vis-à-vis* spherical harmonics.

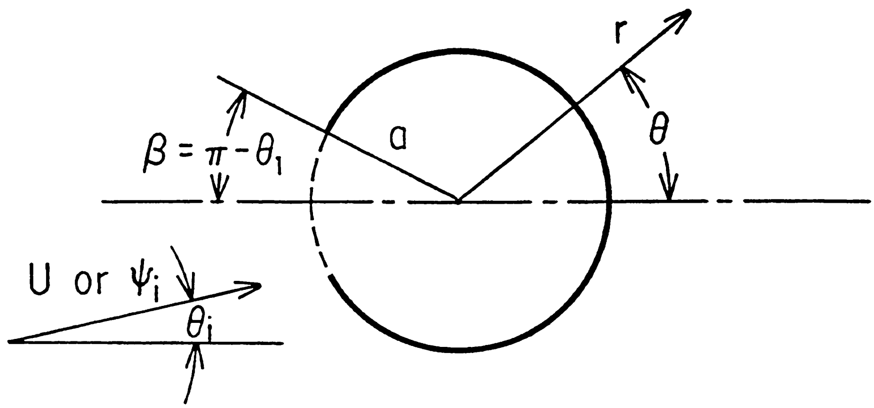

FIG. 1. Spherical bowl.

The principal application of the harmonic functions of (1.1), aside from the aforementioned potential problems, is to Rayleigh scattering by a spherical cap, including the limiting case of a Helmholtz resonator ($\theta_1 \to \pi$). By *Rayleigh scattering*, we imply

$$k \equiv 2\pi a/\lambda \ll 1, \tag{1.2}$$

where λ is the length of the incident wave. Resonance occurs at $k = k_0$, where [7]

$$k_0^2 = (3\beta/2\pi) + O(\beta^2) \qquad (\beta \equiv \pi - \theta_1 \to 0). \tag{1.3}$$

This problem has been attacked previously by Sommerfeld [8], whose incomplete analysis is entirely wide of the mark, by Morse and Feshbach [9], who considered only the Helmholtz resonator and whose end results are marred by algebraic errors, and by Collins [10], who overlooked the effect of resonance on diffraction and whose results are not uniformly valid as $\beta \to 0$.

We construct ψ_n^m (in Secs. 2 and 3) for arbitrary m and n by generalizing the solution given by Ferrers [2] and use the results to determine (in Sec. 5) uniformly valid (with respect to θ_1) approximations to the scattering amplitude and scattering cross-section for plane-wave diffraction. We also consider (in Sec. 4) the hydrodynamic problem and calculate the dipole-moment tensor for a bowl in a uniform flow. This last result, although of little direct interest for a real fluid, enters the Rayleigh-scattering problem and also illustrates an interesting theoretical point raised by Taylor [11] in connection with the virtual mass of a body that contains nearly closed cavities.

2. Dirichlet problem. Let ψ be an harmonic function of the form (1.1) for which ψ_n^m satisfies

$$\psi_n^m(r, \theta) = P_n^m(\cos\theta) \qquad (r = 1, 0 \le \theta < \theta_1) \tag{2.1a}$$

on the cap, exhibits the source-like behaviour

$$\psi_n^m(r, \theta) = O(1/r) \qquad (r \to \infty, 0 \le \theta \le \pi) \tag{2.1b}$$

at infinity, and is continuous and differentiable except at the rim of the cap ($r = 1$, $\theta = \theta_1$), where it must be bounded. We seek the continuation of ψ_n^m over the unit sphere, say

$$\psi_n^m(1, \theta) \equiv \mathcal{P}_n^m(\mu, \mu_1) \qquad (\mu \equiv \cos\theta, \mu_1 \equiv \cos\theta_1). \tag{2.2}$$

The solution of the axisymmetric problem (for which we omit the superscript, $m \equiv 0$) is given by [2][2]

$$\psi_n(r,\theta) = \sum_{s=0}^{\infty} \mathcal{S}_{ns}(\mu_1)\begin{Bmatrix} r^{-s-1} \\ r^s \end{Bmatrix} P_s(\mu) \qquad (r \gtrless 1) \tag{2.3}$$

and

$$\mathcal{P}_n(\mu,\mu_1) = (2^{1/2}/\pi)\mathcal{R}\int_0^{\theta_1} (\cos\alpha - \cos\theta)^{-1/2}\cos(n+\tfrac{1}{2})\alpha\, d\alpha \tag{2.4a}$$

$$= \sum_{s=0}^{\infty} \mathcal{S}_{ns}(\mu_1) P_s(\mu) \tag{2.4b}$$

$$= P_n(\mu) \qquad (\mu_1 \leq \mu \leq 1), \tag{2.4c}$$

where $\mathcal{R}$ implies *the real part of*, (2.4c) follows from (2.4a) by virtue of the Mehler–Laplace representation of $P_n(\mu)$,

$$\mathcal{S}_{ns}(\mu_1) = \mathcal{S}_{sn}(\mu_1) = \frac{1}{\pi}\left[\frac{\sin(n-s)\theta_1}{n-s} + \frac{\sin(n+s+1)\theta_1}{n+s+1}\right], \tag{2.5}$$

and the first term in the square brackets reduces to θ_1 for $n = s$. We note the identity

$$\mathcal{P}_n(-\mu,-\mu_1) = (-)^n(2^{1/2}/\pi)\mathcal{R}\int_{\theta_1}^{\pi} (\cos\theta - \cos\alpha)^{-1/2}\sin(n+\tfrac{1}{2})\alpha\, d\alpha \tag{2.6a}$$

$$= (-)^n P_n(\mu) \qquad (-1 \leq \mu < \mu_1) \tag{2.6b}$$

and the particular solutions

$$\mathcal{P}_0(\mu,\mu_1) = \frac{2}{\pi}\tan^{-1}\left(\frac{1-\mu_1}{\mu_1-\mu}\right)^{1/2} \qquad (-1 \leq \mu \leq \mu_1) \tag{2.7}$$

and

$$\mathcal{P}_1(\mu,\mu_1) = \frac{2}{\pi}\left[\mu\tan^{-1}\left(\frac{1-\mu_1}{\mu_1-\mu}\right)^{1/2} + (1-\mu_1)^{1/2}(\mu_1-\mu)^{1/2}\right] \quad (-1 \leq \mu \leq \mu_1), \tag{2.8}$$

where, here and subsequently, the arctangent is in $[0, \frac{1}{2}\pi]$.

We generalize (2.3) and (2.4) by constructing

$$\psi_n^m(r,\theta) = \mathcal{D}_m\left[\psi_n(r,\theta) - \sum_{j=0}^{m-1} A_{nj}^m(\mu_1)\psi_j(r,\theta)\right] \tag{2.9}$$

and

$$\mathcal{P}_n^m(\mu,\mu_1) = \mathcal{D}_m\left[\mathcal{P}_n(\mu,\mu_1) - \sum_{j=0}^{m-1} A_{nj}^m(\mu_1)\mathcal{P}_j(\mu,\mu_1)\right] \tag{2.10a}$$

$$= \sum_{s=m}^{\infty} \mathcal{S}_{ns}^m(\mu_1) P_s^m(\mu) \quad \left(\mathcal{S}_{ns}^m \equiv \mathcal{S}_{ns} - \sum_{j=0}^{m-1} A_{nj}^m \mathcal{S}_{js}\right) \tag{2.10b}$$

$$= P_n^m(\mu) \qquad (\mu_1 \leq \mu \leq 1), \tag{2.10c}$$

[2] The expansion of (2.3) may be summed to obtain an integral representation of $\psi_n(r,\theta)$, but the result is of limited interest in the present context.

where the operator $\mathfrak{D}_m$ is defined by

$$\mathfrak{D}_m P_n(\mu) \equiv (-)^m(1-\mu^2)^{m/2}(\partial/\partial\mu)^m P_n(\mu) = P_n^m(\mu), \tag{2.11}$$

and the A_{nj}^m are determined by the requirement that ψ_n^m be bounded as $r \to 1$ and $\mu \uparrow \mu_1$. Substituting (2.4a) into (2.10a), we find that this last requirement implies

$$(\partial/\partial\alpha)^p\left[\cos(n+\tfrac{1}{2})\alpha - \sum_{j=0}^{m-1} A_{nj}^m(\mu_1)\cos(j+\tfrac{1}{2})\alpha\right] = 0$$

$$(\alpha = \theta_1,\ p = 0, 1, \cdots, m-1). \tag{2.12}$$

Setting $m = 1$ in (2.10)–(2.12), we obtain

$$\mathcal{S}_{ns}^1 = \mathcal{S}_{ns} - \mathcal{S}_{0s}\sec\tfrac{1}{2}\theta_1\cos(n+\tfrac{1}{2})\theta_1 \tag{2.13a}$$

and

$$\mathcal{P}_n^1(\mu,\mu_1) = -(1-\mu^2)^{1/2}(\partial/\partial\mu)[\mathcal{P}_n(\mu,\mu_1) - \mathcal{P}_0(\mu,\mu_1)\sec\tfrac{1}{2}\theta_1\cos(n+\tfrac{1}{2})\theta_1]. \tag{2.13b}$$

The simultaneous equations implied by (2.12) for $m > 1$ may be circumvented by invoking Collins's [6] general solution to obtain an integral representation of $\mathcal{S}_{ns}^m$; however, the foregoing results suffice for the subsequent investigation.

3. Neumann problem. Let ψ be an harmonic function of the form (1.1) for which ψ_n^m satisfies

$$\partial\psi_n^m(r,\theta)/\partial r = P_n^m(\cos\theta) \qquad (r = 1, 0 \le \theta < \theta_1, n \ge 1) \tag{3.1a}$$

on the cap, exhibits the dipole-like behaviour

$$\psi_n^m(r,\theta) = O(1/r^2) \qquad (r \to \infty, 0 \le \theta \le \pi) \tag{3.1b}$$

at infinity, and is continuous and differentiable except at the rim of the cap, where it must be bounded. We seek the continuation of $\partial\psi_n^m/\partial r$ over the unit sphere, say

$$(\partial\psi_n^m/\partial r)_{r=1} \equiv \mathbf{P}_n^m(\mu,\mu_1). \tag{3.2}$$

The potential ψ_n^m is of direct interest only for $n \ge 1$, but we consider also the function $\mathbf{P}_0(\mu,\mu_1)$ in anticipation of the Helmholtz-resonator problem (see Sec. 5).

The solution of the Dirichlet problem, (2.3), together with the consideration that $\partial\psi/\partial r$ may be singular like $(\mu_1-\mu)^{-1/2}$ as $r \to 1$ and $\mu \uparrow \mu_1$, suggests that the axisymmetric function $\mathbf{P}_n(\mu,\mu_1)$ may be constructed by combining $\mathcal{P}_n(\mu,\mu_1)$ and

$$\mathcal{R}(\cos\alpha - \cos\theta)^{-1/2} = 2^{1/2}\sum_{s=0}^{\infty}\cos(s+\tfrac{1}{2})\alpha\, P_s(\mu) \tag{3.3}$$

in such a way as to render $\partial\psi_n/\partial r$ continuous across $r = 1$. This last consideration requires the elimination of the source ($s = 0$) term in the expansion of $\mathbf{P}_n(\mu,\mu_1)$ in $P_s(\mu)$, as anticipated in (3.1b); accordingly, we consider

$$\mathbf{P}_n(\mu,\mu_1) = \mathcal{P}_n(\mu,\mu_1) - \mathcal{S}_{n0}(\mu_1)(1+\mu_1)^{-1/2}\mathcal{R}(\mu_1-\mu)^{-1/2} \tag{3.4a}$$

$$= \sum_{s=1}^{\infty}\mathrm{S}_{ns}(\mu_1)P_s(\mu) \tag{3.4b}$$

$$= P_n(\mu) \qquad (\mu_1 \le \mu \le 1), \tag{3.4c}$$

where

$$\mathrm{S}_{ns}(\mu_1) = \mathcal{S}_{ns}(\mu_1) - \mathcal{S}_{n0}(\mu_1) \sec \tfrac{1}{2}\theta_1 \cos (s + \tfrac{1}{2})\theta_1 \tag{3.5a}$$

$$= \mathcal{S}^1_{sn} . \tag{3.5b}$$

The corresponding potential is given by

$$\psi_n(r, \theta) = \sum_{s=1}^{\infty} \mathrm{S}_{ns}(\mu_1)\begin{Bmatrix} -(s+1)^{-1}r^{-s-1} \\ s^{-1}r^s \end{Bmatrix} P_s(\mu) + \begin{Bmatrix} 0 \\ \Psi_n \end{Bmatrix} \qquad (r \gtrless 1), \tag{3.6}$$

where the additive constant in $r > 1$ vanishes in consequence of (3.1b), and the additive constant Ψ_n is determined by the requirement that

$$\pi_n(\mu, \mu_1) \equiv \psi_n(1-, \theta) - \psi_n(1+, \theta) \tag{3.7a}$$

$$= \sum_{s=1}^{\infty} [(2s + 1)/s(s + 1)]\mathrm{S}_{ns}(\mu_1)P_s(\mu) + \Psi_n \tag{3.7b}$$

must vanish for $-1 \leq \mu < \mu_1$. Relegating the detailed calculation to the appendix, we obtain

$$\Psi_n = \mathcal{S}_{n0}\theta_1 \tan \tfrac{1}{2}\theta_1 + (\partial\, \mathcal{S}_{ns}/\partial s)_{s=0} \tag{3.8a}$$

$$= n^{-1}(n + 1)^{-1}\mathrm{S}_{0n} \qquad (n \geq 1) \tag{3.8b}$$

$$= \pi^{-1}(\theta_1^2 \tan \tfrac{1}{2}\theta_1 + \theta_1 - \sin \theta_1) \qquad (n = 0). \tag{3.8c}$$

An integral expression for π_n, $\mu_1 \leq \mu \leq 1$, is given by (A11) in the appendix; however, the representation (3.7b) is more useful in typical applications. We note the particular solutions

$$\mathbf{P}_0(\mu, \mu_1) = \frac{2}{\pi} \tan^{-1}\left(\frac{1 - \mu_1}{\mu_1 - \mu}\right)^{1/2} - \left(\frac{\theta_1 + \sin \theta_1}{\pi}\right)(1 + \mu_1)^{-1/2}(\mu_1 - \mu)^{-1/2}$$
$$(-1 \leq \mu < \mu_1) \tag{3.9}$$

and

$$\mathbf{P}_1(\mu, \mu_1) = \frac{2}{\pi}\left[\mu \tan^{-1}\left(\frac{1 - \mu_1}{\mu_1 - \mu}\right)^{1/2} - \left(\frac{1 - \mu_1 + 2\mu}{2}\right)\left(\frac{1 - \mu_1}{\mu_1 - \mu}\right)^{1/2}\right] \qquad (-1 \leq \mu < \mu_1). \tag{3.10}$$

Referring to (2.9)–(2.11), we construct

$$\mathbf{P}^m_n(\mu, \mu_1) = \mathfrak{D}_m\left[\mathcal{P}_n(\mu, \mu_1) - \sum_{j=0}^{m-1} B^m_{nj}(\mu_1)\mathcal{P}_j(\mu, \mu_1)\right] \qquad (m \geq 1) \tag{3.11a}$$

$$= \sum_{s=m}^{\infty} \mathrm{S}^m_{ns}(\mu_1)P^m_s(\mu) \qquad \left(\mathrm{S}^m_{ns} \equiv \mathcal{S}_{ns} - \sum_{j=0}^{m-1} B^m_{nj}\, \mathcal{S}_{js}\right) \tag{3.11b}$$

$$= P^m_n(\mu) \qquad (\mu_1 \leq \mu \leq 1) \tag{3.11c}$$

and

$$\psi^m_n(r, \theta) = \sum_{s=m}^{\infty} \mathrm{S}^m_{ns}(u_1)\begin{Bmatrix} -(s+1)^{-1}r^{-s-1} \\ s^{-1}r^s \end{Bmatrix} P^m_s(\mu) \qquad (r \gtrless 1), \tag{3.12}$$

where the B^m_{nj} are determined by (3.1b) and the requirement that the singularity in

$\partial\psi_n^m/\partial r$ as $r \to 1$ and $\mu \uparrow \mu_1$ must be integrable. Invoking the latter requirements, we obtain

$$\mathrm{S}_{n0}^m \equiv \mathcal{S}_{n0} - \sum_{j=0}^{m-1} B_{nj}^m \mathcal{S}_{j0} = 0 \qquad (m \geq 1) \tag{3.13a}$$

and

$$(\partial/\partial\alpha)^p \left[\cos(n + \tfrac{1}{2}\alpha) - \sum_{j=0}^{m-1} B_{nj}^m \cos(j + \tfrac{1}{2})\alpha \right] = 0$$
$$(p = 0, 1, \cdots, m-2, m \geq 2). \tag{3.13b}$$

Setting $m = 1$ in (3.10) and (3.12a), we obtain

$$\mathrm{S}_{ns}^1 = \mathcal{S}_{ns} - (\mathcal{S}_{n0}/\mathcal{S}_{00})\mathcal{S}_{0s} \tag{3.14a}$$

and

$$\mathrm{P}_n^1(\mu, \mu_1) = -(1-\mu^2)^{1/2}(\partial/\partial\mu)[\mathcal{P}_n(\mu, \mu_1) - (\mathcal{S}_{n0}/\mathcal{S}_{00})P_0(\mu, \mu_1)]. \tag{3.14b}$$

We show in the appendix that

$$\pi_n^m(\mu, \mu_1) \equiv \psi_n^m(1-, \theta) - \psi_n^m(1+, \theta) \tag{3.15a}$$

$$= \sum_{s=m}^{\infty} [(2s+1)/s(s+1)]\mathrm{S}_{ns}^m(\mu_1)P_s^m(\mu) \tag{3.15b}$$

vanishes for $-1 \leq \mu < \mu_1$.

The simultaneous equations implied by (3.13) for $m > 1$ may be circumvented by invoking Collins's [6] general solution; however, the resulting integral representation of S_{ns}^m is rather complicated.

4. Dipole moment and virtual mass. We now suppose the bowl to be moving in an unbounded, inviscid liquid with the uniform velocity $\mathbf{U}$ directed along $\theta = \theta_i$ and $\phi = 0$. The corresponding velocity potential (defined such that the particle velocity at a given point is $\nabla\psi$) may be posed in the form

$$\psi = U[P_1(\mu_i)\psi_1(r, \theta) + P_1^1(\mu_i)\psi_1^1(r, \theta)\cos\phi], \tag{4.1}$$

where ψ_1 and ψ_1^1 are given by (3.6) and (3.12). Letting $r \to \infty$ in (3.12), we obtain

$$\psi \sim -\tfrac{1}{2}Ur^{-2}(\mathrm{S}_{11}\cos\theta_i\cos\theta + \mathrm{S}_{11}^1\sin\theta_i\sin\theta\cos\phi) \qquad (r \to \infty), \tag{4.2}$$

where, from (3.5a) and (3.14a),

$$\pi\mathrm{S}_{11} = \theta_1 + \tfrac{1}{2}\sin\theta_1 - \tfrac{1}{2}\sin 2\theta_1 - \tfrac{1}{6}\sin 3\theta_1 \tag{4.3a}$$

$$= \tfrac{4}{3}\theta_1^3 - \tfrac{7}{15}\theta_1^5 + O(\theta_1^7) \qquad (\theta_1 \to 0) \tag{4.3b}$$

$$= \pi - \tfrac{1}{5}\beta^5 + O(\beta^7) \qquad (\beta \equiv \pi - \theta_1 \to 0). \tag{4.3c}$$

and

$$\pi\mathrm{S}_{11}^1 = \theta_1 + \tfrac{1}{3}\sin 3\theta_1 - (\theta_1 + \sin\theta_1)^{-1}(\sin\theta_1 + \tfrac{1}{2}\sin 2\theta_1)^2 \tag{4.4a}$$

$$= \tfrac{8}{45}\theta_1^5 + O(\theta_1^7) \qquad (\theta_1 \to 0) \tag{4.4b}$$

$$= \pi - \tfrac{3}{2}\beta^3 + O(\beta^5) \qquad (\beta \to 0). \tag{4.4c}$$

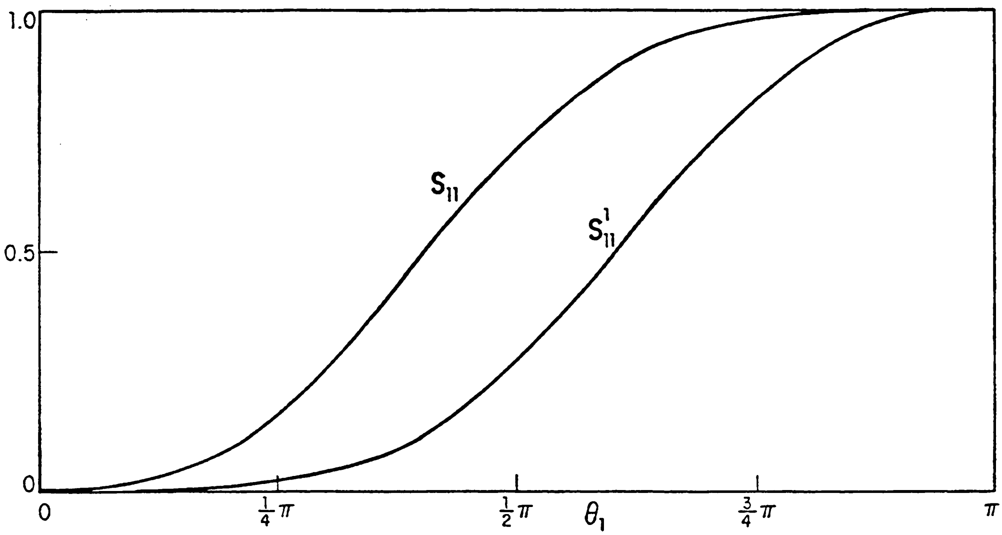

FIG. 2. Dipole-moment parameters for spherical bowl, as given by (4.3) and (4.4).

We infer from (4.2) that the dipole-moment tensor of the bowl is diagonal (as is directly evident from symmetry) and has the Cartesian components $\frac{1}{2}a^3\{S_{11}, S_{11}^1, S_{11}^1\}$, where a is the dimensional radius of the sphere. The corresponding kinetic energy of the fluid motion is (cf. Lamb [14, Sec. 121a])

$$T = \pi\rho U^2 a^3(S_{11} \cos^2\theta_i + S_{11}^1 \sin^2\theta_i) \tag{4.5a}$$

$$= \tfrac{1}{2}\rho(U \cos\theta_i)^2(\tfrac{8}{3}a^3\theta_1^3)[1 + O(\theta_1^2)] \qquad (\theta_1 \to 0) \tag{4.5b}$$

$$= \tfrac{1}{2}\rho U^2(2\pi a^3)[1 + O(\beta^3)] \qquad (\beta \to 0). \tag{4.5c}$$

The limiting result (4.5b) corresponds to a circular disk of radius $a\theta_1$. The limiting result (4.5c) implies that the virtual mass of a sphere containing a small hole is approximately three times that of a closed sphere, although (or because) their dipole moments are approximately equal (Taylor [11] gives a qualitative discussion of this paradox).

5. Rayleigh scattering. Let the acoustical plane wave

$$\psi_i = \exp[ikr(\cos\theta_i \cos\theta + \sin\theta_i \sin\theta \cos\phi)] \tag{5.1}$$

be incident upon the bowl and let $\psi(r, \theta)$ be the scattered wave; then

$$\nabla^2\psi + k^2\psi = 0, \tag{5.2}$$

$$\partial(\psi_i + \psi)/\partial r = 0 \qquad (r = 1, 0 \leq \theta < \theta_1, 0 \leq \phi \leq 2\pi), \tag{5.3}$$

and

$$\psi \sim f(\theta, \phi)r^{-1}e^{ikr} \qquad (r \to \infty, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi), \tag{5.4}$$

where $f(\theta, \phi)$ is the scattering amplitude. We seek the limiting form of $f(\theta, \phi)$ as $k \to 0$ (Rayleigh scattering). We omit the factor $\exp(-ikct)$ from ψ_i and ψ, which must be regarded as complex amplitudes in the conventional sense; in particular, $f(\theta, \phi)$ may be complex. (It must be recalled, in interpreting the subsequent results, that many writers—notably Rayleigh and Lamb—use the time dependence exp ($ikct$).)

Rayleigh's [15] treatment of scattering by small ($k^2 \ll 1$) obstacles reveals that the spherical-harmonic representation of the scattered wave is dominated by the source ($n = 0$) and dipole ($n = 1$) components. The result for a closed, axisymmetric obstacle is

$$f(\theta, \phi) = -\tfrac{1}{3}k^2 \mathrm{S}_0 + f_1(\theta, \phi), \tag{5.5a}$$

where

$$f_1(\theta, \phi) = \tfrac{1}{2}k^2(\mathrm{S}_{11} \cos \theta_i \cos \theta + \mathrm{S}_{11}^1 \sin \theta_i \sin \theta \cos \phi) \tag{5.5b}$$

is the dipole component (in which S_{11} and S_{11}^1) are defined as in Sec. 4 above), $4\pi a^3 \mathrm{S}_0/3$ is the volume of the obstacle, and all lengths are referred to a. The corresponding, total scattering cross-section is given by

$$\sigma = a^2 \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} |f(\theta, \phi)|^2 \, d\phi \tag{5.6a}$$

$$= \tfrac{1}{3}\pi a^2 k^4 [\tfrac{4}{3} |\mathrm{S}_0|^2 + \mathrm{S}_{11}^2 \cos^2 \theta_i + (\mathrm{S}_{11}^1)^2 \sin^2 \theta_i] \tag{5.6b}$$

and reduces to $7\pi a^2 k^4/9$ for a sphere of radius a (for which $\mathrm{S}_0 = \mathrm{S}_{11} = \mathrm{S}_{11}^1 = 1$).

The results for a spherical cap would appear to follow from (5.5) and (5.6) by setting $\mathrm{S}_0 = 0$ and substituting S_{11} and S_{11}^1 from (4.3) and (4.4). In fact, the results so obtained are not uniformly valid for $\beta = \pi - \theta_1 \to 0$, and the effective value of $|\mathrm{S}_0|^2$, *qua* normalized intensity of the spherically symmetric scattered wave, increases to a resonant peak at, say, $k = k_0$ and then decreases to unity at $\beta = 0$. The value of k_0 for a spherical bowl, as calculated by Rayleigh [7], is

$$k_0 = (3\beta/2\pi)^{1/2}[1 + (9\beta/20\pi) + O(\beta^2)] \qquad (\beta \to 0). \tag{5.7}$$

The value of S_0 for $k_0 = O(k)$, as inferred from an heuristic combination of Rayleigh's results with Lamb's [16, Sec. 88] analysis of plane-wave diffraction by a resonator, is

$$\mathrm{S}_0 = k^2(k^2 - k_0^2 + \tfrac{1}{3}ik_0^2k^3)^{-1} \qquad [k_0 = O(k), \, k \to 0], \tag{5.8a}$$

$$\to 1 \qquad (k_0/k \to 0) \tag{5.8b}$$

$$= O(k^2) \qquad (k_0 \gg k). \tag{5.8c}$$

There are, however, discrepancies between the results implied by (5.5)–(5.8) and those given by Morse and Feshbach [9] and by Collins [10]. Morse and Feshbach's results agree qualitatively with those of (5.5)–(5.8) but appear to contain algebraic errors. Collins arrives at the surprising and, it appears, erroneous conclusion that "the scattering cross section of the cap [is] discontinuous as [θ_1] tends to π." It therefore appears worthwhile to offer a more systematic derivation of the above results that is not only uniformly valid with respect to θ_1, but also retains all terms consistent with the basic approximation, which imposes an error factor of $1 + O(k^2)$ in consequence of the approximation of the Helmholtz equation, (5.2), by Laplace's equation in the neighborhood of the obstacle.

We construct the solution of (5.2) and (5.3) by invoking the known expansion of ψ_i in spherical harmonics, posing a similar expansion for ψ, and satisfying (5.3) term by term:

$$\psi_i + \psi = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (2n+1)i^n(2 - \delta_m^0)[(n-m)!/(n+m)!]P_n^m(\cos\theta_i)$$
$$\cdot [j_n(kr)P_n^m(\cos\theta) - kj_n'(k)\psi_n^m(r,\theta)] \cos m\phi, \tag{5.9}$$

where

$$\partial\psi_n^m/\partial r = P_n^m(\cos\theta) \qquad (r = 1, 0 \le \theta < \theta_1). \tag{5.10}$$

Following the development of Sec. 3, we pose the solution of (5.2) and (5.10) in the form

$$\psi_n^m(r,\theta) = \sum_{s=m}^{\infty} \mathfrak{S}_{ns}^m \chi_s(r) P_s^m(\mu), \tag{5.11}$$

where

$$\chi_s(r) = [kh_s'(k)]^{-1}h_s(kr) \qquad (r > 1) \tag{5.12a}$$
$$= [kj_s'(k)]^{-1}j_s(kr) \qquad (r < 1), \tag{5.12b}$$

and $h_s \equiv h_s^{(1)}$ is a spherical Hankel function. Invoking (5.10) and the requirement that ψ_n^m be continuous across the remainder of the unit sphere, we obtain

$$\sum_{s=m}^{\infty} \mathfrak{S}_{ns}^m P_s^m(\mu) = P_n^m(\mu) \qquad (\mu_1 < \mu \le 1) \tag{5.13a}$$

and

$$\sum_{s=m}^{\infty} \zeta_s \mathfrak{S}_{ns}^m P_s^m(\mu) = 0 \qquad (-1 \le \mu < \mu_1), \tag{5.13b}$$

where

$$\zeta_s = \chi_s(1-) - \chi_s(1+) \tag{5.14a}$$
$$= i[k^3 j_s'(k) h_s'(k)]^{-1} \tag{5.14b}$$
$$= -(3/k^2)(1 - \tfrac{2}{5}k^2 - \tfrac{1}{3}ik^3) + O(k^2) \qquad (s = 0) \tag{5.14c}$$
$$= s^{-1}(s+1)^{-1}(2s+1) + O(k^2) \qquad (s \ge 1). \tag{5.14d}$$

We now invoke the restriction $k \ll 1$. Transferring the constant ($s = 0$) terms from the left- to the right-hand sides of (5.13a, b), invoking (5.14d) for $s \ge 1$, and comparing the results to (3.4) and (3.7) for $m = 0$ and to (3.11) and (3.15) for $m \ge 1$, we obtain

$$\mathfrak{S}_{ns}^m = \mathrm{S}_{ns}^m + O(k^2) \qquad (m \ge 1), \tag{5.15a}$$
$$\mathfrak{S}_{ns} = \mathrm{S}_{ns} - \mathrm{S}_n \mathrm{S}_{0s} + O(k^2) \qquad (m = 0, s \ge 1), \tag{5.15b}$$

and

$$\mathfrak{S}_{n0} \equiv \mathrm{S}_n = [\Psi_n + O(k^2)]/[\zeta_0 + \Psi_0 + O(k^2)]. \tag{5.15c}$$

Substituting ζ_0 from (5.14c) and Ψ_n from (3.8b) into (5.15c), we obtain

$$k_0 = (\tfrac{1}{3}\Psi_0 + \tfrac{2}{5})^{-1/2} \tag{5.16}$$

and

$$S_0 = k^2(1 - \tfrac{2}{5}k_0^2)(k^2 - k_0^2 + \tfrac{1}{3}ik_0^2k^3)^{-1}, \tag{5.17}$$

which reduce to (5.7) and (5.8) for $k_0 \ll 1$, and

$$S_n = \tfrac{1}{3}n^{-1}(n+1)^{-1}S_{0n}k_0^2k^2(k^2 - k_0^2 + \tfrac{1}{3}ik_0^2k^3)^{-1}. \tag{5.18}$$

The error factors for (5.17) and (5.18) are of the form $1 + O(k^2)$, uniformly with respect to θ_i (the uniform validity of the error estimate in the neighborhood of $k = k_0$ depends on the readily established fact that the real and imaginary parts of the error in the denominator of (5.17) are $O(k^4)$ and $O(k^5)$, respectively). We omit the error estimates throughout most of the subsequent development with the implicit understanding that they are of this form except as explicitly noted to the contrary.

Substituting (5.15) into (5.11), letting $kr \to \infty$ and $k \to 0$, in which limit

$$\chi_s(r) \sim (-i)^{s+2}k^s[(s+1)\cdot 1\cdot 3 \cdots (2s-1)]^{-1}r^{-1}e^{ikr}, \tag{5.19}$$

and neglecting terms that are definitely small in the sense of the preceding paragraph, we obtain

$$\psi_n^m \sim r^{-1}e^{ikr}\{-\delta_0^m S_n[1 + \tfrac{1}{2}ikS_{01}P_1(\mu)] + \tfrac{1}{2}ikS_{n1}^m P_1^m(\mu) + \tfrac{1}{9}k^2S_{n2}^m P_2^m(\mu)\}. \tag{5.20}$$

Substituting (5.20) into the corresponding approximation to the ψ component of (5.9),

$$\begin{aligned}\psi = \tfrac{1}{3}k^2\psi_0 - ik \sum_{m=0}^{1} P_1^m(\mu_i)\psi_1^m \cos m\phi \\ + \tfrac{2}{3}k^2 \sum_{m=0}^{2} (2 - \delta_m^0)[(2-m)!/(2+m)!]P_2^m(\mu_i)\psi_2^m \cos m\phi,\end{aligned} \tag{5.21}$$

substituting S_0 and S_1 from (5.17) and (5.18), observing that $S_{21} = \frac{1}{3}S_{12}$ and $S_{21}^1 = S_{12}^1$, and omitting the factor $\exp(ikr)/r$ in accord with the definition (5.4), we obtain the scattering amplitude in the form

$$\begin{aligned}f(\theta, \phi) = \tfrac{1}{3}k^2[1 - (k/k_0)^2 - \tfrac{1}{3}ik^3]^{-1}[(k/k_0)^2 + \tfrac{1}{2}ikS_{01}(\mu - \mu_i)] \\ + f_1(\theta, \phi) + \tfrac{1}{9}ik^3S_{12}[P_2(\mu_i)P_1(\mu) - P_1(\mu_i)P_2(\mu)] \\ + \tfrac{1}{9}ik^3S_{12}^1[P_2^1(\mu_i)P_1^1(\mu) - P_1^1(\mu_i)P_2^1(\mu)] \cos\phi,\end{aligned} \tag{5.22}$$

where f_1, the first approximation to the dipole component, is given by (5.5b).

Retaining only the dominant terms in (5.22), we obtain

$$f(\theta, \phi) = [\tfrac{1}{3}k^4(k_0^2 - k^2 - \tfrac{1}{3}ik_0^2k^3)^{-1} + f_1(\theta, \phi)][1 + O(k)], \tag{5.23}$$

which is identical with the approximation provided by (5.5) and (5.8). The total scattering cross-section obtained by substituting (5.22) into (5.6a) is identical with that given by (5.6b) and (5.8), namely

$$\sigma = \tfrac{4}{3}\pi a^2k^4\{\tfrac{4}{3}k^4[(k^2 - k_0^2)^2 + \tfrac{1}{9}k_0^4k^6]^{-1} + S_{11}^2\cos^2\theta_i + (S_{11}^1)^2\sin^2\theta_i\}[1 + O(k^2, k_0^2)]. \tag{5.24}$$

We further simplify (5.22) and (5.24) for the Helmholtz resonator, for which $k_0 = O(k)$, $S_{01} = 3 + O(k_0^2)$, $S_{12} = O(k_0^6)$, $S_{12}^1 = O(k_0^2)$, $S_{11} = 1 + O(k_0^2)$, $S_{11}^1 = 1 + O(k_0^2)$,

$$f(\theta, \phi) = \{\tfrac{1}{3}k^2[1 - (k/k_0)^2 - \tfrac{1}{3}ik^3]^{-1}[(k/k_0)^2 + \tfrac{3}{2}ik(\mu - \mu_s)] + f_1(\theta, \phi)\}[1 + O(k^2, k_0^2)], \qquad (5.25)$$

and

$$\sigma = \tfrac{1}{3}\pi a^2 k^4\{\tfrac{4}{3}k^4[(k^2 - k_0^2)^2 + \tfrac{1}{9}k_0^4k^6]^{-1} + 1\}[1 + O(k^2, k_0^2)]. \qquad (5.26)$$

The approximation (5.25) differs from Morse and Feshbach's [9, (11.3.81)] result (after allowing for the fact that their bowl is defined by $\theta_1 < \theta \leq \pi$) only in their approximation for k_0, (1.3) rather than (5.7), and in their basic dipole component, which they give as $2f_1$; however, the latter discrepancy appears to represent a minor error in their analysis. The approximation (5.26) differs from Morse and Feshbach's result [9, (11.3.82)],

$$\sigma_{MF} = 4\pi a^2 k^4\{k^4[(k^2 - k_0^2)^2 + \tfrac{1}{9}k_0^4k^6]^{-1} + 3\}[1 + O(k^2, k_0^2)], \qquad (5.27)$$

both because of the aforementioned error and because of (what appears to be) an additional slip.

The maximum scattering cross-section implied by (5.26) is

$$\sigma_{\max} = 4\pi a^2 k_0^{-2} \equiv \lambda_0^2/\pi \qquad (k = k_0), \qquad (5.28)$$

where λ_0 is the resonant wavelength. This last result also may be inferred directly from Lamb's analysis [16, p. 279] of resonant scattering, which provides further support for the correctness of (5.26) *vis-à-vis* (5.27). We also note that (5.26) implies that σ achieves a minimum of $26.22a^2k_0^4$ at $k = 1.358\,k_0$; however this minimum is still much larger than the corresponding value of σ for a sphere, namely $8.31a^2k_0^4$. The ratio of σ to its value for a sphere at $k = k_0$, namely $7\pi a^2k_0^4/9$, is plotted in Fig. 3 for $k_0 \ll 1$ (such that the damping term, $\frac{1}{9}k_0^4k^6$, is negligible in the numerical range of the plot).

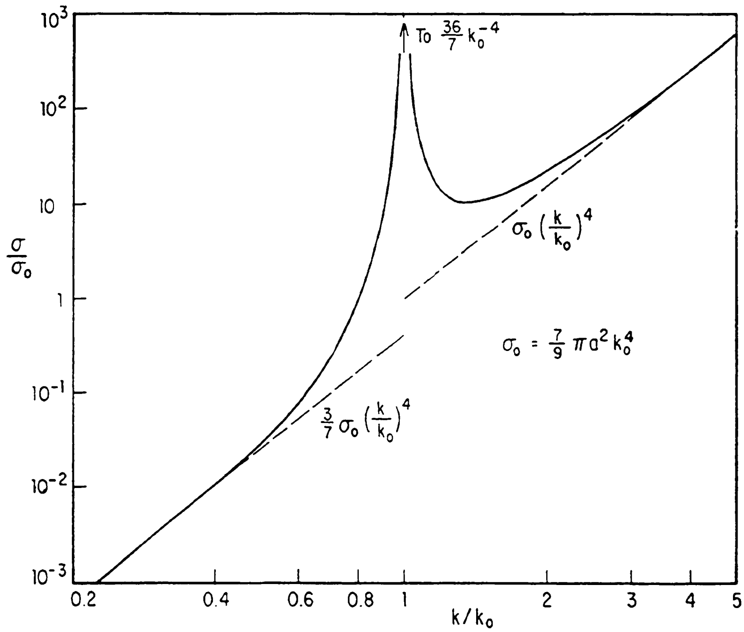

FIG. 3. Variation of scattering cross-section with k/k_0, as given by (5.26) with $k_0 \ll 1$. The reference value, σ_0, is the scattering cross section of a sphere at $k = k_0$. The upper and lower dashed lines give the total value and the dipole component, respectively, of σ/σ_0 for a sphere.

We conclude by calculating the radial velocity in the aperture ($r = 1$, $\theta_1 < \theta \leq \pi$), say v. Differentiating (5.9) with respect to r, invoking

$$(\partial \psi_n^m / \partial r)_{r=1} = \sum_{s=m}^{\infty} \mathfrak{S}_{n,}^m \, P_s^m(\mu) \tag{5.29a}$$

$$= \delta_0^m \mathrm{S}_n[1 - \mathbf{P}_0(\mu, \mu_1)] + \mathbf{P}_n^m(\mu, \mu_1), \tag{5.29b}$$

where (5.29a) follows from (5.11) and (5.12) and (5.29b) follows from (5.29a) through (5.15) and (3.11b), and retaining only the dominant terms, we obtain

$$v = -\tfrac{1}{3}k^2[1 - (k/k_0)^2 - \tfrac{1}{3}ik^3]^{-1}[1 - \mathbf{P}_0(\mu, \mu_1)]$$
$$+ ik \sum_{m=0}^{1} P_1^m(\mu_1)[P_1^m(\mu) - P_1^m(\mu, \mu_1)] \cos m\phi \tag{5.30}$$

within $1 + O(k)$. Invoking the additional restriction $\beta \ll 1$ in the formulae for $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_1^1$, we obtain

$$1 - \mathbf{P}_0 = 2^{1/2}\beta^{-1}(\mu_1 - \mu)^{-1/2}, \tag{5.31a}$$

$$P_1 - \mathbf{P}_1 = (2^{-1/2}\pi)^{-1}[\beta^2(\mu_1 - \mu)^{-1/2} - 6(\mu_1 - \mu)^{1/2}], \tag{5.31b}$$

and

$$P_1^1 - \mathbf{P}_1^1 = -3(2^{1/2}\pi)^{-1}(1 - \mu^2)^{1/2}(\mu_1 - \mu)^{-1/2}, \tag{5.31c}$$

all within $1 + O(\beta^2)$. Substituting (5.3 into (5.30) and expressing β in terms of k_0, we obtain

$$v = (2^{1/2}\pi)^{-1}k^2(k^2 - k_0^2 + \tfrac{1}{3}ik_0^2k^3)^{-1}(\mu_1 - \mu)^{-1/2}[1 + O(k, k_0^2)], \tag{5.32}$$

which is essentially the approximation invoked by Morse and Feshbach [9].

The velocity in the aperture of a Helmholtz resonator for normal incidence also has been calculated by Sommerfeld [8] by an *ad hoc* extension of the method of least squares to the dual equations of (5.13). Converting his result to the present notation, we obtain

$$v = \tfrac{1}{2}k^3 \sum_{n=0}^{\infty} (2n + 1)\zeta_n P_n(\mu) \sum_{m=0}^{\infty} (2m + 1)i^m j_m'(k)\zeta_m \int_{-1}^{\mu_1} P_m(\nu)P_n(\nu)\, d\nu \tag{5.33a}$$

$$= -\tfrac{3}{4}\beta^2[1 - \tfrac{3}{2}ik + O(k^2, k_0^4)], \tag{5.33b}$$

which bears very little resemblance to (5.32). The discrepancy appears to result both from the rather arbitrary weighting accorded to the two dual equations by Sommerfeld and from deficiencies in his order-of-magnitude estimates of the expansion coefficients.

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Appendix. We wish to show that

$$\pi_n(\mu, \mu_1) = \sum_{s=1}^{\infty} [(2s + 1)/s(s + 1)]\mathrm{S}_{ns}(\mu_1)P_s(\mu) + \Psi_n \tag{A1}$$

vanishes in $-1 \leq \mu < \mu_1$ for the appropriate choice of the constant Ψ_n. The coefficient S_{ns} is given by (3.5) and (2.5).

Differentiating (A1) with respect to μ and invoking

$$(1 - \mu^2)(dP_n/d\mu) = n(n+1)(2n+1)^{-1}(P_{n-1} - P_{n+1}), \tag{A2}$$

we obtain

$$(1 - \mu^2)(d\pi_n/d\mu) = \sum_{s=1}^{\infty} \mathrm{S}_{ns}(P_{s-1} - P_{s+1}) \tag{A3a}$$

$$= \sum_{s=0}^{\infty} (\mathrm{S}_{n,s+1} - \mathrm{S}_{n,s-1})P_s \ , \tag{A3b}$$

where (A3b) follows from (A3a) by virtue of the identities $\mathrm{S}_{n0} \equiv \mathrm{S}_{n,-1} \equiv 0$. Substituting S_{ns} into (A3b) from (3.5a) and invoking (2.4) and (2.6), we obtain

$$(1 - \mu^2)(d\pi_n/d\mu) = P_{n+1}(\mu) - P_{n-1}(\mu) + (-)^n[\mathcal{P}_{n+1}(-\mu, -\mu_1) - \mathcal{P}_{n-1}(-\mu, -\mu_1)] + 2\,\mathcal{S}_{n0}(1 - \mu_1)^{1/2}\mathcal{R}(\mu - \mu_1)^{-1/2} \tag{A4a}$$

$$= 0 \qquad (-1 \leq \mu < \mu_1), \tag{A4b}$$

from which we infer that π_n is constant in $-1 \leq \mu < \mu_1$. Setting $\mu = -1$ in (A1) and requiring $\pi_n(-1, \mu_1)$ to vanish, we obtain

$$\Psi_n = \sum_{s=1}^{\infty} (-)^{s-1}[(2s+1)/s(s+1)]\mathrm{S}_{ns}(\mu_1) \tag{A5a}$$

$$\equiv C_1 + C_2 \ , \tag{A5b}$$

where

$$C_1 = \sum_{s=1}^{\infty} (-)^{s-1}[(2s+1)/s(s+1)]\,\mathcal{S}_{ns} \ , \tag{A6}$$

and

$$C_2 = \mathcal{S}_{n0} \sec \tfrac{1}{2}\theta_1 \sum_{s=1}^{\infty} (-)^s[(2s+1)/s(s+1)] \cos (s + \tfrac{1}{2})\theta_1 \ . \tag{A7}$$

Substituting $\mathcal{S}_{ns}$ into (A6) from (2.5), invoking the partial-fraction expansion

$$(2s+1)s^{-1}(s+1)^{-1} = s^{-1} + (s+1)^{-1}, \tag{A8}$$

and rearranging the coefficients of like reciprocal integers in the summation, we obtain

$$C_1 = (2/\pi) \sum_{s=1}^{\infty} (-)^{s-1}[s^{-1} + (s+1)^{-1}] \int_0^{\theta_1} \cos (n + \tfrac{1}{2})\alpha \cos (s + \tfrac{1}{2})\alpha \, d\alpha \tag{A9a}$$

$$= (2/\pi) \int_0^{\theta_1} \cos (n + \tfrac{1}{2})\alpha \left[\cos \tfrac{1}{2}\alpha - 2 \sin \tfrac{1}{2}\alpha \sum_{s=1}^{\infty} (-)^{s-1}s^{-1} \sin \alpha s \right] d\alpha \tag{A9b}$$

$$= (2/\pi) \int_0^{\theta_1} \cos (n + \tfrac{1}{2})\alpha(\cos \tfrac{1}{2}\alpha - \alpha \sin \tfrac{1}{2}\alpha) \, d\alpha \tag{A9c}$$

$$= \mathcal{S}_{n0} + (\partial\, \mathcal{S}_{ns}/\partial s)_{s=0} \ , \tag{A9d}$$

where (A9d) follows from (A9c) through (2.5). Turning to (A7), we obtain

$$C_2 = 2\,\mathcal{S}_{n0} \sec \tfrac{1}{2}\theta_1(\partial/\partial\theta_1) \sum_{s=1}^{\infty} (-)^s s^{-1}(s+1)^{-1} \sin (s + \tfrac{1}{2})\theta_1 \tag{A10a}$$

$$= 2\,\mathcal{S}_{n0} \sec \tfrac{1}{2}\theta_1 (\partial/\partial\theta_1)(\sin \tfrac{1}{2}\theta_1 - \theta_1 \cos \tfrac{1}{2}\theta_1) \tag{A10b}$$

$$= \mathcal{S}_{n0}(\theta_1 \tan \tfrac{1}{2}\theta_1 - 1). \tag{A10c}$$

Substituting (A9d) and (A10c) into (A5b), we obtain (3.8a).

Substituting $\mathcal{S}_{n0}$ from (2.5) and $\mathcal{P}_n$ from (2.6a) into (A4a) and integrating $(d\pi_n/d\mu)$ from $\mu = \mu_1$, we obtain

$$\pi_n(\mu, \mu_1) = 2 \int_{\mu_1}^{\mu} (1 - \mu^2)^{-1}\, d\mu \Big[(2^{1/2}/\pi) \int_{\theta_1}^{\theta} (\mu - \cos \alpha)^{-1/2} \sin \alpha \cos (n + \tfrac{1}{2})\alpha \, d\alpha + \mathcal{S}_{n0}(1 + \mu_1)^{1/2}(\mu - \mu_1)^{-1/2} \Big] \quad (\mu_1 \le \mu \le 1). \tag{A11}$$

Turning to π_n^m, we rewrite (3.14) in the form

$$\pi_n^m = \mathfrak{D}_m \sum_{s=1}^{\infty} [2s + 1)/s(s + 1)] \mathrm{S}_{ns}^m(\mu_1) P_s(\mu) \equiv \mathfrak{D}_m \Omega_n^m , \tag{A12}$$

where the operator $\mathfrak{D}_m$ is defined by (2.11). Differentiating Ω_n^m as in (A3), we obtain

$$(1 - \mu^2)(d\Omega_n^m/d\mu) = \sum_{s=1}^{\infty} (\mathrm{S}_{n,s+1}^m - \mathrm{S}_{s-1}^m) P_s \tag{A13}$$

by virtue of (A2) and $\mathrm{S}_{n0}^m \equiv \mathrm{S}_{n,-1}^m \equiv 0$. Substituting S_{ns}^m into (A13) from (3.11b) and proceeding as in (A4), we obtain

$$(1 - \mu^2)(d\Omega_n^m/d\mu) = 0 \qquad (-1 \le \mu < \mu_1), \tag{A14}$$

from which we infer that Ω_n^m is constant, and π_n^m vanishes, in $-1 \le \mu < \mu_1$.

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