ON DISPLACEMENT DERIVATIVES*

BY

RAY M. BOWEN AND C. -C. WANG

Rice University

1. Introduction. The concept of the *displacement derivative* of a function or a vector field relative to a moving surface was introduced by Hayes [1] and Thomas [2] and [4] in discussions of the propagation of waves in solids and fluids. Both Hayes [1, Sec. 3] and Thomas [2, Sec. 4] and [4, Chap. II] stated, in effect, that the displacement derivative is the time derivative following the normal trajectory of a moving surface. Using this concept, Hayes and Thomas obtained a formula for the displacement derivative of the unit normal $\mathbf{n}$ of a moving surface:

$$\delta n^k/\delta t = -a^{\Gamma\Delta}x^k_{,\Gamma}u_{n,\Delta}\,, \tag{1.1}$$

where $\delta/\delta t$ denotes the displacement derivative, u_n denotes the normal speed of the moving surface, $a^{\Gamma\Delta}$ denotes the components of the induced surface metric, and x^k denotes the spatial coordinates of the moving surface. Of course, x^k is given by functions of the surface coordinates y^Γ and the time t, viz.

$$x^k = x^k(y^1, y^2, t). \tag{1.2}$$

As usual, the partial derivative with respect to a spatial coordinate or a surface coordinate is denoted by a comma followed by the index of the coordinate—the index of a spatial coordinate is a Latin minuscule having range from 1 to 3, while the index of a surface coordinate is a Greek majuscule having range from 1 to 2.

Neither Thomas nor Hayes gave a formula for the displacement derivative of a surface vector field or a surface tensor field. However, following Hayes and Thomas' concept, Truesdell and Toupin [3, Sec. 179] presented a generalization to displacement derivatives for arbitrary mixed spatial-surface tensor fields. Specifically, Truesdell and Toupin gave the formula [3, Eq. (179.5)]:

$$\delta_d\Psi^{k\ldots m\Gamma\ldots\Delta}_{p\ldots q\Lambda\ldots\Sigma}/\delta t = \partial\Psi^{k\ldots m\Gamma\ldots\Delta}_{p\ldots q\Lambda\ldots\Sigma}/\partial t + \Psi^{k\ldots m\Gamma\ldots\Delta}_{p\ldots q\Lambda\ldots\Sigma,i}u_n n^i + \mathfrak{L}_{\mathbf{u}}\Psi^{k\ldots m\Gamma\ldots\Delta}_{p\ldots q\Lambda\ldots\Sigma}\,, \tag{1.3}$$

where $\delta_d/\delta t$ denotes the Truesdell and Toupin generalization of the displacement derivative, $\partial\Psi^{k\ldots m\Gamma\ldots\Delta}_{p\ldots q\Lambda\ldots\Sigma}/\partial t$ denotes the usual partial derivative of the components of Ψ with respect to t, $\Psi^{k\ldots m\Gamma\ldots\Delta}_{p\ldots q\Lambda\ldots\Sigma,i}$ denotes the components of the spatial gradient of Ψ, and $\mathfrak{L}_{\mathbf{u}}\Psi$ denotes the Lie derivative of Ψ with respect to the tangential velocity $\mathbf{u}$. For any function or any spatial vector field, the Truesdell–Toupin formula reduces to a formula for the displacement derivative as defined by Hayes and Thomas. In particular, Truesdell and Toupin [3, Eq. (179.19)] recovered the Hayes–Thomas formula (1.1) for the displacement derivative of the unit normal of the moving surface.

* Received December 17, 1969.

The aims of this paper are two: (i) to determine the geometric meaning, the usefulness, and the limitation of the Truesdell–Toupin formula (1.3), and (ii) to give a new generalization of the concept of displacement derivative different from that of Truesdell and Toupin for mixed spatial-surface tensor fields. Of course, for functions and spatial vector fields, our generalization, like that of Truesdell and Toupin, reduces again to the original concept of Hayes and Thomas. In particular, we shall give also a simple proof of the Hayes–Thomas formula (1.1).

In order to motivate our generalization of the Hayes–Thomas displacement derivative for surface tensor fields, we consider the following example: Let S_t be a moving cylindrical surface characterized by the condition

$$x^1 = t \sin \theta, \qquad x^2 = t \cos \theta, \qquad x^3 = z, \tag{1.4}$$

where x^i denotes the Cartesian spatial coordinates of S_t, and where (r, θ, z) denotes the usual cylindrical coordinate system whence (θ, z) is a surface coordinate system on the cylinder. Clearly S_t moves in such a way that

$$r = t \tag{1.5}$$

for all (θ, z). Hence the unit normal $\mathbf{n}$ has the components $(\sin \theta, \cos \theta, 1)$ relative to (x^i), the normal speed u_n is equal to 1, and the tangential velocity $\mathbf{u}$ vanishes.

Now we consider a vector field $\mathbf{v}$ whose components in (x^i) are given by

$$v^1 = -x^2, \qquad v^2 = x^1, \qquad v^3 = 0, \tag{1.6}$$

for all t. Relative to the cylindrical coordinate system (r, θ, z), the components of $\mathbf{v}$ are given by

$$v^r = 0, \qquad v^\theta = 1, \qquad v^z = 0, \tag{1.7}$$

so that $\mathbf{v}$ is tangent to S_t for all t. Indeed, $\mathbf{v}$ coincides with the natural basis vector in the θ-direction of the surface coordinate system (θ, z). Now if we regard $\mathbf{v}$ as a spatial vector, then the Truesdell–Toupin displacement derivative of $\mathbf{v}$ with respect to S_t is given by

$$\delta_d v^k/\delta t = (\partial v^k/\partial x^i)n^i, \tag{1.8}$$

or more specifically

$$\delta_d v^1/\delta t = -\cos \theta, \qquad \delta_d v^2/\delta t = \sin \theta, \qquad \delta_d v^3/\delta t = 0. \tag{1.9}$$

On the other hand, if we regard $\mathbf{v}$ as a surface vector, then we have

$$\delta_d v^\Gamma/\delta t = 0, \tag{1.10}$$

since the surface components v^Γ of $\mathbf{v}$ relative to (θ, z) are constant fields. This example shows clearly that the Truesdell–Toupin displacement derivative of $\mathbf{v}$ depends critically on the status of $\mathbf{v}$ as a spatial vector or as a surface vector.

In the next section, we shall determine the geometric meaning of the Truesdell–Toupin displacement derivative corresponding to the two possible cases illustrated in the preceding example. Then in Sec. 3, we introduce the new concept of the *total displacement derivative* of a tensor field relative to a moving surface. We require that the total displacement derivative of a tensor field Ψ be the time derivative of Ψ along the normal trajectory of the moving surface, the parallel transport along the normal trajectory being the spatial one. Clearly, this definition reduces to the Hayes–Thomas original definition when Ψ is a function or a spatial vector field, since the total displacement derivative in general does not depend on the status of its argument as a spatial tensor field or as a surface tensor field. Further, for a spatial tensor field the total displacement derivative coincides with the Truesdell–Toupin displacement derivative, but for a surface tensor field the two displacement derivatives are generally different.

In application to mechanics, especially to the consideration of wave propagation, the total displacement derivative seems to be a more useful concept than the Truesdell–Toupin displacement derivative, since it always gives the time rate of *spatial* change of a tensor relative to the moving surface even if the tensor is a surface tensor. For instance, it is the total displacement derivative, not the Truesdell–Toupin displacement derivative, of the surface amplitude vector of a transverse wave that determines the growth or decay of the wave.

For simplicity, we carry out the analysis for the total displacement derivative of tensor fields relative to a two-dimensional moving surface in a three-dimensional Euclidean space only. The generalization to the total displacement derivative relative to a moving hypersurface in an n-dimensional Riemannian space is given in the Appendix.

2. The geometric meaning of the Truesdell-Toupin displacement derivative. As explained in the introduction, a moving surface S_t in space can be characterized by the relations

$$x^k = x^k(y^1, y^2, t), \qquad k = 1, 2, 3, \tag{1.2}$$

where x^k denotes the Cartesian coordinates and y^Γ the surface coordinates of a generic point in S_t. We write $\mathbf{e}_k$ and $\mathbf{f}_\Gamma$ for the natural basis vectors of x^k and y^Γ respectively. Then the component form of $\mathbf{f}_\Gamma$ relative to $\mathbf{e}_k$ is

$$\mathbf{f}_\Gamma = (\partial x^k/\partial y^\Gamma)\mathbf{e}_k\,. \tag{2.1}$$

Since $\{\mathbf{f}_1, \mathbf{f}_2\}$ spans the tangent space of S_t, the unit normal $\mathbf{n}$ of S_t is given by

$$\mathbf{n} = \mathbf{f}_1\times\mathbf{f}_2/|\mathbf{f}_1\times\mathbf{f}_2|\,. \tag{2.2}$$

Clearly, $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{n}\}$ forms a basis in space. We denote the dual basis of $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{n}\}$ by $\{\mathbf{f}^1, \mathbf{f}^2, \mathbf{n}\}$, so that

$$\mathbf{f}_\Gamma\cdot\mathbf{f}^\Delta = \delta_\Gamma^\Delta\,, \qquad \Gamma, \Delta = 1, 2, \tag{2.3}$$

and

$$\mathbf{f}^\Delta\cdot\mathbf{n} = 0, \qquad \Delta = 1, 2. \tag{2.4}$$

We define the velocity of S_t relative to (y^Γ) by

$$\mathbf{c} \equiv (\partial x^i/\partial t)\mathbf{e}_i\,. \tag{2.5}$$

Then $\mathbf{c}$ can be decomposed uniquely into a normal component $u_n\mathbf{n}$ and a tangential component $-\mathbf{u}$, viz.

$$\mathbf{c} = u_n\mathbf{n} - \mathbf{u}, \tag{2.6}$$

where u_n is called the normal speed and is given by

$$u_n = \mathbf{c}\cdot\mathbf{n}. \tag{2.7}$$

It can be verified easily that the normal speed is, but the tangent velocity $\mathbf{u}$ is not, independent of the choice of the surface coordinate system. Further, relative to a change of coordinates from (y^Γ) to $(\bar{y}^\Gamma)$, the corresponding tangential velocities $\mathbf{u}$ and $\bar{\mathbf{u}}$ satisfy the relation

$$\mathbf{u} - \bar{\mathbf{u}} = (\partial y^\Gamma/\partial t)\mathbf{f}_\Gamma \ , \tag{2.8}$$

where the coordinate transformation from $(\bar{y}^\Gamma)$ to (y^Γ) is given by

$$y^\Gamma = y^\Gamma(\bar{y}^1, \bar{y}^2, t). \tag{2.9}$$

A coordinate system (z^Γ) on S_t is a *convected system* if the tangential velocity relative to (z^Γ) vanishes. Clearly, the trace of a surface point with constant coordinates in a convected system is a *normal trajectory* of S_t . From (2.8), we see that the tangential velocity $\mathbf{u}$ of S_t relative to (y^Γ) is equal to the velocity of the (y^Γ)-surface points relative to the convected surface points. We denote the natural basis of the convected coordinate system (z^Γ) by $\{\mathbf{g}_\Gamma\}$ and its surface dual basis by $\{\mathbf{g}^\Gamma\}$. As usual, we have the transformation rules

$$\mathbf{g}_\Gamma = (\partial y^\Delta/\partial z^\Gamma)\mathbf{f}_\Delta \ , \tag{2.10}$$

and

$$\mathbf{f}^\Gamma = (\partial y^\Gamma/\partial z^\Delta)\mathbf{g}^\Delta. \tag{2.11}$$

Now let Ψ be a surface vector field. Then we can express Ψ in component form relative to $\mathbf{g}_\Gamma$, say

$$\Psi = \Psi^\Gamma(z^1, z^2, t)\mathbf{g}_\Gamma \ . \tag{2.12}$$

Substituting this component form into the general formula (1.3), we see that the Truesdell–Toupin displacement derivative of Ψ is given by

$$\delta_d\Psi/\delta t = (\partial\Psi^\Gamma(z, t)/\partial t)\mathbf{g}_\Gamma \ . \tag{2.13}$$

Naturally, we call Ψ a *convected surface vector* if its displacement derivative vanishes. From (2.13), we see that Ψ is a convected surface vector if and only if it has constant components relative to a convected coordinate system. In particular, the basis vectors $\mathbf{g}_\Gamma$ are *ipso facto* convected surface vectors.

The same analysis applies to surface covectors. In this case, the convected basis is the dual basis $\{\mathbf{g}^\Gamma\}$ of $\{\mathbf{g}_\Gamma\}$. In general, the natural product basis of a convected coordinate system forms the convected basis for surface tensors, and the components of the displacement derivative of a surface tensor are equal to the partial derivative with respect to the t of the components of the tensor relative to a convected coordinate system.

It should be noted, however, that the induced surface metric is not necessarily a convected surface tensor, since the spatial inner product of the convected basis vectors $\mathbf{g}_\Gamma$,

$$a_{\Gamma\Delta} = \mathbf{g}_\Gamma \cdot \mathbf{g}_\Delta \ , \tag{2.14}$$

may depend explicitly on t. Of course, if the surface metric $\mathbf{a}$ is not a convected surface tensor, then the usual operations of raising and lowering of indices with respect to $\mathbf{a}$ do *not* commute with the operation of the displacement derivative. In particular, the dual basis $\{\mathbf{g}^\Gamma\}$ of $\{\mathbf{g}_\Gamma\}$ *is* a convected surface *covector basis* but is generally *not* a convected surface *vector basis*.

Now we derive the general formula for the displacement derivative in component form relative to an arbitrary surface coordinate system (y^Γ). Let Ψ be a surface vector field as before. Then from (2.10) the components $\Psi^\Gamma(y^1, y^2, t)$ of Ψ relative to (y^Γ) are related to $\Psi^\Gamma(z^1, z^2, t)$ by

$$\Psi^\Gamma(y^1, y^2, t) = \Psi^\Delta(z^1, z^2, t)(\partial y^\Gamma/\partial z^\Delta). \tag{2.15}$$

Taking the partial derivative with respect to t holding z^Γ constant, we obtain

$$\frac{\partial\Psi^\Gamma(y, t)}{\partial t} + \frac{\partial\Psi^\Gamma(y, t)}{\partial y^\Delta}\frac{\partial y^\Delta}{\partial t} = \frac{\partial\Psi^\Delta(z, t)}{\partial t}\frac{\partial y^\Gamma}{\partial z^\Delta} + \Psi^\Delta(z, t)\frac{\partial^2 y^\Gamma}{\partial z^\Delta\,\partial t}. \tag{2.16}$$

Substituting this relation into (2.13), we see that the displacement derivative $\delta_d\Psi/\delta t$ has the component form

$$\frac{\delta_d\Psi}{\delta t} = \left(\frac{\partial\Psi^\Gamma(y, t)}{\partial t} + \frac{\partial\Psi^\Gamma(y, t)}{\partial y^\Delta}\frac{\partial y^\Delta}{\partial t} - \Psi^\Lambda(y, t)\frac{\partial z^\Delta}{\partial y^\Lambda}\frac{\partial^2 y^\Gamma}{\partial z^\Delta\,\partial t}\right)\mathbf{f}_\Gamma \tag{2.17}$$

relative to (y^Γ). But from (2.8), the tangential velocity $\mathbf{u}$ relative to (y^Γ) has the component form

$$\mathbf{u} = \frac{\partial y^\Gamma}{\partial t}\mathbf{f}_\Gamma\ . \tag{2.18}$$

Consequently we can rewrite the component form of $\delta_d\Psi/\delta t$ as

$$\frac{\delta_d\Psi}{\delta t} = \left(\frac{\partial\Psi^\Gamma(y, t)}{\partial t} + \mathfrak{L}_\mathbf{u}\Psi^\Gamma(y, t)\right)\mathbf{f}_\Gamma\ , \tag{2.19}$$

where the components of the Lie derivative $\mathfrak{L}_\mathbf{u}\Psi$ relative to (y^Γ) are given by

$$\mathfrak{L}_\mathbf{u}\Psi^\Gamma(y, t) = \frac{\partial\Psi^\Gamma(y, t)}{\partial y^\Delta}u^\Delta(y, t) - \Psi^\Delta(y, t)\frac{\partial u^\Gamma(y, t)}{\partial y^\Delta}. \tag{2.20}$$

Eq. (2.20) is a special version of the Truesdell–Toupin formula (1.1). If we select for Ψ the surface vector field $\mathbf{f}_\Gamma$, then (2.19) and (2.20) yield

$$\delta_d\mathbf{f}_\Gamma/\delta t = -(\partial u^\Delta/\partial y^\Gamma)\mathbf{f}_\Delta\ . \tag{2.21}$$

Given (2.21) and (1.1) it is possible to use (2.3) and (2.4) to show that

$$\delta_d\mathbf{f}^\Gamma/\delta t = (\partial u^\Gamma/\partial y^\Delta)\mathbf{f}^\Delta. \tag{2.22}$$

Clearly, if we are given (2.21) and (2.22), the Truesdell–Toupin formula for surface tensor fields in general can be derived in a standard way.

Next we consider the geometric meaning of the displacement derivative of a spatial vector field. According to (1.3), if Ψ has components $\Psi^i(x, t)$ relative to (x^i), then $\delta_d\Psi/\delta t$ has the component form

$$\delta_d\Psi/\delta t = (\partial\Psi^i(x, t)/\partial t + (\partial\Psi^i(x, t)/\partial x^k)u_n n^k)\mathbf{e}_i\ . \tag{2.23}$$

This formula shows that the displacement derivative of a spatial vector field is equal to the time derivative of the vector field along the normal trajectory of the moving surface. In particular, Ψ is a convected spatial vector with respect to S_t if and only if it remains constant on each normal trajectory of S_t . Clearly the same holds for convected spatial tensor fields in general.

Comparing (2.23) with (2.13), we see that there is a basic difference between a convected spatial vector and a convected surface vector. Indeed, the example in the introduction shows that a convected surface vector is not necessarily a convected spatial vector. Conversely, we can give examples of convected spatial vectors that can be regarded as surface vectors but which are not convected surface vectors.

For spatial-surface mixed tensors, the convected mixed tensor basis relative to the moving surface, of course, is formed by the tensor products of the convected spatial basis vectors and the convected surface basis vectors. Thus the displacement derivative of a mixed tensor is strictly a two-point tensor which is not invariant under the spatial-surface conversion operations on the mixed tensor.

As explained in the introduction, in consideration of wave propagation it is desirable to know the spatial derivative of some surface vectors or tensors along the normal trajectory of the wave. The lack of the spatial-surface convertibility of the Truesdell–Toupin general formula (1.3) prompts us to introduce the new concept of the *total displacement derivative* in the next section.

3. The total displacement derivative. Let S_t be a moving surface with convected surface point z as explained in the preceding section, and let Ψ be a spatial, surface, or spatial-surface mixed tensor field whose domain at any time t contains the surface S_t. Then the restriction of Ψ to S_t can be expressed as a field $\bar{\Psi}(z, t)$. We define the total displacement derivative of Ψ relative to S_t by

$$\delta\Psi/\delta t \equiv \partial\bar{\Psi}(z, t)/\partial t, \tag{3.1}$$

where the partial derivative on the right-hand side is based upon the usual Euclidean parallel transport. Since the definition (3.1) does not depend upon the many possible component representations Ψ might have, $\delta\Psi/\delta t$ does not depend upon the choices. For convenience only, it is often desirable to apply (3.1) when Ψ is represented by spatial components, since the components of $\partial\bar{\Psi}(z, t)/\partial t$ are then equal to partial derivatives of the components of $\bar{\Psi}(z, t)$. For example, if Ψ is a vector field, then

$$\delta\bar{\Psi}(z, t)/\delta t = (\partial\bar{\Psi}^i(z, t)/\partial t)\mathbf{e}_i\,, \tag{3.2}$$

where $\bar{\Psi}^i(z, t)$ are the components of $\bar{\Psi}(z, t)$ relative to the rectangular Cartesian spatial coordinate system (x^i), viz.

$$\bar{\Psi}(z, t) = \bar{\Psi}^i(x, t)\mathbf{e}_i\,. \tag{3.3}$$

Notice that we can always use the components of $\bar{\Psi}$ relative to the spatial coordinate system (x^i) in (3.2) *regardless* of the status of $\bar{\Psi}$ as a spatial vector or as a surface vector. If $\bar{\Psi}(z, t)$ is a surface vector with components $\bar{\Psi}^\Gamma(z, t)$ relative to the surface coordinate system (z^Γ), then its spatial components $\bar{\Psi}^i(z, t)$ can be determined by

$$\bar{\Psi}^i(z, t) = \bar{\Psi}^\Gamma(z, t)\,\partial x^i/\partial z^\Gamma, \tag{3.4}$$

where

$$x^i = x^i(z^1, z^2, t) \tag{3.5}$$

characterizes the relation between the spatial coordinates x^i and the surface coordinates z^Γ of a generic convected surface point $z \in S_t$. In particular, if $\bar{\Psi}(z, t)$ is equal to the surface vector $\mathbf{g}_\Gamma$, then

$$\delta\mathbf{g}_\Gamma/\delta t = (\partial/\partial t)(\partial x^i/\partial z^\Gamma)\mathbf{e}_i\,. \tag{3.6}$$

For a smooth moving surface, the partial derivative on the right-hand side can be expressed as

$$\frac{\partial}{\partial t}\left(\frac{\partial x^i}{\partial z^\Gamma}\right) = \frac{\partial}{\partial z^\Gamma}\left(\frac{\partial x^i}{\partial t}\right) = \frac{\partial u_n n^i}{\partial z^\Gamma}. \tag{3.7}$$

Hence by Weingarten's formula (cf. [3, App. Eq. (21.6)])

$$\partial \mathbf{n}/\partial z^\Gamma = -b_{\Gamma\Delta}\mathbf{g}^\Delta, \tag{3.8}$$

where $b_{\Gamma\Delta}$ denotes the components of the second fundamental form $\mathbf{b}$ of S_t , we obtain

$$\delta \mathbf{g}_\Gamma/\delta t = (\partial u_n/\partial z^\Gamma)\mathbf{n} - b_{\Gamma\Delta}u_n\mathbf{g}^\Delta. \tag{3.9}$$

This formula shows that the total displacement derivative of the surface basis vector $\mathbf{g}_\Gamma$ need not vanish in general. More specifically, the *normal* component of $\delta\mathbf{g}_\Gamma/\delta t$ is produced by the *nonuniformity* of the normal speed u_n on S_t and the *tangential* component of $\delta\mathbf{g}_\Gamma/\delta t$ is produced by the *curvature* of S_t .

From (3.2), it is easily seen that the total displacement derivative satisfies the usual product rule with respect to the spatial inner product. Hence from the condition

$$\mathbf{n}\cdot\mathbf{g}_\Gamma = 0, \tag{3.10}$$

we obtain

$$(\delta\mathbf{n}/\delta t)\cdot\mathbf{g}_\Gamma + \mathbf{n}\cdot(\delta\mathbf{g}_\Gamma/\delta t) = 0. \tag{3.11}$$

Now since $\mathbf{n}$ is the unit tangent of the normal trajectory of S_t , its total displacement derivative is necessarily a tangential vector, i.e.,

$$(\delta\mathbf{n}/\delta t)\cdot\mathbf{n} = 0. \tag{3.12}$$

Combining (3.11) and (3.12) and using the formula (3.9), we then obtain

$$\delta\mathbf{n}/\delta t = -(\partial u_n/\partial z^\Gamma)\mathbf{g}^\Gamma. \tag{3.13}$$

This is the Hayes–Thomas formula (cf. (1.2)) relative to the coordinate system (z^Γ).

Finally, from the conditions

$$\mathbf{g}_\Gamma\cdot\mathbf{g}^\Delta = \delta_\Gamma^\Delta, \qquad \mathbf{n}\cdot\mathbf{g}^\Delta = 0, \tag{3.14}$$

we obtain

$$(\delta\mathbf{g}_\Gamma/\delta t)\cdot\mathbf{g}^\Delta + \mathbf{g}_\Gamma\cdot(\delta\mathbf{g}^\Delta/\delta t) = 0, \tag{3.15}$$

and

$$(\delta\mathbf{n}/\delta t)\cdot\mathbf{g}^\Delta + \mathbf{n}\cdot(\delta\mathbf{g}^\Delta/\delta t) = 0. \tag{3.16}$$

Substituting (3.9) and (3.13) into (3.15) and (3.16), we obtain the formula

$$\delta\mathbf{g}^\Delta/\delta t = a^{\Delta\Gamma}(\partial u_n/\partial z^\Gamma)\mathbf{n} + b^{\Delta\Gamma}u_n\mathbf{g}_\Gamma . \tag{3.17}$$

The formulae (3.13), (3.9) and (3.13), (3.17) give completely the total displacement derivatives of the basis $\{\mathbf{g}_1 , \mathbf{g}_2 , \mathbf{n}\}$ and the dual basis $\{\mathbf{g}^1, \mathbf{g}^1, \mathbf{n}\}$.

Now we consider the total displacement derivative of the bases $\{\mathbf{f}_1 , \mathbf{f}_2 , \mathbf{n}\}$ and $\{\mathbf{f}^1, \mathbf{f}^2, \mathbf{n}\}$ corresponding to an arbitrary surface coordinate system (y^Γ) on S_t . Since the normal speed u_n does not depend on the choice of the surface coordinate system, we obtain from (3.13)

$$\delta \mathbf{n}/\delta t = -(\partial u_{\mathbf{n}}/\partial y^{\Gamma})\mathbf{f}^{\Gamma}. \tag{3.18}$$

This is the Hayes–Thomas formula relative to the coordinate system (y^{Γ}). Next, from (2.10) we have

$$\frac{\delta \mathbf{g}_{\Gamma}}{\delta t} = \frac{\delta}{\delta t}\left(\frac{\partial y^{\Delta}}{\partial z^{\Gamma}}\right)\mathbf{f}_{\Delta} + \frac{\partial y^{\Delta}}{\partial z^{\Gamma}}\frac{\delta \mathbf{f}_{\Delta}}{\delta t}. \tag{3.19}$$

Since $\partial y^{\Delta}/\partial z^{\Gamma}$ is a function of (z^1, z^2, t), its total displacement derivative is equal to its partial time derivative, viz.

$$\frac{\delta}{\delta t}\left(\frac{\partial y^{\Delta}}{\partial z^{\Gamma}}\right) = \frac{\partial}{\partial t}\left(\frac{\partial y^{\Delta}}{\partial z^{\Gamma}}\right) = \frac{\partial}{\partial z^{\Gamma}}\left(\frac{\partial y^{\Delta}}{\partial t}\right). \tag{3.20}$$

Hence (3.19) implies

$$\frac{\delta \mathbf{f}_{\Delta}}{\delta t} = \frac{\partial z^{\Gamma}}{\partial y^{\Delta}}\frac{\delta \mathbf{g}_{\Gamma}}{\delta t} - \frac{\partial u^{\Gamma}(y, t)}{\partial y^{\Delta}}\mathbf{f}_{\Gamma}, \tag{3.21}$$

where $u^{\Gamma}(y, t)$ denotes the tangential velocity relative to (y^{Γ}) (cf. (2.18)). Substituting (3.9) into (3.21), we obtain

$$\frac{\delta \mathbf{f}_{\Delta}}{\delta t} = \frac{\partial u_{\mathbf{n}}}{\partial y^{\Delta}}\mathbf{n} - b_{\Delta\Gamma}(y, t)u_{\mathbf{n}}\mathbf{f}^{\Gamma} - \frac{\partial u^{\Gamma}(y, t)}{\partial y^{\Delta}}\mathbf{f}_{\Gamma}, \tag{3.22}$$

where $b_{\Delta\Gamma}(y, t)$ denotes the components of the second fundamental form **b** relative to (y^{Γ}). By a similar argument, we have also

$$\frac{\delta \mathbf{f}^{\Delta}}{\delta t} = a^{\Delta\Gamma}(y, t)\frac{\partial u_{\mathbf{n}}}{\partial y^{\Gamma}}\mathbf{n} + b^{\Delta\Gamma}(y, t)u_{\mathbf{n}}\mathbf{f}_{\Gamma} + \frac{\partial u^{\Delta}(y, t)}{\partial y^{\Gamma}}\mathbf{f}^{\Gamma}, \tag{3.23}$$

where $a^{\Gamma\Delta}(y, t)$ denotes the components of the surface metric **a** relative to (y^{Γ}). The formulae (3.18), (3.22) and (3.18), (3.23) give completely the total displacement derivative of the bases $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{n}\}$ and $\{\mathbf{f}^1, \mathbf{f}^2, \mathbf{n}\}$.

Having determined the total displacement derivatives of the basis vectors, we can now derive a general formula for the total displacement derivative of an arbitrary vector field Ψ. First, we express Ψ in component form relative to the basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{n}\}$, viz.

$$\Psi = \Psi_{\mathbf{n}}(y, t)\mathbf{n} + \Psi^{\Gamma}(y, t)\mathbf{f}_{\Gamma}. \tag{3.24}$$

Computing the total displacement derivative and using the product rule, we find

$$\frac{\delta \Psi}{\delta t} = \frac{\delta}{\delta t}(\Psi_{\mathbf{n}}(y, t)\mathbf{n}) + \frac{\delta}{\delta t}(\Psi^{\Gamma}(y, t))\mathbf{f}_{\Gamma} + \Psi^{\Gamma}(y, t)\frac{\delta \mathbf{f}_{\Gamma}}{\delta t}. \tag{3.25}$$

The total displacement derivative of the function $\Psi^{\Gamma}(y, t)$ can be calculated in the usual way by

$$\frac{\delta}{\delta t}(\Psi^{\Gamma}(y, t)) = \frac{\partial \Psi^{\Gamma}(y, t)}{\partial t} + \frac{\partial \Psi^{\Gamma}(y, t)}{\partial y^{\Delta}}u^{\Delta}(y, t). \tag{3.26}$$

It follows from (1.1), (1.3) and (3.18) that for the vector $\Psi_{\mathbf{n}}(y, t)\mathbf{n}$

$$(\delta/\delta t)(\Psi_{\mathbf{n}}(y, t)\mathbf{n}) = (\delta_d/\delta t)(\Psi_{\mathbf{n}}(y, t)\mathbf{n}). \tag{3.27}$$

Substituting (3.26), (3.27) and (3.22) into (3.25) we find

$$\frac{\delta\Psi}{\delta t} = \frac{\delta_d}{\delta t}(\Psi_n(y, t)\mathbf{n}) + \left(\frac{\partial\Psi^\Gamma(y, t)}{\partial t} + \frac{\partial\Psi^\Gamma(y, t)}{\partial y^\Delta}u^\Delta(y, t) - \Psi^\Delta(y, t)\frac{\partial u^\Gamma(y, t)}{\partial y^\Delta}\right)\mathbf{f}_\Gamma + \Psi^\Gamma(y, t)\left(\frac{\partial u_n}{\partial y^\Gamma}\mathbf{n} - b_{\Gamma\Delta}(y, t)u_n\mathbf{f}^\Delta\right). \tag{3.28}$$

If we now use (2.19), (2.20), and (3.24), (3.28) can be written

$$\frac{\delta\Psi}{\delta t} = \frac{\delta_d\Psi}{\delta t} + \Psi^\Gamma(y, t)\left(\frac{\partial u_n}{\partial y^\Gamma}\mathbf{n} - b_{\Gamma\Delta}(y, t)u_n\mathbf{f}^\Delta\right), \tag{3.29}$$

which shows clearly the difference between the total displacement derivative and the displacement derivative in Truesdell–Toupin's sense.

The total displacement derivative of a tensor field in general, of course, can be determined in a similar way. The basic rules are the following four:

(i) The total displacement derivative obeys the product rule with respect to the tensor product and the spatial inner product.

(ii) The total displacement derivatives of the spatial basis vectors $\mathbf{e}_i$ of the Cartesian coordinate system (x^i) vanish identically.

(iii) The total displacement derivatives of the bases $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{n}\}$ and $\{\mathbf{f}^1, \mathbf{f}^2, \mathbf{n}\}$ are given by (3.18), (3.22) and (3.18), (3.23), respectively.

(iv) The total displacement derivative of a function $f(x, y, t)$ is given by

$$\frac{\delta f}{\delta t} = \frac{\partial f(x, y, t)}{\partial t} + \frac{\partial f(x, y, t)}{\partial x^i}u_n n^i + \frac{\partial f(x, y, t)}{\partial y^\Gamma}u^\Gamma(y, t). \tag{3.30}$$

In fact, (3.30) is a valid equation even if f is a spatial-surface tensor field.

As an application of the above rules we consider the total displacement derivative of the surface metric $\mathbf{a}$ whose component form relative to (y^Γ) may be written either as

$$\mathbf{a} = a_{\Gamma\Delta}(y, t)\mathbf{f}^\Gamma \otimes \mathbf{f}^\Delta \tag{3.31}$$

or as

$$\mathbf{a} = \mathbf{f}_\Gamma \otimes \mathbf{f}^\Gamma. \tag{3.32}$$

As we have mentioned earlier, the total displacement derivative of $\mathbf{a}$ does not depend on the choice of the component form of $\mathbf{a}$. Applying the basic rules (i)–(iv) successively on (3.32), we obtain

$$\delta\mathbf{a}/\delta t = (\partial u_n/\partial y^\Delta)(\mathbf{n} \otimes \mathbf{f}^\Delta + \mathbf{f}^\Delta \otimes \mathbf{n}). \tag{3.33}$$

Of course, this result is also a consequence of the identity

$$\mathbf{a} = \mathbf{1} - \mathbf{n} \otimes \mathbf{n}. \tag{3.24}$$

Indeed, from (3.18) we have

$$\frac{\delta\mathbf{a}}{\delta t} = -\left(\frac{\delta\mathbf{n}}{\delta t} \otimes \mathbf{n} + \mathbf{n} \otimes \frac{\delta\mathbf{n}}{\delta t}\right) = \frac{\partial u_n}{\partial y^\Delta}(\mathbf{f}^\Delta \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{f}^\Delta). \tag{3.35}$$

We can deduce a formula for the Truesdell–Toupin displacement derivative of the surface metric a from the formula (3.33). First, we write the surface metric in the component form

$$\mathbf{a} = a_{\Gamma\Delta}(z, t)\mathbf{g}^{\Gamma} \otimes \mathbf{g}^{\Delta}. \tag{3.36}$$

The tensorial status of a now is important. By definition (cf. (2.13)) we have

$$\frac{\delta_d \mathbf{a}}{\delta t} = \frac{\partial a_{\Gamma\Delta}(z, t)}{\partial t}\mathbf{g}^{\Gamma} \otimes \mathbf{g}^{\Delta}. \tag{3.37}$$

But from (3.36) we have also

$$\frac{\delta \mathbf{a}}{\delta t} = \frac{\partial a_{\Gamma\Delta}(z, t)}{\partial t}\mathbf{g}^{\Gamma} \otimes \mathbf{g}^{\Delta} + \frac{\delta \mathbf{g}^{\Gamma}}{\delta t} \otimes \mathbf{g}_{\Gamma} + \mathbf{g}_{\Gamma} \otimes \frac{\delta \mathbf{g}^{\Gamma}}{\delta t}. \tag{3.38}$$

Hence

$$\frac{\delta \mathbf{a}}{\delta t} = \frac{\delta_d \mathbf{a}}{\delta t} + \frac{\delta \mathbf{g}^{\Gamma}}{\delta t} \otimes \mathbf{g}_{\Gamma} + \mathbf{g}_{\Gamma} \otimes \frac{\delta \mathbf{g}^{\Gamma}}{\delta t}. \tag{3.39}$$

Substituting (3.33) and (3.17) into this equation, we then obtain

$$\delta_d \mathbf{a}/\delta t = -2b_{\Gamma\Delta}(z, t)u_n\mathbf{g}^{\Gamma} \otimes \mathbf{g}^{\Delta}. \tag{3.40}$$

This formula agrees with [3, Eq. $(179.10)_1$].

Appendix. Total displacement derivative relative to a moving hypersurface in an n-dimensional Riemannian manifold. Let M be an n-dimensional Riemannian manifold and let S_t be a moving hypersurface in M. Then for any tensor field Ψ (spatial, surface, or spatial-surface mixed) whose domain at any time t contains the surface S_t we define the total displacement derivative of Ψ by

$$\delta\Psi/\delta t \equiv D\bar{\Psi}(z, t)/Dt, \tag{A.1}$$

where z denotes a generic convected surface point on S_t, and $\bar{\Psi}(z, t)$ denotes the value of Ψ along the normal trajectory of S_t at the point z. The operation D/Dt is the usual covariant derivative along a curve with respect to the Riemannian connection of M. Hence if $\bar{\Psi}(z, t)$ is a vector field with components $\bar{\Psi}^i(z, t)$ relative to a spatial coordinate system $(x^i, i = 1, \cdots, n)$, then

$$\frac{\delta\Psi}{\delta t} = \left(\frac{\partial\bar{\Psi}^i(z, t)}{\partial t} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\bar{\Psi}^j(z, t)u_n n^k\right)\mathbf{e}_i \tag{A.2}$$

where $\{{}^{\,i}_{jk}\}$ denotes the Christofell's symbols of the Riemannian metric relative to (x^i), and where $\mathbf{e}_i$ denotes the natural basis vectors of (x^i). Notice that $\mathbf{e}_i$ is not necessarily a convected field in this general case.

According to the general formula (A.2), the total displacement derivative of the basis vector $\mathbf{g}_\Gamma$ is still given by a formula of the form (3.9), viz. .

$$\delta\mathbf{g}_\Gamma/\delta t = (\partial u_n/\partial z^\Gamma)\mathbf{n} - b_{\Gamma\Delta}u_n\mathbf{g}^{\Delta}. \tag{A.3}$$

Consequently the formulae (3.13), (3.17), (3.18), (3.22), and (3.23) are also valid in general, since the spatial Riemannian metric is a convected field with respect to the Riemannian parallel transport along the normal trajectory of S_t.

By the same argument as before, we can compute the total displacement derivative of a tensor field in general by applying the four basic rules successively as explained in Sec. 3, except that the total displacement derivatives of the spatial basis vectors $\mathbf{e}_i$ no longer vanish in general but are given by

$$\frac{\delta \mathbf{e}_i}{\delta t} = \left\{ \begin{matrix} j \\ ik \end{matrix} \right\} u_{\mathrm{n}} n^k \mathbf{e}_i \ . \tag{A.4}$$

Acknowledgment. The research reported here was supported by the U. S. National Science Foundation under Grant GP-9492 for R. M. Bowen and Grant GP-9358 for C.-C. Wang.

References

[1] W. Hayes, J. Fluid Mech. 2, 595–600 (1957)

[2] T. Y. Thomas, *Extended compatibility conditions for the study of surfaces of discontinuity in continuum mechanics,* J. Math. Mech. 6, 311–322 (1957)

[3] C. Truesdell and R. Toupin, *The classical field theory,* Handbuch der Physik, Band III/1, Springer-Verlag, Berlin, 1960

[4] T. Y. Thomas, *Plastic flow and fracture in solids,* Mathematics in Science and Engineering, vol. 2, Academic Press, New York, 1961