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# ORTHOGONALITY, POSITIVE OPERATORS, AND THE FREQUENCY-POWER FORMULAS*

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**I. Introduction.** In their investigation into the properties of nonlinear capacitors, J. M. Manley and H. E. Rowe introduced a class of relationships which have now come to be known as frequency-power formulas. Basically, frequency-power formulas constrain certain weighted sums, which involve the Fourier coefficients of a function $x(\cdot)$ and $y(\cdot) = f(x(\cdot))$, to be either zero or positive. For example, the results of Manley and Rowe [1] can be stated roughly as follows: Suppose one has a nonlinear capacitor with the voltage $v$ across the capacitor related to the stored charge $q$ by $v = f(q)$ where $f(\cdot)$ is a continuous function of the reals into themselves. If $v$ and $q$ have the Fourier series (a $\sim$ is used to denote the correspondence between the formal series and the function)

$$v(t) \sim \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} V_{m,n} \exp [j(m\omega_1 + n\omega_2)t]$$

and

$$q(t) \sim \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} Q_{m,n} \exp [j(m\omega_1 + n\omega_2)t],$$

then

$$\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{nP_{m,n}}{m\omega_1 + n\omega_2} = 0$$

and

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{mP_{m,n}}{m\omega_1 + n\omega_2} = 0,$$

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where $P_{m,n} = \frac{1}{2}$ Re $[V_{m,n} I^*_{m,n}]$, and $I_{m,n} = j(m\omega_1 + n\omega_2)Q_{m,n}$ . The quantity $P_{m,n}$ represents a real power quantity at the frequency $m\omega_1 + n\omega_2$ , motivating the name "frequency-power" formulas for these relations.

Frequency-power formulas which constrain certain weighted sums to be positive were first obtained by C. H. Page [2], and these formulas were obtained in connection with frequency conversion using nonlinear resistors. A rather extensive listing of known frequency-power formulas is given in [3].

Since the original papers of Manley and Row [1] and Page [2], a number of authors have given alternative proofs and generalizations [3]–[8]. However, the precise mathematical framework into which these problems fall has never been made completely clear. As a result, the unity that exists between the various formulas has been obscured, and also there has developed a certain amount of confusion in the literature as to the precise conditions under which the frequency-power formulas are valid. In particular, one of the main difficulties which has been ignored in previous works on frequency-power formulas is the fact that the Fourier series of a function need not converge pointwise to that function. In fact, there exist *continuous* periodic functions having Fourier series which diverge on an uncountable set of points [9] and [10].

Although differentiability is enough to guarantee convergence of the Fourier series for periodic functions, the situation is much more complicated in the case of functions having a multiple Fourier series such as $v(\cdot)$ and $q(\cdot)$ above. For even the existence of a bounded continuous derivative of such functions does not guarantee any kind of regularity which is known to be sufficient for the convergence of its Fourier series (cf. [11]). This is why a $\sim$ has been used above to denote a correspondence and not necessarily equality between a function and its Fourier series.

Since the style of proof used in previous works was dependent on pointwise equality between the Fourier series and the function, the validity of these formulas can justifiably be questioned when this equality does not hold; and it is a difficult problem indeed to establish if in fact equality does hold.[1] The method of proof to be used here circumvents these convergence problems by introducing the theory of quasiperiodic functions. As a result, it is possible to prove the formulas even for cases in which the Fourier series are divergent. In every case, however, the infinite sums involved in the frequency-power formulas will be shown to be absolutely convergent.

In this paper the frequency-power formulas will be proven in a rather simple mathematical framework in which three basic ingredients will be used in the proofs of the general formulas. These are:

(i) An orthogonality relation in a proper inner product setting.

(ii) An identification of positive operators mapping a certain inner product space into itself.

(iii) For quasiperiodic functions $x(\cdot)$ and continuous nonlinearities $f(\cdot)$ the Fourier coefficients of $y(\cdot) = f(x(\cdot))$ are independent of the "basic frequencies."

The frequency-power formulas which constrain certain sums to be zero will be obtained from (i) and (iii) above, and those which constrain certain sums to be positive will

[1] It should be noted that even when a function, say $v(\cdot)$, has only a finite number of nonzero terms in its Fourier series, $y(\cdot) = f(v(\cdot))$ will in general have an infinite number of nonzero terms. Thus, although there are no problems with convergence in regards to $v(\cdot)$, there certainly will be with regards to $y(\cdot)$.

arise from (ii) and (iii). The proofs for time-varying nonlinearities are obtained by reducing the problem to an equivalent time-invariant case. From this viewpoint, the effect of time variation in the nonlinearity can be seen quite clearly.

The results to be presented include the previously known frequency-power formulas, and in fact more general results are obtained for the positivity type formulas. It is important to note that in every case only one frequency-power formula is given, and this can be applied to capacitors, resistors, inductors, or any device being described by a relationship $y(t) = f(x(t), t)$. In this way the basic mathematics underlying all of these problems is brought out, and thereby a certain economy of thought is achieved.

In the next section some basic facts concerning almost periodic and quasiperiodic functions will be stated, and statement (iii) above will be proven. In Sec. III an orthogonality relation will be proven and this will lead to the frequency-power formulas constraining certain sums to be zero. Sec. IV is concerned with frequency-power formulas yielding positivity constraints. These will be obtained by first characterizing a rather large class of positive operators, and from this characterization the frequency-power formulas are easily obtained by utilizing (iii) above. Finally, in Sec. IV these results are used to prove a nonoscillation theorem for nonlinear feedback systems.

**II. Basic facts about almost periodic and quasiperiodic functions.** The almost periodic functions which were introduced by H. Bohr [12] are defined as follows:

*Definition* 1. A complex valued function $f(\cdot)$ of the real variable $t$ on $(-\infty, \infty)$ is said to be *almost periodic* if for every $\epsilon > 0$ there exists a real number $l(\epsilon) > 0$ such that every interval of length $l(\epsilon)$ of the real line contains at least one number $\tau$ such that

$$|f(t+\tau) - f(t)| < \epsilon \quad \text{for all} \quad t \in (-\infty, \infty).$$

The almost periodic functions are a generalization of the continuous periodic functions, and in fact they consist of all uniform limits of real finite trigonometric polynomials $\sum_{n=1}^{N} a_n e^{i\lambda_n t}$.

The space of real-valued almost periodic functions will be denoted by $\mathfrak{A}$, and the following basic properties of $\mathfrak{A}$ will be needed (see [11] and [13] for proofs).

(i) The elements of $\mathfrak{A}$ are bounded and uniformly continuous. Also, $\mathfrak{A}$ forms a real vector space and is closed under (pointwise) multiplication.

(ii) $\mathfrak{A}$ is a complete vector space under the uniform topology.

(iii) The mean value functional $M$ on $\mathfrak{A}$ is defined by

$$M\{f\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, dt \tag{1}$$

and the above limit exists for all $f \in \mathfrak{A}$.

(iv) Using (ii) and (iii) it follows that for all $f \in \mathfrak{A}$ the *Fourier coefficients*

$$a(\lambda) = M\{f \exp[-i\lambda t]\} \tag{2}$$

exist, and it is a basic result that there is at most a countable set of $\lambda$'s for which $a(\lambda)$ is nonzero. These values of $\lambda$, called the *Fourier exponents* of $f$, will be denoted by $\Lambda_1, \Lambda_2, \cdots$. Then $a(\Lambda_n) = F_n$ are called the *Fourier coefficients* of $f$. The formal series $\sum_n F_n \exp[i\Lambda_n t]$ is called the *Fourier series* of $f$, and the correspondence between $f$ and its Fourier series will be denoted by

$$f(t) \sim \sum_n F_n \exp[i\Lambda_n t].$$

The set $\{\Lambda_n\}_{n=1}^{\infty}$ is called the *spectrum* of $f$.

(v) Let $f, g \in \mathfrak{A}$ have the Fourier series

$$f(t) \sim \sum_n F_n \exp[i\Lambda_n t],$$

$$g(t) \sim \sum_n G_n \exp[i\Lambda_n t],$$

where some of the $F_n$ and $G_n$ may be zero. Then $fg \in \mathfrak{A}$ and

$$(fg)(t) \sim \sum_n \Big[ \sum_{\Lambda_k + \Lambda_q = \Lambda_n} F_k G_q \Big] \exp[i\Lambda_n t], \tag{4}$$

and the sum in brackets is absolutely convergent.

(vi) Suppose that $f \in \mathfrak{A}$ and has a uniformly continuous derivative, denoted by $\dot{f}$. Then $\dot{f} \in \mathfrak{A}$ and if $f$ has the Fourier series given in (3), then

$$\dot{f}(t) \sim \sum_n i\Lambda_n F_n \exp[i\Lambda_n t]. \tag{5}$$

(vii) Let $H(\cdot)$ be a continuous function from the real line into itself, and let $f \in \mathfrak{A}$. Then $H(f(\cdot))$ is an element of $\mathfrak{A}$.

(viii) $\mathfrak{A}$ is a (noncomplete, nonseparable) inner product space with $\langle f_1, f_2 \rangle \triangleq M\{f_1 f_2\}$.

A subspace of $\mathfrak{A}$ with which we will be mainly concerned is the space of quasiperiodic functions. To motivate their definition, consider the function $f$ given by

$$f(t) = \cos \omega_1 t + \cos \omega_2 t \tag{6}$$

where $\omega_1/\omega_2$ is irrational, and define $F: R^2 \to R$ by

$$F(x_1, x_2) = \cos x_1 + \cos x_2 . \tag{7}$$

Then clearly

$$f(t) = F(\omega_1 t, \omega_2 t). \tag{8}$$

More generally, consider the finite trigonometric polynomial

$$f(t) = \sum_{n_1} \cdots \sum_{n_q} F_{n_1, \ldots, n_q} \exp[i(n_1\omega_1 + \cdots + n_q\omega_q)t], \tag{9}$$

where only a finite number of the $F_{n_1, \ldots, n_q}$ are nonzero and $\omega_1, \cdots, \omega_q$ are real numbers. Note that since (9) contains only a finite sum there is no convergence problem. Now, define $F: R^q \to R$ by

$$F(x_1, \cdots, x_q) = \sum_{n_1} \cdots \sum_{n_q} F_{n_1, \ldots, n_q} \exp[i(n_1 x_1 + \cdots + n_q x_q)], \tag{10}$$

and thus

$$f(t) = F(\omega_1 t, \cdots, \omega_q t). \tag{11}$$

Hence, any finite trigonometric polynomial can be written in the form of (11) with $F(x_1, \cdots, x_q)$ continuous and periodic of period $2\pi$ in each variable $x_k$.

If one were to try to extend the representation of (9) to include infinite sums, a number of difficult convergence problems would arise. On the other hand, the representation of (11) can be used to define a class of functions which have Fourier series in the form of (9) in which an infinite number of the $F_{n_1,\dots,n_q}$ are nonzero. These functions are the quasiperiodic functions, formally defined as follows:

*Definition* 2. A real valued function $f$ is said to be *quasiperiodic* if it can be represented in the form

$$f(t) = F(\omega_1 t, \omega_2 t, \cdots, \omega_n t)$$

where $F(x_1, \cdots, x_n)$ is a continuous function of period $2\pi$ in each of the variables $x_1, \cdots, x_n$. The real numbers $\omega_1, \cdots, \omega_n$ are called the *basic frequencies* of $f$. The space of quasiperiodic functions will be denoted by $Q$.

The space of quasiperiodic functions is properly contained in $\mathfrak{A}$. The spectrum of a quasiperiodic function consists of elements of the form

$$\Lambda_m = k_1^{(m)}\omega_1 + \cdots + k_n^{(m)}\omega_n, \tag{12}$$

with $k_1^{(m)}, \cdots, k_n^{(m)}$ integers. On the other hand, the spectrum of an almost periodic function can consist of numbers of the form

$$\Lambda_m = r_1^{(m)}\omega_1 + \cdots + r_{n_m}^{(m)}\omega_{n_m} \tag{13}$$

where the $r$'s are rational and in general the $\omega_k$'s form an infinite set of basic frequencies. (Each $\Lambda_m$, however, is, obtained by a finite sum.)

Although the quasiperiodic functions form a more restricted class than the almost periodic functions, they are adequate for most purposes, and offer the distinct advantage that from a mathematical point of view are much easier to manage. Furthermore, by using the result of Bogoliubov [14] that $\mathfrak{A}$ is the uniform closure of $Q$, many results concerning functions in $Q$ can be extended to those in $\mathfrak{A}$.

From Definition 2 it is seen that the study of quasiperiodic functions involves nothing more than the study of periodic functions of many variables. As in the case of periodic functions of one variable, a Fourier series corresponding to a periodic function of many variables can be obtained as follows. Let $F(x_1, \cdots, x_n)$ be a continuous function of period $2\pi$ in each variable $x_i$, and define

$$F_k = \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} F(x_1, \cdots, x_n)e^{i(k,x)}\,dx_1 \cdots dx_n \tag{14}$$

where $k = (k_1, \cdots, k_n)$ is an $n$-tuple of integers, $x = (x_1, \cdots, x_n)$, and $(k, x) \triangleq k_1x_1 + k_2x_2 + \cdots + k_nx_n$. Then since $F$ is bounded, it is clear that the integral in (14) exists for all $k$, and thus $F$ can be associated with the Fourier series

$$F(x_1, \cdots, x_n) \sim \sum_{k\in Z^n} F_k e^{i(k,x)}, \tag{15}$$

where $Z^n$ represents all $n$-tuples of integers.

Consider now the quasiperiodic function $f$ given by $f(t) = F(\omega_1 t, \cdots, \omega_n t)$. Since $f$ is also an element of $\mathfrak{A}$ it has a Fourier series as described in (iv) above. The connection between the Fourier series of (15) and that of (3) is provided by the following theorem.

THEOREM 1. *Let $f$ be a quasiperiodic function given by $f(t) = F(\omega_1 t, \cdots, \omega_n t)$ where the $\omega$'s are linearly independent over the rationals, and let $F$ have the Fourier series given by* (15). *Then*

(i) *if $M\{fe^{-i\lambda t}\} \neq 0$, then $\lambda = k_1\omega_1 + \cdots + k_n\omega_n$ for some integers $k_1, \cdots, k_n$, and*

(ii) $M\{fe^{-i(k,\omega)t}\} = F_k$ *where* $\omega = (\omega_1, \cdots, \omega_n)$ *and* $k = (k_1, \cdots, k_n)$. *Thus if* $F$ *has the Fourier series*

$$F(x_1, \cdots, x_n) \sim \sum_{k \in Z^n} F_k e^{i(k,x)},$$

*then* $f$ *has the Fourier series*

$$f(t) \sim \sum_{k \in Z^n} F_k \exp [i(k, \omega)t].$$

This theorem is easily proved by using the Weierstrass approximation theorem for periodic functions of many variables (cf. [13]) and will be omitted. An important consequence of this result is the following.

THEOREM 2. *Let* $\omega = (\omega_1, \cdots, \omega_n)$ *be any n-tuple of real numbers such that the set of* $\omega$*'s are linearly independent over the rationals. Let* $f_\omega$ *be the quasiperiodic function given by*

$$f_\omega(t) = F(\omega_1 t, \cdots, \omega_n t), \tag{16}$$

*and let* $H(\cdot)$ *be a continuous mapping of the reals into themselves. Then* $y_\omega(\cdot) \triangleq H(f_\omega(\cdot))$ *is quasiperiodic. Let the Fourier series (as defined in* (iv)) *for* $f_\omega$ *and* $y_\omega$ *be*

$$f_\omega(t) \sim \sum_{k \in Z^n} F_k \exp [i(k, \omega)t], \tag{17}$$

$$y_\omega(t) \sim \sum_{k \in Z^n} Y_k \exp [i(k, \omega)t]. \tag{18}$$

*Then, the Fourier coefficients of* $f_\omega$ *(the* $F_k$*'s) and the Fourier coefficients of* $y_\omega$ *(the* $Y_k$*'s) are independent of the choice of* $\omega$.

*Proof.* The fact that the $F_k$'s are independent of $\omega$ is an immediate consequence of Theorem 1. To show that $y_\omega$ is quasiperiodic, define $Y(x_1, \cdots, x_n)$ by $Y(x_1, \cdots, x_n) = H(F(x_1, \cdots, x_n))$. Since $H$ is a continuous mapping of the reals into themselves and $F$ is continuous in each $x_i$, it is seen that $Y$ is also continuous in each $x_i$. Furthermore, since $F$ is periodic of period $2\pi$ in each $x_i$, it follows that $Y$ also has this property. Finally, since $f_\omega$ is given by (16) it is seen that

$$y_\omega(t) = Y(\omega_1 t, \cdots, \omega_n t).$$

Hence, $y_\omega$ is quasiperiodic; and since $Y$ depends only on $F$ and not on the choice of $\omega$ it follows from Theorem 1 that the $Y_k$'s depend only on the $F_k$'s and not on the choice of $\omega$. Q.E.D.

This result, showing that $f_\omega$ and $y_\omega$ are quasiperiodic for any choice of $\omega$ and that their Fourier coefficients are independent of $\omega$, will play an important role in obtaining the frequency-power formulas.

**III. Orthogonality and the Manley–Rowe type formulas.** In this section the frequency-power formulas of the Manley–Rowe type will be proven for quasiperiodic functions and continuous nonlinearities. These results can be extended to include discontinuous nonlinearities, but this would require the theory of generalized almost periodic functions. Therefore, in order to minimize the required background material, and to keep the proofs elementary, these generalizations are not to be presented here. (These generalizations follow from appropriate approximations.)

The material which follows is concerned with vector-valued functions of time. Almost periodicity or quasiperiodicity refers to each component. For this case the inner product of (viii) is changed to $\langle f_1, f_2 \rangle = M\{f_1' f_2\}$ where the prime denotes transposition.

The Manley–Rowe type frequency-power relation will be introduced in two steps: first a general orthogonality relationship is established, and this result and Theorem 2 will then be used to obtain the Manley–Rowe formulas in their full generality.

THEOREM 3. *Let $f$ be a continuous mapping of $R^n$ into $R^n$, and assume that $U$ is a differential mapping from $R^n$ into $R$ such that*

$$f(x) = \nabla_x U(x) = \frac{\partial U}{\partial x}(x) = \begin{bmatrix} \dfrac{\partial U}{\partial x_1} \\ \vdots \\ \dfrac{\partial U}{\partial x_n} \end{bmatrix}(x) \quad \textit{for all} \quad x \in R^n. \tag{19}$$

*Further, assume that $x(\cdot)$ is an almost periodic $R^n$-valued function of $t$ with a uniformly continuous derivative. Then $y(\cdot) = f(x(\cdot))$ is almost periodic and the following orthogonality relation holds*

$$\langle \dot{x}, y \rangle = \sum_{\Lambda_n + \Lambda_q = 0} \Lambda_n X_n' Y_q = 0, \tag{20}$$

*where*

$$x(t) \sim \sum_n X_n \exp\,[i\Lambda_n t], \tag{21}$$

*and*

$$y(t) \sim \sum_n Y_n \exp\,[i\Lambda_n t]. \tag{22}$$

*Proof.* This result will be proved by first showing that $\langle \dot{x}, y \rangle = 0$, and the remaining part of (20) will be obtained by writing this inner product in terms of the Fourier coefficients. Using $y(t) = f(x(t))$ it follows that

$$\langle \dot{x}, y \rangle = M\{\dot{x}'y\} = \lim_{t \to \infty} \frac{1}{2T} \int_{-T}^{T} \dot{x}'(t) f(x(t))\, dt, \tag{23}$$

and from (19) it is seen that

$$\frac{dU(x(t))}{dt} = \dot{x}'(t) \frac{\partial U}{\partial x}(x(t)) = \dot{x}'(t) f(x(t)). \tag{24}$$

Using (24) in (23) yields

$$\begin{aligned} M\{\dot{x}'y\} &= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{dU(x(t))}{dt}\, dt \\ &= \lim_{T \to \infty} \frac{1}{2T} [U(x(T)) - U(x(-T))]. \end{aligned} \tag{25}$$

However, since $x(t)$ and thus $U(x(t))$ are bounded functions of $t$, it is clear that the limit in (25) is zero, and thus $M\{\dot{x}'y\} = \langle \dot{x}, y \rangle = 0$.

Since $x(\cdot)$ is almost periodic with a uniformly continuous derivative $\dot{x}(\cdot)$, property (vi) of Sec. II establishes that $\dot{x}(\cdot)$ is almost periodic with the Fourier series

$$\dot{x}(t) \sim \sum_n i\Lambda_n X_n \exp\,[i\Lambda_n t]. \tag{26}$$

Then, property (v) of Sec. II yields

$$\langle \dot{x}, y \rangle = M\{\dot{x}'y\} = \sum_{\Lambda_n+\Lambda_q=0} i\Lambda_n X_n' Y_q . \tag{27}$$

Finally, using the fact that $M\{\dot{x}'y\} = \langle \dot{x}, y \rangle = 0$ with (27 gives (20). Q.E.D.

By combining the results of Theorems 2 and 3 the frequency-power formulas involving a continuous time-invariant nonlinearity and quasiperiodic functions will be obtained in the following.

THEOREM 4. *Let $f$ be as in Theorem 3. Suppose that $x(t) \in R^n$ is quasiperiodic with basic frequencies $\omega_1, \cdots, \omega_k$ which are linearly independent over the rationals. Also assume that $x(\cdot)$ has a uniformly continuous derivative. Then $y(\cdot) = f(x(\cdot))$ is quasiperiodic with the same basic frequencies as $x(\cdot)$ and*

$$\sum_{r \in Z^k} r_j X_r' Y_r^* = 0 \quad \textit{for each} \quad j = 1, 2, \cdots, k, \tag{28}$$

*where the asterisk denotes complex conjugate,*

$$x(t) \sim \sum_{r \in Z^k} X_r \exp [i(r, \omega)t], \tag{29}$$

$$y(t) \sim \sum_{r \in Z^k} Y_r \exp [j(r, \omega)t], \tag{30}$$

$r = (r_1, \cdots, r_k)$ *an n-tuple of integers, and* $\omega = (\omega_1, \cdots, \omega_k)$.

*Proof.* Since $x(\cdot)$ is quasiperiodic with basic frequencies $\omega_1, \cdots, \omega_k$ there exists a continuous function $X(\sigma_1, \cdots, \sigma_k) \in R^n$ periodic of period $2\pi$ in each $\sigma_i$ such that

$$x(t) = X(\omega_1 t, \cdots, \omega_k t). \tag{31}$$

Let $\xi = (\xi_1, \cdots, \xi_k)$ be any $k$-tuple of real numbers which are independent over the rationals, and let $x_\xi(\cdot)$ be defined as

$$x_\xi(t) = X(\xi_1 t, \cdots, \xi_k t). \tag{32}$$

Thus $x_\xi(\cdot)$ is a quasiperiodic function for each $\xi$, and $x(\cdot) = x_\omega(\cdot)$. Further, let $y_\xi(\cdot) = f(x_\xi(\cdot))$ and thus $y(\cdot) = y_\omega(\cdot)$. It then follows from Theorem 2 that $y_\xi(\cdot)$ is quasiperiodic for each $\xi$, and also that $x_\xi(\cdot)$ and $y_\xi(\cdot)$ have the Fourier series

$$x_\xi(t) \sim \sum_{r \in Z^k} X_r \exp [i(r, \xi)t], \tag{33}$$

$$y_\xi(t) \sim \sum_{r \in Z^k} Y_r \exp [i(r, \xi)t], \tag{34}$$

where the $X_r$'s and $Y_r$'s are those of (29) and (30).

From Theorem 3, and in particular (20), it follows that for each $\xi$

$$\sum_{r \in Z^k} (r, \xi) X_r' Y_{-r} = 0 \tag{35}$$

in which $-r = (-r_1, \cdots, -r_k)$. Eq. (35) is obtained from (20) by observing that the set of Fourier exponents $\{\Lambda_n\}_{n=1}^{\infty}$ of $x_\xi(\cdot)$ and $y_\xi(\cdot)$ is the set $\{(r, \xi)\}_{r \in Z^k}$ and that, for $r^1, r^2 \in Z^k$, $(r^1, \xi) + (r^2, \xi) = 0$ implies that $r^1 = -r^2$ since the $\xi_i$'s are linearly independent over the rationals.

Since $(r, \xi) = \sum_{i=1} r_i \xi_i$, (35) can be rewritten as

$$\sum_{r \in Z^k} \sum_{j=1}^{k} r_j \xi_j X_r' Y_{-r} = 0, \tag{36}$$

and since the sum in (35) is absolutely convergent (see property (v), Sec. II) the order of summation in (36) can be reversed to obtain

$$\sum_{j=1}^{k} \xi_j \sum_{r \in Z^k} r_j X_r' Y_{-r} = 0. \tag{37}$$

It is now asserted that each $P_j = \sum_{r \in Z^k} r_j X_r' Y_{-r} = 0, j = 1, 2, \cdots, k$. Indeed, (37) must hold for each set of $\xi_j$'s independent over the rationals, and the $X_r$'s and $Y_{-r}$'s do not depend on the $\xi_j$'s by virtue of Theorem 2. Thus, let $\xi_j = P_j + \epsilon_j$; then (37) becomes

$$\sum_{j=1}^{k} (P_j^2 + \epsilon_j P_j) = 0. \tag{38}$$

Suppose $P_j \neq 0$ for some $j$'s. The $\epsilon_j$'s can be chosen sufficiently small so that $P_j^2 + \epsilon_j P_j > 0$ if $P_j \neq 0$ and, at the same time, the $\xi_j$'s remain independent over the rationals. This, however, contradicts (38), and thus all $P_j = 0$, that is

$$\sum_{r \in Z^k} r_j X_r' Y_{-r} = 0, \qquad j = 1, 2, \cdots, k. \tag{39}$$

Finally, since $y(\cdot)$ is real it follows that $Y_{-r} = Y_r^*$. Using this in (39) yields (28). Q.E.D.

As an application of Theorem 4, consider a nonlinear capacitor defined by

$$v = f(q) \tag{40}$$

where $v$ is the voltage across the capacitor and $q$ is the charge stored in the capacitor. Assume $q(\cdot)$ is quasiperiodic with basic frequencies $\omega_1$ and $\omega_2$. Then with

$$q(t) \sim \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} Q_{m,n} \exp [i(m\omega_1 + n\omega_2)t], \tag{41}$$

$$v(t) \sim \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} V_{m,n} \exp [i(m\omega_1 + n\omega_2)t], \tag{42}$$

Eq. (28) gives

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} mQ_{m,n}V_{m,n}^* = 0 \tag{43}$$

and

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} nQ_{m,n}V_{m,n}^* = 0. \tag{44}$$

The current $i(\cdot)$ in the capacitor is given by $i(t) = q(t)$. Thus,

$$i(t) \sim \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} I_{m,n} \exp [i(m\omega_1 + n\omega_2)t], \tag{45}$$

where $I_{m,n} = i(n\omega_1 + n\omega_2)Q_{m,n}$ (see property (vi) in Sec. II). Using this relation in (43) and (44) results in

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} m \frac{I_{m,n}V_{m,n}^*}{m\omega_1 + n\omega_2} = 0, \tag{46}$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} n \frac{I_{m,n}V_{m,n}^*}{m\omega_1 + n\omega_2} = 0. \tag{47}$$

Observing that $i(\cdot)$ and $v(\cdot)$ being real implies that $I^*_{m,n} = I_{-m,-n}$ and $V^*_{m,n} = V_{-m,-n}$, it is seen that (46) and (47) can be rewritten as

$$\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} m \frac{P_{m,n}}{m\omega_1 + n\omega_2} = 0, \tag{48}$$

$$\sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} n \frac{P_{m,n}}{m\omega_1 + n\omega_2} = 0, \tag{49}$$

where $P_{m,n} = \text{Re}\,[I_{m,n}V^*_{m,n}]$. These are the familiar formulas obtained by Manley and Rowe [1]. Analogous formulas for nonlinear resistors and inductors can be obtained from (20) in a similar manner.

It is perhaps worthwhile at this point to discuss the question of what happens when the basic frequencies are linearly dependent over the rationals. This question has been raised previously [15], and seems to have caused some confusion in the literature. Suppose that one has a nonlinearity $f(\cdot)$ and two "generators" with respective outputs $x_1(t)$ and $x_2(t)$. Suppose further that $x_1(\cdot)$ and $x_2(\cdot)$ are periodic with frequencies $\omega_1$ and $\omega_2$ respectively. Let the input $x(\cdot)$ to the nonlinearity $f(\cdot)$ be any continuous function, say $g(\cdot,\cdot)$ of $x_1$ and $x_2$. Thus

$$x(t) = g(x_1(t), x_2(t)). \tag{50}$$

Define now $X_i(\sigma)$ by

$$X_i(\sigma) = x_i(\sigma/\omega_i) \qquad i = 1, 2, \tag{51}$$

so $X_i$ is periodic of period $2\pi$ and $x_i(t) = X(\omega_i t)$. Hence the input $x(t)$ is given by

$$x(t) = X(\omega_1 t, \omega_2 t) \tag{52}$$

where

$$X(\sigma_1, \sigma_2) = g(X_1(\sigma_1), X_2(\sigma_2)). \tag{53}$$

The important point to note here is that if one constructs an input function $x(\cdot)$ from two other periodic functions $x_1(\cdot)$ and $x_2(\cdot)$, this construction of $x(\cdot)$ defines a continuous function $X(\sigma_1, \sigma_2)$ periodic of period $2\pi$ in each $\sigma_i$, and this function does not depend on $\omega_1$ and $\omega_2$.

Now, defining the ensemble of inputs $x_\xi(\cdot)$ by

$$x_\xi(t) = X(\xi_1 t, \xi_2 t), \tag{54}$$

and the corresponding outputs $y_\xi(\cdot)$ by

$$y_\xi(t) = f(x_\xi(t)), \tag{55}$$

we have from Theorem 2 that

$$x_\xi(t) \sim \sum_{n,m=-\infty}^{\infty} X_{n,m} \exp\,[i(n\xi_1 + m\xi_2)t], \tag{56}$$

$$y_\xi(t) \sim \sum_{n,m=-\infty}^{\infty} Y_{n,m} \exp\,[i(n\xi_1 + m\xi_2)t], \tag{57}$$

where the $X_{n,m}$'s and $Y_{n,m}$'s do *not* depend on the $\xi$'s. Theorem 4 can now be applied to $x_\xi(\cdot)$ and $y_\xi(\cdot)$ to obtain the frequency-power formulas. Note that the formulas given in

(28) involve only the Fourier coefficients, and since $x(t) = x_\omega(t)$ and $y(t) = y_\omega(t)$,

$$x(t) \sim \sum X_{n,m} \exp [i(n\omega_1 + m\omega_2)t], \tag{58}$$

$$y(t) \sim \sum Y_{n,m} \exp [i(n\omega_1 + m\omega_2)t], \tag{59}$$

and thus the frequency power formulas for $X_{n,m}$ and $Y_{n,m}$ are valid for $x(\cdot)$ and $y(\cdot)$ regardless of the values of $\omega_1$ and $\omega_2$.

Obviously the above arguments can be extended to any number of frequencies $\omega_1, \cdots, \omega_k$. The essential ingredient which has been used in the above is that one knows how $x_1(\cdot)$ and $x_2(\cdot)$ are combined to obtain $x(\cdot)$. If only $x(\cdot)$ were known and $\omega_1$ and $\omega_2$ were commensurate, then one could *not* determine the Fourier coefficients $X_{n_1,m_1}$ and $X_{n_2,m_2}$ separately whenever $(n_1\omega_1 + m_1\omega_2) = (n_2\omega_1 + m_2\omega_2)$. For a given $n_1$ and $m_1$ the only quantity available from a knowledge of $x(\cdot)$ is

$$\sum_{(n\omega_1 + m\omega_2) = (n_1\omega_1 + m_1\omega_2)} X_{n,m}.$$

The result of Theorem 4 will now be generalized to the case in which the nonlinearity depends on $t$ explicitly. This explicit time dependence will be assumed to be quasiperiodic, and the proof of this case will be reduced to that of Theorem 4.

THEOREM 5. *Let $f(\cdot,\cdot)$ be a continuous mapping of $R^n \times R$ into $R^n$, and suppose there exists a mapping $U(\cdot,\cdot)$ of $R^n \times R$ into $R$ such that*

$$f(x, t) = \frac{\partial U}{\partial x}(x, t) \qquad \text{for all} \qquad x \in R^n,\ t \in R. \tag{60}$$

*Assume further that for every fixed $x \in R^n$, $f(x, t) \in R^n$ is quasiperiodic in $t$ with basic frequencies $\omega_1, \cdots, \omega_k$. Let $x(t) \in R^n$ be quasiperiodic with basic frequencies $\omega_1, \cdots, \omega_k$, $\omega_{k+1}, \cdots, \omega_n$ and have a uniformly continuous derivative. Then $y(\cdot) = f(x(\cdot),\cdot)$ is quasiperiodic with the same basic frequencies as $x(\cdot)$, and*

$$\sum_{r \in Z^N} r_j X_r' Y_r^* = 0 \quad \text{for} \quad j = k + 1, \cdots, N, \tag{61}$$

*where*

$$x(t) \sim \sum_{r \in Z^N} X_r \exp [i(r, \omega)t], \tag{62}$$

$$y(t) \sim \sum_{r \in Z^N} Y_r \exp [i(r, \omega)t], \tag{63}$$

*$r = (r_1, \cdots, r_N)$, and $\omega = (\omega_1, \cdots, \omega_N)$.*

*Proof.* From the quasiperiodicity assumptions it is known that there exist functions $F(x, \alpha_1, \cdots, \alpha_k)$, $\bar{U}(x, \alpha_1, \cdots, \alpha_k)$, and $X(\alpha_1, \cdots, \alpha_N)$ continuous and periodic of period $2\pi$ in each $\alpha$ such that

$$f(x, t) = F(x, \omega_1 t, \cdots, \omega_k t), \tag{64}$$

$$F(x, \alpha_1, \cdots, \alpha_k) = \frac{\partial U}{\partial x}(x, \alpha_1, \cdots, \alpha_k), \tag{65}$$

and

$$x(t) = X(\omega_1 t, \cdots, \omega_N t). \tag{66}$$

Now let

$$Y(\alpha_1, \cdots, \alpha_N) = F(X(\alpha_1, \cdots, \alpha_N), \alpha_1, \cdots, \alpha_k). \tag{67}$$

Clearly $Y(\alpha_1, \cdots, \alpha_N)$ is continuous and periodic of period $2\pi$ in each $\alpha_i$, and since

$$y(t) = f(x(t), t) = Y(\omega_1 t, \cdots, \omega_N t), \tag{68}$$

it is seen that $y(\cdot)$ is quasiperiodic.

At this point it is convenient to introduce the following notation. For any $N$-tuple of real numbers, say $q = (q_1, \cdots, q_N)$, define $\pi$ by

$$\pi q = (q_1, \cdots, q_k) \tag{69}$$

and $\sigma$ by

$$\sigma q = (q_{k+1}, \cdots, q_N). \tag{70}$$

Now, for $\alpha = (\alpha_1, \cdots, \alpha_N)$, define $\hat{x}(t, \pi\alpha)$ by

$$\hat{x}(t, \pi\alpha) = X(\alpha_1, \alpha_2, \cdots, \alpha_k, \omega_{k+1}t, \cdots, \omega_N t). \tag{71}$$

Then $\hat{x}(t, \pi\alpha)$ and $\hat{y}(t, \pi\alpha) = F(\hat{x}(t, \pi\alpha), \alpha_1, \cdots, \alpha_k)$ satisfy the hypothesis of Theorem 4 for each fixed $\pi\alpha$ since $F(\cdot, \alpha_1, \cdots, \alpha_k)$ is a nonlinearity independent of $t$ and satisfies (65). Therefore, as in the proof of Theorem 3,

$$M\left\{\frac{\partial \hat{x}'}{\partial t}(t, \pi\alpha)\hat{y}(t, \pi\alpha)\right\} = 0 \quad \text{for all} \quad \alpha. \tag{72}$$

Let the Fourier series of $X$ and $Y$ be

$$X(\alpha_1, \cdots, \alpha_N) \sim \sum_{r \in Z^n} X_r e^{i(r,\alpha)}, \tag{73}$$

$$Y(\alpha_1, \cdots, \alpha_N) \sim \sum_{r \in Z^N} Y_r e^{(r,\alpha)}. \tag{74}$$

Then, letting

$$\Lambda_r = (r, \omega) = \sum_{j=1}^{N} r_j \omega_j \quad \text{and} \quad \sigma\Lambda_r = (\sigma r, \sigma\omega) = \sum_{i=k+1}^{N} r_i \omega_i ,$$

the Fourier series for $\hat{x}(\cdot, \pi\alpha)$, $(\partial\hat{x}/\partial t)(\cdot, \pi\alpha)$, and $\hat{y}(\cdot, \pi\alpha)$ are

$$\hat{x}(t, \pi\alpha) \sim \sum_{r \in Z^N} X_r \exp[i(\pi r, \pi\alpha)] \exp[i(\sigma\Lambda_r)t], \tag{75}$$

$$\frac{\partial \hat{x}}{\partial t}(t, \pi\alpha) \sim \sum_{r \in Z^N} i(\sigma\Lambda_r)X_r \exp[i(\pi r, \pi\alpha)] \exp[i(\sigma\Lambda_r)t], \tag{76}$$

and

$$\hat{y}(t, \pi\alpha) \sim \sum_{r \in Z^N} Y_r \exp[i(\pi r, \pi\alpha)] \exp[i(\sigma\Lambda_r)t]. \tag{77}$$

Thus, from property (v), Sec. II,

$$M\left\{\frac{\partial \hat{x}}{\partial t}\hat{y}\right\} = \sum_{r,m \in Z^N; \sigma\Lambda_r + \sigma\Lambda_m = 0} i(\sigma\Lambda_r)X_r' Y_m \exp[i(\pi(r + m), \pi\alpha)], \tag{78}$$

and using (72) there results

$$S(\alpha_1, \cdots, \alpha_k) = \sum_{\sigma\Lambda_r+\sigma\Lambda_m=0;\, r,m\in Z^N} (\sigma\Lambda_r)X_r'Y_m \exp[i(\pi, r+m), \pi\alpha)] = 0 \quad (79)$$

for all $\pi\alpha = (\alpha_1, \cdots, \alpha_k)$. It is now easily seen that

$$\frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi} S(\alpha_1, \cdots, \alpha_k)\, d\alpha, \cdots, d\alpha_k = \sum_{\sigma\Lambda_r+\sigma\Lambda_m=0;\, \pi r+\pi m=0} i(\sigma\Lambda_r)X_r'Y_m . \quad (80)$$

But $\sigma\Lambda_r + \sigma\Lambda_m = 0$ and $\pi r + \pi m = 0$ implies that $r = -m$ by virtue of the linear independence of the $\omega_i$'s. Using $Y_{-r} = Y_r^*$, (79) and (80) yield

$$\sum_{r\in Z^N} (\sigma\Lambda_r)X_r'Y_r^* = 0. \quad (81)$$

Using $\sigma\Lambda_r = r_{k+1}\omega_{k+1} + \cdots + r_N\omega_N$ in (81) results in

$$\sum_{j=k+1}^{N} \omega_j \sum_{r\in Z^N} r_jX_r'Y_r^* = 0. \quad (82)$$

From the same argument as used in Theorem 4 it is seen that (82) implies that

$$\sum_{r\in Z^N} r_jX_rY_r^* = 0 \quad \text{for} \quad j = k+1, \cdots, N, \quad (83)$$

which yields (61).

Note that when the nonlinearity is time-varying, one only has frequency-power formulas for the basic frequencies of $x(\cdot)$ which are independent of those of $f(\cdot, t)$. However, if one uses $X(\alpha_1, \cdots, \alpha_N)$ for computing the Fourier coefficients, the frequency-power formulas hold for any choice of $\omega_{k+1}, \cdots, \omega_N$. This is again due to the result of Theorem 2 which shows that the Fourier coefficients of $y(\cdot)$ are determined by those of $X(\alpha_1, \cdots, \alpha_N)$, and thus do not depend on the choice of the $\omega_i$'s.

**IV. Some positive operators and associated frequency-power formulas.** Consider the space $\mathfrak{A}$ of real-valued almost periodic functions. As mentioned in (viii), Sec. II, this space becomes an inner product space upon defining the inner product $\langle\cdot,\cdot\rangle$ of any $x, y \in \mathfrak{A}$ by

$$\langle x, y\rangle \triangleq M\{xy\} \quad (84)$$

where $M$ is the mean value functional defined in (iii), Sec. II. It is easily varified that this is indeed an inner product. Having this inner product, the following standard definition of a positive operator can be made.

*Definition* 3. An operator $T$ mapping $\mathfrak{A}$ into itself is called *nonnegative*[2] on $\mathfrak{A}$ (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathfrak{A}$, and it is called *positive* on $\mathfrak{A}$ if $T - \epsilon I \geq 0$ for some $\epsilon > 0$ ($I$ denotes the identity operator on $\mathfrak{A}$).

The *crosscorrelation* function of $x, y \in \mathfrak{A}$ is defined as the function $R_{xy}(\tau) = \langle x(t), y(t+\tau)\rangle$. It can be shown that $R_{xy} \in \mathfrak{A}$. It is well known that the autocorrelation function $R_{xx}(\cdot)$ attains its maximum at the origin. The following result is a generalization of this fact to a class of crosscorrelation functions.

THEOREM 6. *Let $f$ be a continuous monotone nondecreasing function (i.e., $(\sigma_1 - \sigma_2)\cdot(f(\sigma_1) - f(\sigma_2)) \geq 0$ for all $\sigma_1, \sigma_2 \in R$), and let $y(\cdot) = f(x(\cdot))$ with $x \in \mathfrak{A}$. Then $y \in \mathfrak{A}$, and*

$$R_{xy}(0) \geq R_{xy}(\tau) \quad \text{for all} \quad \tau \in R. \quad (85)$$

[2] This terminology is standard in the stability theory literature. Positive operators are on occasion referred to as passive, dissipative, or accretive. Note that no symmetry is required in this definition.

*If in addition f is odd (i.e.,* $f(-\sigma) = -f(\sigma)$) *then*

$$R_{xy}(0) \geq |R_{xy}(\tau)| \quad \textit{for all} \quad \tau \in R. \tag{86}$$

*Proof.* Consider the integral

$$I(\sigma_1, \sigma_2) = \int_{\sigma_1}^{\sigma_2} [f(\sigma) - f(\sigma_1)]\, d\sigma. \tag{87}$$

Since $f$ is monotone nondecreasing it follows that $I(\sigma_1, \sigma_2) \geq 0$, and thus

$$(\sigma_1 - \sigma_2)f(\sigma_1) \geq F(\sigma_1) - F(\sigma_2) \quad \text{for all} \quad \sigma_1, \sigma_2 \in R \tag{88}$$

where $F(\sigma) = \int_0^\sigma f(u)\, du$. Letting $\sigma_1 = x(t + \tau)$ and $\sigma_2 = x(t)$ in (88) gives

$$x(t + \tau)y(t + \tau) - x(t)y(t + \tau) \geq F(x(t + \tau)) - F(x(t)), \tag{89}$$

for all $t, \tau \in R$. Applying $M\{\cdot\}$ to both sides of (89) gives

$$R_{xy}(0) - R_{xy}(\tau) \geq M\{F(x(t + \tau)\} - M\{F(x(t))\} = 0. \tag{90}$$

The mean values of $F(x(t))$ and of $F(x(t + \tau))$ exist since for $x \in \mathcal{A}$ and $F$ continuous, $F(x(t)) \in \mathcal{A}$.

For $f(\cdot)$ odd, (88) can be rewritten as $(\sigma_1 - (-\sigma_2))f(\sigma_1) \geq F(\sigma_1) - F(-\sigma_2)$, and since for $f$ odd $F(-\sigma) = F(\sigma)$, this gives $(\sigma_1 + \sigma_2)f(\sigma_1) \geq F(\sigma_1) - F(\sigma_2)$. Using the same argument as above this leads to $R_{xy}(0) + R_{xy}(\tau) \geq 0$ for all $\tau \in R$. Thus for this case there also results $R_{xy}(0) \geq -R_{xy}(\tau)$, thereby giving (86). Q.E.D.

One more preliminary result will be required.

THEOREM 7. *If* $x \in \mathcal{A}$, *and* $g(\cdot)$ *is of bounded total variation, then the function* $y(\cdot)$ *defined by*

$$y(t) = \int_{-\infty}^{\infty} x(t - \tau)\, dg(\tau) \tag{91}$$

*belongs to* $\mathcal{A}$. *Furthermore, if*

$$x(t) \sim \sum_n X_n \exp[i\Lambda_n t], \tag{92}$$

*then*

$$y(t) \sim \sum_n G(i\Lambda_n)X_n \exp[i\Lambda_n t] \tag{93}$$

*where*

$$G(i\omega) = \int_{-\infty}^{\infty} \exp[-i\omega t]\, dg(t). \tag{94}$$

This result is well known, and is easily proved using the definition of almost periodic functions The proof is omitted.

Denote by $F$ the mapping of $\mathcal{A}$ into itself defined by

$$(Fx)(t) = f(x(t)) \tag{95}$$

where $f$ is continuous. The next theorem characterizes a rather large class of nonnegative operators on $\mathcal{A}$.

THEOREM 8. *Let $F$ be defined as above with $f$ monotone nondecreasing (odd monotone nondecreasing) and $xf(x) \geq 0$. Let $G$ be the mapping of $\mathcal{A}$ into itself defined by*

$$(Gx)(t) = x(t) - \int_{-\infty}^{\infty} x(t - \tau)\, dg(\tau) \tag{96}$$

*where $g(\cdot)$ is any monotone nondecreasing function (any function of bounded variation) having a total variation less than or equal to unity. Then $GF$ is a nonnegative operator on $\mathcal{A}$.*

*Proof.* Consider first the case where $f$ is monotone nondecreasing. Let $x \in \mathcal{A}$ and $y(t) = (Fx)(t)$. Then

$$\langle GFx, x \rangle = \langle Gy, x \rangle. \tag{97}$$

But

$$(Gy)(t) = y(t) - \int_{-\infty}^{\infty} y(t - \tau)\, dg(\tau), \tag{98}$$

so

$$\langle GFx, x \rangle = R_{yx}(0) - \int_{-\infty}^{\infty} R_{yx}(\tau)\, dg(\tau), \tag{99}$$

where the use of Fubini's theorem and the Lebesgue-dominated convergence theorem in obtaining the integral term in (99) is easily justified. Let $C^2 = 1 - \int_{-\infty}^{\infty} dg(t)$; then (99) can be rewritten as

$$\langle GFx, x \rangle = C^2 R_{yx}(0) + \int_{-\infty}^{\infty} [R_{yx}(0) - R_{yx}(\tau)]\, dg(\tau). \tag{100}$$

From Theorem 6 it is known that $R_{yx}(0) - R_{yx}(\tau) \geq 0$, and since $xf(x) \geq 0$ and $g(\cdot)$ is monotone nondecreasing it follows that $R_{yx}(0) \geq 0$. Hence (100) implies that $\langle GFx, x \rangle \geq 0$ for all $x \in \mathcal{A}$.

For the odd monotone case, it follows from (99) that

$$\langle GFx, x \rangle \geq R_{yx}(0) - \int_{-\infty}^{\infty} |R_{yx}(\tau)|\, |dg(\tau)|. \tag{101}$$

Let $C^2 = 1 - \int_{-\infty}^{\infty} |dg(\tau)|$, and rewrite (101) as

$$\langle GFx, x \rangle \geq C^2 R_{yx}(0) + \int_{-\infty}^{\infty} [R_{yx}(0) - |R_{yx}(\tau)|]\, |dg(\tau)|. \tag{102}$$

Theorem 6 shows that $R_{yx}(0) - |R_{yx}(\tau)| \geq 0$, and thus it follows from (102) that $\langle GFx, x \rangle \geq 0$ for all $x \in \mathcal{A}$. Q.E.D.

Note that $GF$ will be a positive operator if $F - \epsilon I$ for some $\epsilon > 0$ also satisfies the hypothesis for $F$ and if the total variation of $g$ is strictly less than unity.

Let $\mathcal{A}^1$ denote the subspace of $\mathcal{A}$ consisting of all $x \in \mathcal{A}$ such that $dx/dt = \dot{x}$ exists everywhere and $\dot{x} \in \mathcal{A}$. From Sec. II it is known that $\mathcal{A}^1$ thus consists of all $x \in \mathcal{A}$ having a uniformly continuous derivative. The following result extends the class of positive operators obtained in Theorem 8 when the operators are restricted to $\mathcal{A}^1$.

THEOREM 9. *Let $F$ and $G$ be as in Theorem 8, and assume that the function $f$ defining $F$ is continuously differentiable* (i.e., its derivative $f'$ exists and is continuous). *Then $(G + \alpha(d/dt))F$ is a nonnegative operator on $\mathcal{A}^1$ for all $\alpha \in R$.*

*Proof.* Let $x \in \mathcal{A}^1$ and $y(t) = f(x(t))$. Then $\dot{y}(t) = f'(x(t))\dot{x}(t)$, and since $x \in \mathcal{A}^1$ and $f'$ is continuous it follows that $\dot{y} \in \mathcal{A}$ and hence $y \in \mathcal{A}^1$. Thus $F$ maps $\mathcal{A}^1$ into itself, and $\langle x, (d/dt)Fx\rangle = 0$ for all $x \in \mathcal{A}^1$. Indeed,

$$\left\langle x, \frac{d}{dt} Fx \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) \frac{d}{dt} f(x(t))\, dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} [x(T)f(x(T)) - x(-T)f(x(-T))] - \lim_{T\to\infty} \frac{1}{2T} \int_{x(-T)}^{x(T)} f(x)\, dx = 0, \tag{102}$$

since $x(t)$ and $f(x(t))$ are both bounded for all $t \in R$.

The fact that $(G + \alpha(d/dt))F$ is nonnegative on $\mathcal{A}^1$ now follows immediately from Theorem 8 and (102). Q.E.D.

From Theorems 8 and 9 and the results concerning quasiperiodic functions in Sec. II, frequency-power formulas generalizing those of Page [2] and Pantell [4] are easily obtained in the next two theorems.

THEOREM 10. *Let $f$ be a continuous (odd) monotone nondecreasing function with $xf(x) \geq 0$ for all $x \in R$, and let $G(i\omega)$ be defined by*

$$G(i\omega) = 1 - \int_{-\infty}^{\infty} e^{-i\omega t}\, dg(t) \tag{103}$$

*where $g(\cdot)$ is any monotone nondecreasing function (any function of bounded variation) having a total variation less than or equal to unity. Suppose that $x(\cdot)$ is quasiperiodic with basic frequencies $\omega_1, \cdots, \omega_k$. Then $y(\cdot) = f(x(\cdot))$ is quasiperiodic with the same basic frequencies as $x(\cdot)$ and*

$$\sum_{r\in Z^k} G[i(r, \xi)] Y_r X_r^* \geq 0 \tag{104}$$

*where*

$$x(t) \sim \sum_{r\in Z^k} X_r \exp[i(r, \omega)t], \tag{105}$$

$$y(t) \sim \sum_{r\in Z^k} Y_r \exp[i(r, \omega)t], \tag{106}$$

*$\omega = (\omega_1, \cdots, \omega_k)$, $r = (r_1, \cdots, r_k)$ an n-tuple of integers, and $\xi = (\xi_1, \cdots, \xi_k)$ is any n-tuple of real numbers.*

*Proof.* Since from Theorem 8 the operator $GF$ is nonnegative, $\langle GFx, x\rangle = \langle Gy, x\rangle \geq 0$. From Theorem 7 it is seen that

$$(Gy)(t) \sim \sum_{r\in Z^k} G[i(r, \omega)] Y_r \exp[i(r, \omega)t], \tag{107}$$

and using property (v) of Sec. II

$$\langle Gy, x\rangle = M\{(Gy)x\} = \sum_{r\in Z^k} G[i(r, \omega)] Y_r X_{-r}\,. \tag{108}$$

Then since $x(t)$ is real, $X_{-r} = X_r^*$, using (104) with $\langle Gy, x\rangle \geq 0$ gives

$$\sum_{r\in Z^k} G[i(r, \omega)] Y_r X_r^* \geq 0. \tag{109}$$

Now from Theorem 3 it is known that $Y_r$ and $X_r$ do not depend on $\omega_1, \cdots, \omega_k$ and thus (109) must hold for all $\omega_1, \cdots, \omega_k$ independent over the rationals. That is,

$$\sum_{r \in Z^k} G[i(r, \xi)] Y_r X_r^* \geq 0 \tag{110}$$

for all $\xi = (\xi_1, \cdots, \xi_k)$ for which the $\xi_i$'s are independent over the rationals.

To show that (110) must hold also for a linearly dependent set of $\xi_i$'s, it will be shown that the sum in (110) is continuous in $\xi$. Note first of all that the sum in (110) is absolutely convergent (property (v), Sec. II). Therefore, for a given $\epsilon > 0$ there exists a bounded subset $D$ of $Z^k$ such that

$$\Big|\sum_{r \in D^c} G[i(r, \xi)] Y_r X_r^*\Big| < \epsilon \quad \text{for all} \quad \xi \tag{111}$$

where $D^c$ = complement of $D$ in $Z^k$. Secondly, note that since $g(\cdot)$ is of bounded total variation $G(i\omega)$ is uniformly continuous in $\omega$ [16, p. 15]. Thus

$$\begin{aligned} \Big|\sum_{r \in Z^k} G[i(r, \xi^1)] Y_r X_r^* &- \sum_{r \in Z^k} G[i(r, \xi^2)] Y_r X_r^*\Big| \\ &\leq \sum_{r \in D} |G[i(r, \xi^1)] - G[i(r, \xi^2)]| \, |Y_r X_r^*| + 2\epsilon \\ &\leq \max_{r \in D} |G[i(r, \xi^1)] - G[i(r, \xi^2)]| \, \Big|\sum_{r \in D} |Y_r X_r^*| + 2\epsilon. \end{aligned} \tag{112}$$

Thus, due to the continuity of $G(i\omega)$, $\|\xi^1 - \xi^2\| = \max_{1 \leq i \leq k} |\xi_i^1 - \xi_i^2|$ can be chosen sufficiently small that

$$\Big|\sum_{r \in Z^k} G[i(r, \xi^1)] Y_r X_r^* - \sum_{r \in Z^k} G[i(r, \xi^2)] Y_r X_r^*\Big| \leq 3\epsilon, \tag{113}$$

which establishes the desired continuity, since $\epsilon$ is arbitrary. Thus, since the set of $\xi$'s having linearly independent components is dense in $R^k$, it follows that (106) must hold for all $\xi \in R^k$.

THEOREM 11. *Let $f$, $G$, $x(\cdot)$ and $y(\cdot)$ be as in Theorem* 10, *and assume in addition that $f$ is continuously differentiable, and $x \in \mathfrak{A}^1$. Then with $M(i\omega)$ defined by*

$$M(i\omega) = G(i\omega) + \alpha i\omega, \tag{114}$$

*the following frequency-power relation holds for any $\xi \in R^k$ and $\alpha \in R$:*

$$\sum_{r \in Z^k} M[i(r, \xi)] X_r^* Y_r \geq 0. \tag{115}$$

The proof of this result follows that of Theorem 10 and is omitted.

As in the case of the frequency-power formulas obtained in the previous section, the results of Theorems 10 and 11 can be extended to time-varying nonlinearities. This extension is given in the following theorem.

THEOREM 12. *Let $f(\cdot, \cdot)$ be a mapping of $R \times R \rightarrow R$ such that $f(\cdot, t)$ is a continuous (odd) monotone nondecreasing function for each fixed $t$, $xf(x, t) \geq 0$ for all $x, t \in R$, and for every fixed $x \in R$, $f(x, t)$ is quasiperiodic in $t$ with basic frequencies $\omega_1, \cdots, \omega_k$. Let $G(i\omega)$ be defined by*

$$G(i\omega) = 1 - \int_{-\infty}^{\infty} e^{-i\omega t} \, dg(t) \tag{116}$$

*where $g(\cdot)$ is any monotone nondecreasing function (any function of bounded variation)* *having a total variation less than or equal to unity. Further, let $x(\cdot)$ be quasiperiodic with basic frequencies $\omega_1, \cdots, \omega_k, \cdots, \omega_r$ and let $y(t) = f(x(t), t)$. Then $y(t)$ is also quasiperiodic, and for any $\xi \in R^{n-k}$*

$$\sum_{r \in Z^n} G[i(\sigma r, \xi)] Y_r X_r^* \geq 0, \tag{117}$$

*where $\sigma r = (r_{k+1}, \cdots, r_n)$ and $x(t)$ and $y(t)$ have the Fourier series*

$$x(t) \sim \sum_{r \in Z^n} X_r \exp [j(r, \omega)t], \tag{118}$$

$$y(t) \sim \sum_{r \in Z^n} Y_r \exp [j(r, \omega)t], \tag{119}$$

*with $\omega = (\omega_1, \cdots, \omega_n)$.*

*Proof.* Since $f(x, t)$ is quasiperiodic in $t$, there exists a function $F(x, \alpha_1, \cdots, \alpha_k)$ such that $f(x, t) = F(x, \omega_1 t, \cdots, \omega_k t)$. As in the proof of Theorem 5 we can define $\hat{x}(t, \pi\alpha)$ and $\hat{y}(t, \pi\alpha)$ having the Fourier series

$$\hat{x}(t, \pi\alpha) \sim \sum_{r \in Z^n} X_r \exp [i(\pi r, \pi\alpha)] \exp [i(\sigma r, \sigma\omega)t], \tag{120}$$

$$\hat{y}(t, \pi\alpha) \sim \sum_{r \in Z^n} Y_r \exp [i(\pi r, \pi\alpha)] \exp [i(\sigma r, \sigma\omega)t]. \tag{121}$$

Applying the result of Theorem 10 to $F(\cdot, \alpha_1, \cdots, \alpha_k)$, $\hat{x}(t, \pi\alpha)$ and $\hat{y}(t, \pi\alpha)$, there results for any $\xi \in R^{n-k}$

$$\sum_{r, m \in Z^n; \sigma r + \sigma m = 0} G[i(\sigma r, \xi)] Y_r X_m \exp [i(\pi r + \pi m, \pi\alpha)] \geq 0. \tag{122}$$

Averaging with respect to $\sigma_1, \cdots, \sigma_k$ from 0 to $2\pi$ as in (80), this becomes

$$\sum_{r \in Z^n} G[i(\sigma r, \xi)] Y_r X_{-r} \geq 0. \tag{123}$$

Finally using $X_{-r} = X_r^*$, (113) results. Q.E.D.

Clearly the result of Theorem 11 can be generalized as well to the time-varying case by restricting $f(x, t)$ to be continuously differentiable in $x$, assuming $x(\cdot)$ to be in $\mathfrak{A}^1$, and replacing $G(i\omega)$ by $M(i\omega)$ in (117) (where $M(i\omega)$ is defined by (114)).

By suitably restricting $x(\cdot)$, the frequency-power formulas of Theorem 11 can be extended to include functions $M(i\omega)$ given by

$$M(i\omega) = \int_{-\infty}^{\infty} \frac{e^{i\omega\tau} - 1 - i\omega g(\tau)}{\tau^2} \, dV(\tau), \tag{124}$$

where $V(\cdot)$ is a monotone nondecreasing function such that

$$\int_x^{\infty} (1/\tau^2) \, dV(\tau) \quad \text{and} \quad \int_{-\infty}^{-x} (1/\tau^2) \, dV(\tau)$$

exist for $x > 0$, and $g(\cdot)$ is any bounded real-valued function of $\tau$ which is continuous at the origin with $g(0) = 1$. Then, if $x(\cdot)$ is quasiperiodic having a second derivative which is also quasiperiodic, the frequency-power relation of Theorem 11 holds when $M(i\omega)$ is of the form given by (124).

This extension of Theorem 11 is easily proved from a result in probability theory

[17, p. 528] which states that an $M(i\omega)$ in the form of (124) is the pointwise limit of a sequence

$$M_n(i\omega) = c_n[G_n(i\omega) + i\beta_n\omega], \tag{125}$$

where the $G_n(i\omega)$ are as defined in (103). The additional smoothness conditions on $x(\cdot)$ are required to ensure the convergence of the sum in (115).

Using particular functions for $V(\cdot)$, one can obtain large classes of specific frequency-power formulas. Some functions of particular interest are:

(a) $$M(i\omega) = 1 - \cos \alpha\omega \quad \alpha \text{ any real number}$$

(b) $$M(i\omega) = |\omega|^\tau [1 - i\delta \tan (\pi\tau/2)], \quad \omega \geq 0, \; |\omega|^\tau [1 + i\delta \tan (\pi\tau/2)], \quad \omega < 0,$$

where $0 \leq \tau \leq 2$, $\tau \neq 1$, and $|\delta| \leq 1$.

(c) $$M(i\omega) = \begin{cases} |\omega| [1 + i\delta \log |\omega|], & \omega \geq 0 \\ |\omega| [1 - i\delta \log |\omega|], & \omega < 0 \end{cases}$$

where $|\delta| \leq 1$.

(d) $$M(i\omega) = 1 - \gamma e^{-|\omega|\tau}$$

where $\gamma$ and $\tau$ are real $0 \leq \gamma \leq 1$, and $0 \leq \tau \leq 2$.
For details in the calculations see [17] or [18].

Note that (a) gives a result of Page [2] and in (b) with $\tau = 2$ and $\delta = 0$ the result of Pantell [4] is obtained.

**V. An application: a nonoscillation theorem.** The above results will now be used to obtain a criterion for the nonexistence of quasiperiodic solutions in the feedback system shown in Fig. 1 when $f$ satisfies the conditions of Theorem 10 and $H(s) = \int_{-\infty}^{\infty} e^{-st} dh(t)$ with $h$ of bounded variation. For suppose $x(\cdot)$ is a quasiperiodic solution. Then $y(\cdot)$ is also quasiperiodic and from Theorem 10

$$\sum_{r \in Z^k} G[i(r, \omega)] Y_r X_r^* \geq 0. \tag{124}$$

Also, using Theorem 7, it is seen that $X_r = -H[i(r, \omega)] Y_r$ and thus (120) becomes

$$-\sum_{r \in Z^k} G[i(r, \omega)] H^*[i(r, \omega)] \; |Y_r|^2 \geq 0. \tag{125}$$

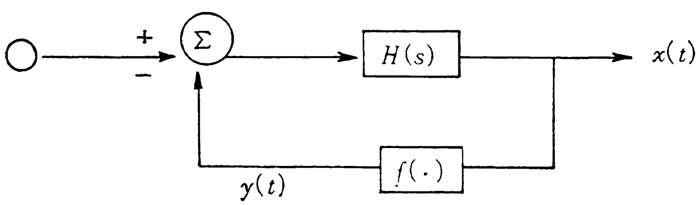


FIG. 1.

Now suppose there exists a $G(i\lambda)$ in the form of (103) such that

$$\text{Re } [G(i\lambda)H^*(i\lambda)] \geq 0 \quad \text{for all } \lambda \in R.$$

Then, in this choice of $G(i\lambda)$, (125) implies that $|Y_r| = 0$ for all $r \in Z^k$. Hence

THEOREM 13. *Consider the feedback system of Fig.* 1 *with* $f(\cdot)$ *monotone and nondecreasing,* $H(s) = \int_{-\infty}^{\infty} e^{-st} dh(t)$ *with h of bounded variation. If there exists a* $G(i\omega)$ *in the form of* (103) *such that* $\mathrm{Re}\,[G(i\omega)H^*(i\omega)] \geq 0$ *for all* $\omega \in R$ *then there will exist no quasiperiodic solution* $y(\cdot)$ *and thus the solutions cannot asymptotically approach a quasiperiodic function.*

Although this result is subsumed by the stability results presented in [19], [20], and [21], it is interesting to note that frequency-power formulas can be utilized in obtaining these stability criteria.

**VI. Conclusions.** In this paper a precise mathematical framework has been given in which to view frequency-power relations. More specifically, by restricting consideration to quasiperiodic functions, it has been shown that frequency-power formulas result from

(i) an orthogonality relation on the space of almost periodic functions; and
(ii) positive operators defined on the space of almost periodic functions.

The type of formulas resulting from (i) are commonly called the Manley-Rowe type formulas, and all previously known formulas of this type have been subsumed by the one relationship given in Theorem 5 (Theorem 4 is specialized to the time-invariant case). The formulas resulting from (ii) are of the type obtained by Page and Pantell, and these results are contained in Theorems 8 through 12. These theorems subsume the previously known formulas, and in fact give considerably more general results.

It has also been shown (Theorem 13) that the frequency-power relations can be used to obtain criteria for the absence of oscillations in certain feedback systems. Although these criteria have been proven elsewhere in a somewhat stronger form, the results given here provide a nice interpretation of these stability criteria, namely, that if the criteria are satisfied, no solution can asymptotically approach a quasiperiodic function. This can also be thought of as an extension of the describing function method in which one excludes the possibility of a periodic solution.

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