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AN IMPROVED ESTIMATE FOR THE ERROR IN THE CLASSICAL, LINEAR THEORY OF PLATE BENDING*

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Abstract. The relative mean square error in the three-dimensional stress field predicted by classical plate theory is shown to be $O(h/L_*)^2$, where h is the plate thickness and L_* is a mean square measure of the wavelength of the midplane deformation pattern. This improves a recent result of Nordgren who obtained a relative error estimate of $O(h/L_*)$. The improved error estimate, which, like Nordgren's, is based on the Prager–Synge hypercircle theorem in elasticity, is obtained by constructing a kinematically admissible three-dimensional displacement field that depends on the solution of the classical plate equations but which yields an accurate, nonzero distribution of the transverse shearing strain through the thickness.

1. Introduction. In a recent paper [1], Nordgren has considered the errors in the three-dimensional stress field predicted by the classical, linear, two-dimensional theory of plate bending. Starting from the (presumably known) two-dimensional stress couple and midplane displacement fields, Nordgren constructs a statically admissable tensor stress field $\tilde{\mathbf{d}}$ and a kinematically admissable tensor stress field $\hat{\mathbf{d}}$. The Prager–Synge hypercircle theorem in elasticity [2], [3] implies that

$$C[\mathbf{d} - \tfrac{1}{2}(\tilde{\mathbf{d}} + \hat{\mathbf{d}})] = C[\mathbf{d}_D], \tag{1.1}$$

where $\mathbf{d}$ is the actual tensor stress field, $\mathbf{d}_D = \frac{1}{2}(\tilde{\mathbf{d}} - \hat{\mathbf{d}})$, and $C[\cdot]$ a homogeneous, positive definite quadratic functional representing the stress energy. Nordgren shows that $\{C[\mathbf{d}_D]\}^{1/2}$ is proportional to the plate thickness h and remarks that this result "is somewhat surprising since the exact solutions for plates in elasticity theory give a relative error proportional to the square of the thickness."

The reason for the relatively large value of $C[\mathbf{d}_D]$ obtained by Nordgren, as he himself points out, is that his expression for the approximate transverse shearing stress $\frac{1}{2}(\tilde{\mathbf{d}}^\alpha + \hat{\sigma}^\alpha)$ is not a good representation of the actual distribution which varies nearly parabolically through the thickness.

In this paper we obtain, by constructing a somewhat more elaborate three-dimensional displacement field than did Nordgren, the improved error estimate

$$C[\mathbf{d}_D]/C[\hat{\mathbf{d}}] = O(h/L_*)^4, \tag{1.2}$$

where L_* is a mean square measure of the midplane deformation pattern, explicitly computable once the plate theory solution is known. The key idea in obtaining (1.2)

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is the following: even though the transverse shearing strain γ_α is assumed to be zero in the derivation of classical plate bending theory from three-dimensional elasticity, once a plate theory solution has been found, one may go back and construct a nonzero distribution of γ_α through the thickness *that is correct to within a relative error of order* $(h/L_*)^2$. Using such a distribution we are able to improve Nordgren's relative error estimate from $O(h/L_*)$ to $O(h/L_*)^2$.

Working independently of Nordgren, Koiter [4] has just completed a similar analysis in which he shows that the relative mean square error in the three-dimensional displacement field predicted by classical shell theory is $O(h/R + h^2/L^2)^{1/2}$, where R is the minimum over the shell midsurface of the smallest principal radius of curvature and L is the minimum *local* wavelength of the midsurface deformation pattern. Koiter's relative error estimate can be improved to $O(h/R + h^2/L^2)$, as will be shown elsewhere by my colleague D. A. Danielson [5]. The reason for a separate error analysis for plate theory is that its great geometrical simplicity allows for a more explicit, extensive, and precise error analysis than is feasible for general shell theory.

2. Three-dimensional considerations. Let A denote a plane area in three-dimensional Euclidean space and let its boundary ∂A consist of one or more closed, piecewise smooth curves. Let θ^α, $\alpha = 1, 2$, denote an arbitrary set of curvilinear coordinates on A and let $\mathbf{r}(\theta^\alpha)$ denote the position vector of points on A with respect to a fixed origin 0 situated on A. The covariant base vectors on A are defined and denoted by $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$, where the subscript α preceded by the comma denotes partial differentiation with respect to θ^α. The metric tensor associated with the curvilinear coordinates θ^α is defined and denoted by $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$. All indices will be raised or lowered with respect to $a_{\alpha\beta}$. Covariant differentiation based on $a_{\alpha\beta}$ will be denoted by a vertical bar. The unit normal vector on the positive side of A is denoted by $\mathbf{k}$.

A plate of variable thickness $h(\mathbf{r})$ is defined to be a body which, in its undeformed configuration, occupies the volume of space

$$V = \{\mathbf{R} \mid \mathbf{R} = \mathbf{r} + z\mathbf{k}, \mathbf{r} \in A, |z| \leq \tfrac{1}{2}h(\mathbf{r})\}. \tag{2.1}$$

The boundary of V consists of upper and lower faces

$$A_\pm = \{\mathbf{R} \mid \mathbf{R} = \mathbf{r} \pm \tfrac{1}{2}h(\mathbf{r})\mathbf{k}, \mathbf{r} \in A\} \tag{2.2}$$

and an edge

$$E = \{\mathbf{R} \mid \mathbf{R} = \mathbf{r} + z\mathbf{k}, \mathbf{r} \in \partial A, |z| \leq \tfrac{1}{2}h(\mathbf{r})\}. \tag{2.3}$$

Let the components of the three-dimensional stress tensor $\boldsymbol{\sigma}$ in the cylindrical coordinate system (θ^α, z) be denoted by $\sigma^{\alpha\beta}$, σ^α, σ and let the components of the body force vector $\mathbf{b}$ be denoted by b^α, b. Then the linear, three-dimensional static equilibrium equations can be written

$$\sigma^{\alpha\beta}|_\beta + \sigma^\alpha_{,3} + b^\alpha = 0, \tag{2.4}$$

$$\sigma^\alpha|_\alpha + \sigma_{,3} + b = 0, \tag{2.5}$$

$$\sigma^{\alpha\beta} = \sigma^{\beta\alpha}. \tag{2.6}$$

If $\mathbf{U}$ denotes the three dimensional displacement vector with components U_α, W, the components of the linearized three-dimensional strain tensor are

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(U_{\alpha|\beta} + U_{\beta|\alpha}), \tag{2.7}$$

$$\gamma_\alpha = \tfrac{1}{2}(W_{,\alpha} + U_{\alpha,3}), \tag{2.8}$$

$$\gamma = W_{,3} . \tag{2.9}$$

We shall assume that the plate is elastically isotropic, having a strain energy density of the form

$$V = \frac{E}{2(1+\nu)}\left[\gamma^{\alpha\beta}\gamma_{\alpha\beta} + 2\gamma^\alpha\gamma_\alpha + \gamma^2 + \frac{\nu}{1-2\nu}(\gamma^\alpha_\alpha + \gamma)^2\right], \tag{2.10}$$

where E is Young's modulus and ν is Poisson's ratio. The associated stress-strain relations read

$$(1+\nu)(1-2\nu)\sigma_{\alpha\beta} = E[(1-2\nu)\gamma_{\alpha\beta} + \nu a_{\alpha\beta}(\gamma^\lambda_\lambda + \gamma)], \tag{2.11}$$

$$(1+\nu)\sigma_\alpha = E\gamma_\alpha , \tag{2.12}$$

$$(1+\nu)(1-2\nu)\sigma = E[(1-\nu)\gamma + \nu\gamma^\alpha_\alpha]. \tag{2.13}$$

To relate the equations of three-dimensional elasticity to those of the bending theory of plates, we first integrate (2.5) with respect to z from $-\frac{1}{2}h$ to $\frac{1}{2}h$. This yields the normal force equilibrium equation

$$Q^\alpha|_\alpha + p = 0, \tag{2.14}$$

where

$$Q^\alpha = \int_-^+ \sigma^\alpha \, dz, \tag{2.15}$$

$$p = \int_-^+ b \, dz + q, \tag{2.16}$$

$$q = [\mathbf{d}_n(\theta^\alpha, \tfrac{1}{2}h) + \mathbf{d}_n(\theta^\alpha, -\tfrac{1}{2}h)]\cdot\mathbf{k}, \tag{2.17}$$

$$\int_-^+ = \int_{-h/2}^{h/2} , \tag{2.18}$$

and $\mathbf{d}_n$ denotes the stress vector acting on the faces of the plate. (The unit normal vectors to the plate faces $\pm A$ are identically equal to $\pm\mathbf{k}$ only if $h = $ const.)

Multiplying (2.4) by z and integrating from $-\frac{1}{2}h$ to $\frac{1}{2}h$, we obtain the moment equilibrium equation

$$M^{\alpha\beta}|_\beta - Q^\alpha + l^\alpha = 0, \tag{2.19}$$

where

$$M^{\alpha\beta} = \int_-^+ z\sigma^{\alpha\beta} \, dz, \tag{2.20}$$

$$l^\alpha = \tfrac{1}{2}hm^\alpha + \int_-^+ zb^\alpha \, dz, \tag{2.21}$$

$$m^\alpha = [\mathbf{d}_n(\theta^\alpha, \tfrac{1}{2}h) - \mathbf{d}_n(\theta^\alpha, -\tfrac{1}{2}h)]\cdot\mathbf{a}^\alpha. \tag{2.22}$$

The standard reduced equilibrium equation of plate bending theory follows upon solving (2.19) for Q^α and inserting the resulting expression into (2.14):

$$M^{\alpha\beta}|_{\alpha\beta} + p + l^\alpha|_\alpha = 0. \tag{2.23}$$

Observe that the effects of the applied face tractions and the distributed body forces enter the reduced equilibrium equation only in the combination

$$p + l^{\alpha}|_{\alpha} \equiv P. \tag{2.24}$$

This leads us to define any two sets of prescribed face and body forces as being *equivalent* if each set yields the same value for P. Reference to (2.16), (2.17), (2.21) and (2.22) shows that, in particular, any given set of prescribed face tractions $\boldsymbol{\sigma}_n(\theta^{\alpha}, \frac{1}{2}h)$, $\boldsymbol{\sigma}_n(\theta^{\alpha}, -\frac{1}{2}h)$ and body forces $\mathbf{b}(\theta^{\alpha}, z)$ is equivalent to a distribution $b_*(\theta^{\alpha})$ of the normal component of the body force vector *alone* given by

$$hb_* = \int_-^+ (b + zl^{\alpha}|_{\alpha})\, dz + q + \tfrac{1}{2}hm^{\alpha}|_{\alpha}\,. \tag{2.25}$$

Thus, any given distribution of face tractions and body forces which produce a state of pure bending may be uniquely decomposed into the sum of a *regular part* consisting of zero face tractions and a body force distribution independent of z given by (2.25), plus an *irregular part*[1] consisting of the "residual" body force distribution $\mathbf{b} - b_*\mathbf{k}$. Clearly, it only makes sense to compare the difference between the three-dimensional stress field predicted by plate bending theory and those associated with *regular*, three-dimensional loads.

Henceforth we assume that the given face and body loads are regular, that is $\mathbf{b} = b(\theta^{\alpha})\mathbf{k}$, $m^{\alpha} = q = 0$, and that the reduced plate bending equation (2.23) is satisfied. This enables us to convert the three-dimensional force equilibrium equations (2.4) and (2.5) into the integral equations

$$\sigma^{\alpha} = \frac{1}{2}\int_-^+ \operatorname{sgn}(\zeta - z)\sigma^{\alpha\beta}|_{\beta}\, d\zeta, \tag{2.26}$$

$$\sigma = \frac{1}{2}\int_-^+ \operatorname{sgn}(\zeta - z)(\sigma^{\alpha}|_{\alpha} + b)\, d\zeta, \tag{2.27}$$

where the signum function is defined by $\operatorname{sgn}(x) = \pm 1$ if $x \gtrless 0$ and $\operatorname{sgn}(0) = 0$. In view of (2.23) and the fact that, for regular loads, $p = hb$, we may rewrite (2.27) in the form

$$\sigma = \frac{1}{2}\int_-^+ \operatorname{sgn}(\zeta - z)\sigma^{\alpha}|_{\alpha}\, d\zeta + (z/h)M^{\alpha\beta}|_{\alpha\beta}\,. \tag{2.28}$$

3. Statically admissible stress field. To construct a statically admissible stress field $\bar{\boldsymbol{\sigma}}$, we make the conventional assumption that $\bar{\sigma}^{\alpha\beta}$ vanishes on the midplane and varies linearly through the thickness. It then follows from (2.20) that

$$\bar{\sigma}^{\alpha\beta} = \frac{12z}{h^3}M^{\alpha\beta}. \tag{3.1}$$

Henceforth, we assume that h is a constant. To compute the remaining components of $\bar{\boldsymbol{\sigma}}$ we evaluate (2.26) and (2.28) successively, so obtaining

$$\bar{\sigma}^{\alpha} = \frac{3}{2h}\left(1 - \frac{4z^2}{h^2}\right)M^{\alpha\beta}|_{\beta}\,, \tag{3.2}$$

$$\bar{\sigma} = -\frac{z}{2h}\left(1 - \frac{4z^2}{h^2}\right)M^{\alpha\beta}|_{\alpha\beta}\,. \tag{3.3}$$

[1] This is Koiter's terminology [4].

4. Kinematically admissible stress field. The Prager-Synge equality (1.1) shows that the closer $\tilde{\mathbf{\delta}}$ is to $\hat{\mathbf{\delta}}$, the smaller the error in the three-dimensional stress field predicted by plate theory. To obtain a relative error of $O(h/L_*)^2$, it is essential to make $\hat{\mathbf{\delta}}^\alpha$ close to $\tilde{\mathbf{\delta}}^\alpha$; in fact, we shall construct a displacement field such that $\hat{\mathbf{\delta}}^\alpha = \tilde{\sigma}^\alpha$. To this end assume

$$\hat{U}_\alpha(\theta^\beta, z) = zf_\alpha(\theta^\beta) + z^3 g_\alpha(\theta^\beta), \tag{4.1}$$

$$\hat{W}(\theta^\beta, z) = w(\theta^\beta) + \zeta(\theta^\beta, z), \tag{4.2}$$

where $w(\theta^\beta)$ is the midplane normal displacement of classical plate theory which is related to the components of the stress couple tensor through the stress-strain relations

$$M_{\alpha\beta} = -D[(1 - \nu)w_{,\alpha\beta} + \nu a_{\alpha\beta}\nabla^2 w], \tag{4.3}$$

where ∇^2 is the Laplacian operator. From (2.7)–(2.9), the expressions for the components of the strain tensor associated with (4.1) and (4.2) are

$$\hat{\gamma}_{\alpha\beta} = zf_{(\alpha|\beta)} + z^3 g_{(\alpha|\beta)}\ , \tag{4.4}$$

$$\hat{\gamma}_\alpha = \tfrac{1}{2}(w_{,\alpha} + \zeta_{,\alpha} + f_\alpha + 3z^2 g_\alpha), \tag{4.5}$$

$$\hat{\gamma} = \zeta_{,3}\ , \tag{4.6}$$

where $T_{(\alpha\beta)}$ denotes the symmetric part of any tensor $T_{\alpha\beta}$. The components of $\hat{\mathbf{\delta}}$ follow from the stress-strain relations (2.11)–(2.13), which we rewrite in the form

$$\hat{\sigma}_{\alpha\beta} = \frac{E}{1 - \nu^2}[(1 - \nu)\hat{\gamma}_{\alpha\beta} + \nu a_{\alpha\beta}\hat{\gamma}_\lambda^\lambda] + \frac{\nu}{1 - \nu} a_{\alpha\beta}\hat{\sigma}, \tag{4.7}$$

$$\hat{\sigma}_\alpha = \frac{E}{1 + \nu}\hat{\gamma}_\alpha\ , \tag{4.8}$$

$$\hat{\sigma} = E\hat{\gamma} + \nu\hat{\sigma}_\alpha^\alpha\ . \tag{4.9}$$

Following Nordgren [1] and Koiter [4], we choose ζ so that $\hat{\sigma}$ is approximately zero. For this purpose it is sufficient to take

$$E\hat{\gamma} = -\nu\sigma_\alpha^\alpha\ . \tag{4.10}$$

From (3.1) and (4.6) we have then

$$\zeta = -\frac{6\nu z^2}{Eh^3} M_\alpha^\alpha\ . \tag{4.11}$$

Now substitute (4.11) into (4.5) and choose the functions f_α and g_α so that the left-hand side of (4.8) is equal to the expression for $\tilde{\sigma}_\alpha$ given by (3.2). This yields

$$f_\alpha = -w_{,\alpha} + \frac{3(1 + \nu)}{Eh} M_\alpha^\beta|_\beta\ , \tag{4.12}$$

$$g_\alpha = \frac{2\nu}{Eh^3} M_\beta^\beta|_\alpha - \frac{4(1 + \nu)}{Eh^2} M_\alpha^\beta|_\beta\ . \tag{4.13}$$

Having explicitly determined the functions f_α, g_α and ζ in terms of plate theory solutions, we obtain the following expressions for $\hat{\sigma}_{\alpha\beta}$ and $\hat{\sigma}$ via (4.3)–(4.9):

$$\hat{\sigma}_{\alpha\beta} = \bar{\sigma}_{\alpha\beta} + 3\left(\frac{z}{h}\right)\left[M^{\lambda}_{(\alpha|\lambda\beta)} + \frac{\nu}{1-2\nu}\, a_{\alpha\beta}M^{\lambda\mu}|_{\lambda\mu}\right] - 2\left(\frac{z}{h}\right)^3\left\{2M^{\lambda}_{(\alpha|\lambda\beta)} - \frac{\nu}{1+\nu}\, M^{\lambda}_{\lambda}|_{\alpha\beta} + \frac{\nu a_{\alpha\beta}}{1-2\nu}\left[2M^{\lambda\mu}|_{\lambda\mu} - \frac{\nu}{1+\nu}\, M^{\lambda}_{\lambda}|^{\mu}_{\mu}\right]\right\}, \tag{4.14}$$

$$\hat{\sigma} = \frac{\nu}{1-2\nu}\left(\frac{z}{h}\right)\left[3M^{\lambda\mu}|_{\lambda\mu} - 2\left(\frac{z}{h}\right)^2\left(2M^{\lambda\mu}|_{\lambda\mu} - \frac{\nu}{1+\nu}\, M^{\lambda}_{\lambda}|^{\mu}_{\mu}\right)\right]. \tag{4.15}$$

By use of the plate stress-strain relations (4.3), we may express $\hat{\sigma}_{\alpha\beta}$ and $\hat{\sigma}$ in terms of w as follows:

$$\hat{\sigma}_{\alpha\beta} = \bar{\sigma}_{\alpha\beta} - D\left(\frac{z}{h}\right)\left[3 - 2(2-\nu)\left(\frac{z}{h}\right)^2\right]\left(\nabla^2 w|_{\alpha\beta} + \frac{\nu}{1-2\nu}\, a_{\alpha\beta}\nabla^4 w\right), \tag{4.16}$$

$$\hat{\sigma} = -\frac{\nu D}{1-2\nu}\left(\frac{z}{h}\right)\left[3 - 2(2-\nu)\left(\frac{z}{h}\right)^2\right]\nabla^4 w. \tag{4.17}$$

5. Boundary conditions. On an edge E of the plate, defined by (2.3), the virtual work is given by the expression

$$\int_{\partial A}\int_{-}^{+} \boldsymbol{\sigma}_\nu \cdot \delta\mathbf{U}\, dz\, ds = \int_{\partial A}\int_{-}^{+} (\sigma^{\alpha\beta}\, \delta U_\beta + \sigma^{\alpha}\, \delta W)\nu_\alpha\, dz\, ds, \tag{5.1}$$

where s is arc length along ∂A, and the ν_α are the covariant components of the outward unit vector $\boldsymbol{\nu}$ to ∂A.

Given a set of boundary conditions over E, our task is, first, to determine the appropriate plate boundary conditions along ∂A and, second, to decompose the given boundary conditions into regular and irregular parts.

The plate bending boundary conditions are obtained in the standard way by imposing the Kirchhoff constraint on the kinematically admissible displacement field $\delta\mathbf{U}$. This reduces (5.1) to a boundary integral of the form

$$\int_{\partial A} (V\, \delta w - M\, \delta w^{\cdot})\, ds + \sum\, [M^{\alpha\beta}\nu_\alpha \tau_\beta\, \delta w], \tag{5.2}$$

where

$$V = Q^{\alpha}\nu_\alpha + (M^{\alpha\beta}\nu_\alpha\tau_\beta)', \tag{5.3}$$

$$M = M^{\alpha\beta}\nu_\alpha\nu_\beta\,, \tag{5.4}$$

the τ_β are the covariant components of the unit tangent vector $\boldsymbol{\tau} = \mathbf{k} \times \boldsymbol{\nu}$, dots and primes denote, respectively, differentiation with respect to arc length along $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$, and the last term in (5.2) represents the sum of the jumps of the quantity in brackets at each of the corners of ∂A.

For simplicity we restrict ourselves to problems in which either certain components of the edge stress vector $\boldsymbol{\sigma}_\nu$ or the corresponding components of the edge displacement vector $\mathbf{U}$ are prescribed.[2] From (2.15), (2.20), (5.3) and (5.4) we may compute values

[2] The most general type of boundary conditions arise when the plate is attached to another elastic body of *known elastic response*. In this case one has in place of stress boundary conditions more general relations of the form $\bar{\boldsymbol{\sigma}}_\nu + B\mathbf{U} = \boldsymbol{\sigma}_\nu$, where B is an influence coefficient matrix and $\bar{\boldsymbol{\sigma}}_\nu$ is a prescribed vector. To carry out an error analysis, the strain energy of the body must be supplemented by terms which represent the energy stored by the elastic device at the boundary.

for V or M once certain components of $\mathbf{\sigma}_\nu$ are prescribed. Complementary displacement boundary conditions for the plate are obtained by setting

$$w(\mathbf{r}) = \bar{W}(\mathbf{r}, 0), \qquad \mathbf{r} \in \partial A \tag{5.5}$$

$$w^{\cdot}(\mathbf{r}) = -\left[\int_-^+ \bar{U}^\alpha(\mathbf{r}, z)\, dz\right]\nu_\alpha, \qquad \mathbf{r} \in \partial A, \tag{5.6}$$

whenever the quantities on the right are prescribed.

Once a plate bending solution has been obtained (but not before!), regular boundary conditions may be defined. We take the expressions for $\tilde{\sigma}^{\alpha\beta}$ and $\tilde{\sigma}^\alpha$ given by (3.1) and (3.2) in terms of $M^{\alpha\beta}$ and insert these expressions into the relation

$$\tilde{\sigma}_\nu = (\tilde{\mathbf{\sigma}}^{\alpha\beta}\mathbf{a}_\beta + \tilde{\sigma}^\alpha \mathbf{k})\nu_\alpha\,. \tag{5.7}$$

This determines a particular distribution of $\mathbf{\sigma}_\nu$. On those portions of the edge where $\mathbf{\sigma}_\nu$ is prescribed we define (5.7) to be its *regular part.* Thus any prescribed stress distribution $\bar{\mathbf{\sigma}}_\nu$ is uniquely decomposable into a regular part $\tilde{\mathbf{\sigma}}_\nu$ given by (5.7) plus a residual or irregular part $\bar{\mathbf{\sigma}}_\nu - \tilde{\mathbf{\sigma}}_\nu$. The regular part of the prescribed edge displacement vector $\bar{\mathbf{U}}$ is defined to be that part of $\bar{\mathbf{U}}$ with a thickness distribution given by (4.1) and (4.2) with f_α, g_α, and ζ given in terms of $M^{\alpha\beta}$ by (4.11)–(4.13).

6. The error estimate. Let us first assume that the plate is subject to edge loads only, since the error estimate may then be written in a particularly simple, explicit form. From the plate equilibrium equation (2.23) and the plate stress-strain relations (4.3) it follows that $\nabla^4 w = 0$. Hence $\tilde{\sigma} = \hat{\sigma} = 0$ while (4.16) reduces to

$$\hat{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\alpha\beta} - D\left(\frac{z}{h}\right)\left[3 - 2(2-\nu)\left(\frac{z}{h}\right)^2\right]\nabla^2 w|_{\alpha\beta}\,. \tag{6.1}$$

The stress energy is defined by

$$C[\mathbf{\sigma}] = \frac{1}{2E}\iint_A \int_-^+ [(1+\nu)(\sigma^{\alpha\beta}\sigma_{\alpha\beta} + 2\,\sigma^\alpha\sigma_\alpha + \sigma^2) - \nu(\sigma_\alpha^\alpha + \sigma)^2]\, dA\, dz. \tag{6.2}$$

For the case at hand ($p = 0$), note that

$$\begin{aligned} C[\tilde{\mathbf{\sigma}}] &\geqq \frac{1}{2E}\iint_A \int_-^+ [(1+\nu)\tilde{\sigma}^{\alpha\beta}\tilde{\sigma}_{\alpha\beta} - \nu(\tilde{\sigma}_\alpha^\alpha)^2]\, dA\, dz \\ &= (6/Eh^3)\iint_A [(1+\nu)M^{\alpha\beta}M_{\alpha\beta} - \nu(M_\alpha^\alpha)^2]\, dA \\ &= \tfrac{1}{2}D\iint_A [(1-\nu)w|^{\alpha\beta}w|_{\alpha\beta} + \nu(\nabla^2 w)^2]\, dA \\ &\geq \tfrac{1}{2}(1-\nu)D\iint_A w|^{\alpha\beta}w|_{\alpha\beta}\, dA. \end{aligned} \tag{6.3}$$

Furthermore

$$\begin{aligned} C[\mathbf{\sigma}_D] &= \frac{1}{8E}\iint_A \int_-^+ (1+\nu)(\tilde{\sigma}^{\alpha\beta} - \hat{\sigma}^{\alpha\beta})(\tilde{\sigma}_{\alpha\beta} - \hat{\sigma}_{\alpha\beta})\, dA\, dz \\ &= \frac{h^4}{32}(1-\nu)H(\nu)D\iint \nabla^2 w|^{\alpha\beta}\nabla^2 w|_{\alpha\beta}\, dA, \end{aligned} \tag{6.4}$$

where

$$H(\nu) = \frac{1}{48(1-\nu)^2}\int_0^1 \zeta^2[6-(2-\nu)\zeta^2]^2\,d\zeta \\ < 1, \qquad 0 \le \nu \le \tfrac{1}{2}. \tag{6.5}$$

If we define a mean square midplane deformation wavelength L_* by the expression

$$L_*^4 \iint_A \nabla^2 w|^{\alpha\beta}\nabla^2 w|_{\alpha\beta}\,dA = \iint_A w|^{\alpha\beta}w|_{\alpha\beta}\,dA, \tag{6.6}$$

it follows that

$$\frac{C[\boldsymbol{\sigma}_D]}{C[\tilde{\boldsymbol{\sigma}}]} \le \frac{H(\nu)}{16}\left[\frac{h}{L_*}\right]^4. \tag{6.7}$$

To obtain an estimate of the form (6.7) in the general case ($p \ne 0$), we make use of the inequalities

$$(1+\nu)(\sigma^{\alpha\beta}\sigma_{\alpha\beta} + 2\sigma^{\alpha}_{\alpha}\sigma_{\alpha} + \sigma^2) - \nu(\sigma^{\alpha}_{\alpha} + \sigma)^2 \ge (1-2\nu)(\sigma^{\alpha\beta}\sigma_{\alpha\beta} + 2\sigma^{\alpha}\sigma_{\alpha} + \sigma^2) \\ \le (1-2\nu)\sigma^{\alpha\beta}\sigma_{\alpha\beta}\,, \qquad 0 \le \nu \le \tfrac{1}{2} \tag{6.8}$$

and

$$(\nabla^4 w)^2 = (a^{\alpha\beta}\nabla^2 w|_{\alpha\beta})^2 \le 2\nabla^2 w|^{\alpha\beta}\nabla^2 w|_{\alpha\beta}\,. \tag{6.9}$$

From (3.1) and (6.8) it follows that

$$C[\tilde{\boldsymbol{\sigma}}] \ge \frac{1-2\nu}{2E}\iint_A\int_-^+ \tilde{\sigma}^{\alpha\beta}\tilde{\sigma}_{\alpha\beta}\,dA\,dz \\ = \frac{6(1-2\nu)}{Eh^3}\iint_A M^{\alpha\beta}M_{\alpha\beta}\,dA \\ \ge \frac{(1-2\nu)(1-\nu)}{2(1+\nu)}D\iint_A w|^{\alpha\beta}w|_{\alpha\beta}\,dA, \tag{6.10}$$

and from (3.3), (4.16), (4.17), and (6.9),

$$C[\boldsymbol{\sigma}_D] \le \frac{1+\nu}{8E}\iint_A\int_-^+ [(\tilde{\sigma}^{\alpha\beta} - \hat{\sigma}^{\alpha\beta})(\tilde{\sigma}_{\alpha\beta} - \hat{\sigma}_{\alpha\beta}) + (\tilde{\sigma} - \hat{\sigma})^2]\,dA\,dz \\ \le \frac{h^4}{32}(1-\nu)K(\nu)D\iint_A \nabla^2 w|^{\alpha\beta}\nabla^2 w|_{\alpha\beta}\,dA \tag{6.11}$$

where

$$K(\nu) = \frac{1}{48(1-\nu)^2(1-2\nu)^2}\int_0^1 \zeta^2\{[6-(2-\nu)\zeta^2]^2 + 2[1+4\nu-(1-\nu^2)\zeta^2]^2\}\,d\zeta. \tag{6.12}$$

Hence, in terms of the mean square wavelength defined by (6.6),

$$\frac{C[\boldsymbol{\sigma}_D]}{C[\tilde{\boldsymbol{\sigma}}]} \le \frac{1+\nu}{1-2\nu}\frac{K(\nu)}{16}\left(\frac{h}{L_*}\right)^4, \qquad 0 \le \nu < \tfrac{1}{2}. \tag{6.13}$$

Incompressible bodies ($\nu = \frac{1}{2}$) require a special treatment which will not be considered here.

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