TEMPERATURE OF A NONLINEARLY RADIATING SEMI-INFINITE SOLID*

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1. Introduction. Let $T(x, t)$ be the temperature of a semi-infinite heat-conducting solid occupying the half-space $x \geq 0$. We suppose that its surface radiates energy at a rate proportional to $[T(0, r)]^n$ and that the surface is heated by a source at a rate proportional to a given function $f(t)$. Here n is a positive constant, the value $n = 1$ corresponding to Newton's law of cooling and $n = 4$ to Stefan's radiation law. If $T = 0$ initially, then for $t > 0$, T is determined by the following initial boundary value problem:

$$T_t(x, t) = T_{xx}(x, t), \qquad x > 0, \quad t > 0, \tag{1.1}$$

$$T_x(0, t) = \alpha T^n(0, t) - f(t), \qquad t > 0, \tag{1.2}$$

$$T(x, 0) = 0, \qquad x \geq 0, \tag{1.3}$$

$$T \to 0 \quad \text{as} \quad x \to \infty, \qquad t \geq 0. \tag{1.4}$$

Here $\alpha > 0$ is a given constant.

This problem has been considered by Mann and Wolf [1], Roberts and Mann [2] and Padmavally [3], while Friedman [4] has considered more general problems of a similar kind. From their work we can conclude that if $f(t)$ is a piecewise continuous bounded function then the above problem has a solution and it is unique. In addition Padmavally [3] has shown that if $f(t)$ is nondecreasing in the interval $0 \leq t \leq \tau$ then $T(0, t)$ is also nondecreasing in this interval.

Our aim is to obtain more detailed information about the surface temperature $T(0, t)$ when $f(t) \geq 0$ and $f(t)$ is integrable. First we shall obtain a sequence of upper and lower bounds on $T(x, t)$, which incidentally provide a constructive proof of its existence, and we shall also show its uniqueness. Then we shall show that as $t \to \infty$, $T(0, t) \sim \pi^{1/2}E(\infty)t^{-1/2}$ where $E(\infty)$ is the net energy flux into the solid through the surface. Furthermore, we shall show that $E(\infty) > 0$ for $n \geq 3$ while $E(\infty) = 0$ for $n \leq 2$. Thus for $n \geq 3$ some of the energy which enters the solid remains there, while for $n \leq 2$ it is all ultimately radiated away. We shall also examine the behavior of $T(0, t)$ for small values of t as well as for large and small values of α.

2. Equivalent integral equation. A solution $T(x, t)$ of (1.1)–(1.4) can be represented in terms of $T(0, t)$ by the formula

$$T(x, t) = \int_0^t f(s)G_\rho(x, t, s)\, ds$$

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$$+ \int_0^t [\rho(s) - \alpha T^{n-1}(0, s)]T(0, s)G_\rho(x, t, s)\, ds, \qquad t \geq 0, \quad x \geq 0. \tag{2.1}$$

This formula is obtained by applying Green's theorem to $T(x, t)$ and the Green's function G_ρ defined by the following linear problem:

$$G_{\rho,t} = G_{\rho,xx}, \qquad x > 0, \quad t \geq s \geq 0, \tag{2.2}$$

$$G_{\rho,x}(0, t, s) = \rho(t)G_\rho(0, t, s) - \delta(t - s), \qquad t \geq s, \tag{2.3}$$

$$G_\rho(x, t, s) = 0, \qquad t < s, \quad x \geq 0, \tag{2.4}$$

$$G_\rho(x, t, s) \to 0 \quad \text{as} \quad x \to \infty, \qquad t \geq s. \tag{2.5}$$

The nonnegative function $\rho(t)$ in (2.3) is arbitrary, and can be chosen to facilitate the analysis. Any solution $T(x, t)$ of (2.1) satisfies (1.1)–(1.4).

We now set $x = 0$ in (2.1) to obtain a nonlinear integral equation for $T(0, t)$:

$$T(0, t) = \int_0^t f(s)G_\rho(0, t, s)\, ds$$
$$+ \int_0^t [\rho(s) - \alpha T^{n-1}(0, s)]T(0, s)G_\rho(0, t, s)\, ds, \qquad t \geq 0. \tag{2.6}$$

Once $T(0, t)$ is found from (2.6), it can be used in (2.1) to yield a solution $T(x, t)$ of (1.1)–(1.4). Thus the problem is reduced to solving (2.6).

Let us denote by $u_\rho(x, t)$ the first term on the right side of (2.1), i.e.

$$u_\rho(x, t) = \int_0^t f(s)G_\rho(x, t, s)\, ds. \tag{2.7}$$

It is evident that u_ρ is the solution of the linear problem (2.2)–(2.5) with $\delta(t - s)$ replaced by $f(t)$. Now (2.6) can be written in the form

$$T(0, t) = u_\rho(0, t) + \int_0^t [\rho(s) - \alpha T^{n-1}(0, s)]T(0, s)G_\rho(0, t, s)\, ds. \tag{2.8}$$

When $\rho(t) \equiv 0$, (2.6) and (2.8) become the following simple-looking equation:

$$T(0, t) = \pi^{-1/2} \int_0^t [f(s) - \alpha T^n(0, s)](t - s)^{-1/2}\, ds. \tag{2.9}$$

3. Bounds on $T(x, t)$. Let us define the sequences of functions u_j and ρ_j as follows:

$$u_j(x, t) = u_{\rho_j}(x, t), \qquad j = 1, 2, \cdots,$$
$$\rho_0(t) \equiv 0, \qquad \rho_j(t) = \alpha[u_{j-1}(0, t)]^{n-1}, \qquad j = 1, 2, \cdots. \tag{3.1}$$

By the maximum principle, $G_\rho \geq 0$ and then from (2.7) and the assumption that $f \geq 0$ we have $u_j \geq 0$. Now for any two functions $\rho(t)$ and $\bar{\rho}(t)$, the functions u_ρ and $u_{\bar{\rho}}$ given by (2.7) are related by the integral equation

$$u_{\bar{\rho}}(x, t) = u_\rho(x, t) + \int_0^t [\rho(s) - \bar{\rho}(s)]u_{\bar{\rho}}(0, s)G_\rho(x, t, s)\, ds. \tag{3.2}$$

From (3.2) it follows first that $u_1 \leq u_0$ and then that $u_1 \leq u_2 \leq u_0$. By induction we find

$$0 \leq u_1 \leq u_3 \leq \cdots \leq u_{2j-1} \leq \cdots \leq u_{2j} \leq \cdots \leq u_2 \leq u_0 , \; x \geq 0, \quad t \geq 0. \quad (3.3)$$

The functions u_{2j-1} form a monotone increasing sequence bounded above by u_0, while the u_{2j} form a monotone decreasing sequence bounded below by zero. Thus both sequences converge to limits, u^0 and u^e, defined by

$$\lim_{j\to\infty} u_{2j-1} = u^0, \qquad \lim_{j\to\infty} u_{2j} = u^e. \quad (3.4)$$

By using (3.2) in a suitable way, we can show that $u^e = u^0 = u(x, t)$, say, and that $u(0, t)$ is the unique solution of (2.9). Furthermore, $u(x, t)$ is the unique solution of (1.1)–(1.4). (See Appendix A for details.) Thus the sequence $u_i(0, t)$ converges to the unique solution $T(0, t)$ of (2.9), providing a constructive proof of its existence, as was shown by Mann and Wolf [1] for a different sequence. From (3.4) and (3.3) it follows that the u_{2j-1} form an increasing sequence of lower bounds on $T(x, t)$ while the u_{2j} form a decreasing sequence of upper bounds:

$$0 \leq u_1 \leq u_3 \leq \cdots \leq u_{2j-1} \leq \cdots \leq T \leq \cdots \leq u_{2j} \leq \cdots \leq u_2 \leq u_0 , \quad x \geq 0, t \geq 0, \quad (3.5)$$

In particular, (3.5) yields $T(x, t) \geq 0$.

Another interesting lower bound on $T(0, t)$ can be obtained by choosing $\rho(t) = \rho^*(t)$ in (2.3) where

$$\rho^*(t) = \alpha M t^{-1}, \qquad t > 0, \quad M > 0. \quad (3.6)$$

In Appendix B we show that as $t \to \infty$,

$$u_{\rho^*}(0, t) \sim C^* t^{-1/2}, \qquad C^* > 0. \quad (3.7)$$

We now use ρ^* and u_{ρ^*} in (2.8) to obtain

$$T(0, t) = u_{\rho^*}(0, t) + \alpha \int_0^t \{M s^{-1} - [T(0, s)]^{n-1}\} T(0, s) G_{\rho^*}(0, t, s) \, ds, \qquad t \geq 0. \quad (3.8)$$

Now $T(0, t)$ is positive, bounded, and decays at least as fast as $t^{-1/2}$ as $t \to \infty$, as we see from (3.5) and (4.8). Therefore it is possible to choose M so large that $Mt^{-1} - [T(0, t)]^{n-1} \geq 0$ for all $t > 0$ provided that $n \geq 3$. Then it follows from (3.8) and (3.7) that

$$T(0, t) \geq u_{\rho^*}(0, t) \sim C^* t^{-1/2}, \qquad C^* > 0, \quad n \geq 3. \quad (3.9)$$

We now assume that $0 \leq f(t) \leq C$ where $C > 0$. Then we define μ and K by

$$\mu = \alpha n K^{n-1}, \qquad K = (C/\alpha)^{1/n}. \quad (3.10)$$

Upon setting $\rho = \mu$ in (2.8), we obtain

$$\begin{aligned} T(0, t) &= u_\mu(0, t) + \alpha(n - 1)K^n \int_0^t G_\mu(0, t, s) \, ds \\ &\quad - \alpha \int_0^t [(n - 1)K^n - nK^{n-1}T(0, s) + T^n(0, s)] G_\mu(0, t, s) \, ds. \end{aligned} \quad (3.11)$$

In (3.11) we use the easily proved inequality $(n - 1)K^n - nK^{n-1}T + T^n \geq 0$ if $n \geq 1$, $T \geq 0$, $K \geq 0$. We also use the fact stated above that $G_\mu \geq 0$, and then (3.11) yields

$$T(0, t) \leq u_\mu(0, t) + \alpha(n-1)K^n \int_0^t G_\mu(0, t, s)\, ds$$

$$\leq [C + \alpha(n-1)K^n] \int_0^t G_\mu(0, t, s)\, ds, \qquad n \geq 1. \tag{3.12}$$

In Appendix C we show that the integral in (3.12) is bounded above by μ^{-1}, so (3.12) becomes

$$T(0, t) \leq K = (C/\alpha)^{1/n}, \qquad n \geq 1. \tag{3.13}$$

To obtain another lower bound we define γ by

$$\gamma = \alpha K^{n-1} = \alpha^{1/n} C^{1-1/n}. \tag{3.14}$$

Then we set $\rho = \gamma$ in (2.8) and then use (3.13) to obtain

$$T(0, t) = u_\gamma(0, t)$$

$$+ \alpha \int_0^t \{K^{n-1} - [T(0, s)]^{n-1}\} T(0, s) G_\gamma(0, t, s)\, ds \geq u_\gamma(0, t), \qquad n \geq 1. \tag{3.15}$$

The lower bound u_γ in (3.15) is given by (2.7). For any constant $\gamma > 0$, G_γ is given by

$$G_\gamma(0, t, s) = \pi^{-1}(t-s)^{-1/2} \int_0^\infty \frac{\xi^{1/2} e^{-\xi}}{\xi + \gamma^2(t-s)}\, d\xi, \qquad t > s, \quad \gamma \geq 0. \tag{3.16}$$

We now use (3.16) in (2.7) and evaluate u_γ for t large. Then (3.15) yields

$$T(0, t) \geq u_\gamma(0, t) \sim C_\gamma t^{-3/2}, \qquad C_\gamma > 0, \quad n \geq 1. \tag{3.17}$$

4. Behavior of $T(0, t)$ for $t \to \infty$. By integrating (1.1) with respect to x from 0 to ∞ and with respect to t from 0 to t and using (1.2)–(1.4), we obtain

$$\int_0^t [f(s) - \alpha T^n(0, s)]\, ds = \int_0^\infty T(x, t)\, dx. \tag{4.1}$$

The left side of (4.1) is $E(t)$, the net energy flow into the solid up to time t, while the right side is the energy in the solid at time t. We have shown above that if $f \geq 0$ then $T(x, t) \geq 0$, and thus the right side of (4.1) is nonnegative. Therefore (4.1) yields

$$E(t) \equiv \int_0^t [f(s) - \alpha T^n(0, s)]\, ds \geq 0 \quad \text{if} \quad f \geq 0. \tag{4.2}$$

From (4.2) we obtain

$$\int_0^\infty T^n(0, s)\, ds < \infty \quad \text{if} \quad \int_0^\infty f(s)\, ds < \infty. \tag{4.3}$$

We can now determine the behavior of $T(0, t)$ for $t \to \infty$ by utilizing (4.3) to evaluate the integral in (2.9) asymptotically. We see at once that

$$T(0, t) \sim \pi^{-1/2} \int_0^\infty [f(s) - \alpha T^n(0, s)]\, ds t^{-1/2} \sim \pi^{-1/2} E(\infty) t^{-1/2}. \tag{4.4}$$

Upon using (4.4) in (3.9) we obtain

$$E(\infty) \geq \pi^{1/2} C^* > 0, \qquad n \geq 3. \tag{4.5}$$

By using (4.4) in (4.3), we see that when $E(\infty) > 0$ the integral of T^n is finite only if $n > 2$. It follows that

$$E(\infty) = 0, \qquad n \leq 2. \tag{4.6}$$

Thus (4.4) shows only that $T(0, t) = o(t^{-1/2})$ for $n \leq 2$. On the other hand, (3.17) shows that $T(0, t)$ does not decrease faster than $t^{-3/2}$ for $n \geq 1$.

When $n = 1$ the explicit solution of (2.8) is

$$T(0, t) = u_\alpha(0, t) \sim C_\alpha t^{-3/2}, \qquad C_\alpha > 0, \quad n = 1, \quad \alpha > 0. \tag{4.7}$$

Thus for $n = 1$, $T(0, t)$ decays at the fastest rate permitted by (3.17). However if $\alpha = 0$, which we have hitherto excluded, then (2.9) shows that $T(0, t)$ is independent of n and is given by

$$T(0, t) = u_0(0, t) \sim C_0 t^{-1/2}, \qquad C_0 > 0, \quad \alpha = 0. \tag{4.8}$$

Comparison of (4.4) with (4.8) shows that for $n > 2$, $T(0, t)$ decays at the same slow rate $O(t^{-1/2})$ as if the boundary were not radiating. To understand this we write the radiation rate αT^n as $\alpha'(t)T$ with the effective radiation constant $\alpha'(t) = \alpha T^{n-1}$. Now for $n > 1$, $\alpha'(t)$ tends to zero as $t \to \infty$, so the boundary tends to behave as a nonradiating boundary ($\alpha = 0$) as $t \to \infty$. Evidently for $1 < n < 2$, $\alpha'(t)$ does not tend to zero fast enough to make $T(0, t)$ decay as slowly as $t^{-1/2}$, but for $n > 2$ it does.

5. Perturbation expansions. To find $T(0, t)$ for small values of α, we use (2.9) and solve it by iterations. For α small we can write the results as

$$\begin{aligned} T(0, t) = u_0(0, t) &- \alpha\pi^{-1/2} \int_0^t \frac{u_0^n(0, s)}{(t - s)^{1/2}}\, ds \\ &+ n\alpha^2\pi^{-1} \int_0^t \frac{u_0^{n-1}(0, s)}{(t - s)^{1/2}} \int_0^s \frac{u_0^n(0, r)}{(s - r)^{1/2}}\, dr\, ds + O(\alpha^3). \end{aligned} \tag{5.1}$$

For t small, we require $f(t)$ to be such that $u_0(0, t)$ has the expansion

$$u_0(0, t) = at^h + bt^q + 0(t^q), \qquad t \to 0, \quad q > h. \tag{5.2}$$

Then the iterative solution of (2.9) yields

$$\begin{aligned} T(0, t) = at^h + bt^q + 0(t^q) &- \alpha\pi^{-1/2}a^n\Gamma_{nh}t^{nh+1/2}[1 + 0(t^{q-h})] \\ &+ \alpha^2 n\pi^{-1}a^{2n-1}I_{nh}I_{2nh-h+1/2}t^{(2n-1)h+1}[1 + 0(t^{q-h})], \qquad t \to 0. \end{aligned} \tag{5.3}$$

Here we have introduced I_d, defined by

$$I_d = \int_0^1 \frac{s^d}{(1 - s)^{1/2}}\, ds. \tag{5.4}$$

To find $T(0, t)$ for α large, we first use the Abel inversion formula to solve (2.9) for T^n in the form

$$T^n(0, t) = \frac{f(t)}{\alpha} - \frac{1}{\alpha\pi^{1/2}} \frac{d}{dt} \int_0^t (t - s)^{1/2} T(0, s)\, ds. \tag{5.5}$$

Then we iterate (5.5) to obtain

$$T(0,\ t) = \alpha^{-1/n}[f(t)]^{1/n} - \alpha^{-2/n}n^{-1}\pi^{-1/2}[f(t)]^{1/n-1}\frac{d}{dt}\int_0^t [f(s)]^{1/n}(t-s)^{-1/2}\,ds + O(\alpha^{-3/n}), \qquad t > 0. \quad (5.6)$$

The result (5.6) cannot be valid at $t = 0$ because $f(0)$ may not be zero, whereas $T(0, 0)$ must be zero. It is not valid for t large if $f(t)$ decays too fast. Thus an initial layer expansion is required at and near $t = 0$, and another expansion may be needed for large t, but we shall not determine it.

Appendix A. Existence and uniqueness. To show that $u^e \equiv u^0$, we consider (3.2) with $\rho(t) = \alpha[u_{2j}(0, t)]^{n-1}$ and $\bar{\rho}(t) = \alpha[u_{2j-1}(0, t)]^{n-1}$. Then taking limits as $j \to \infty$ yields the equation

$$u^e(x,\ t) - u^0(x,\ t) = \int_0^t [u^e(0,\ s) - u^0(0,\ s)]\mathfrak{N}(x,\ t,\ s)\,ds, \qquad t \geq 0, \quad x \geq 0, \qquad (A.1)$$

where

$$\mathfrak{N}(x,\ t,\ s) = \frac{[u^e(0,\ s)]^{n-1} - [u^0(0,\ s)]^{n-1}}{u^e(0,\ s) - u^0(0,\ s)}\,u^e(0,\ s)G_\rho(x,\ t,\ s) \geq 0. \qquad (A.2)$$

By setting $x = 0$ in (A.1) we obtain

$$u^e(0,\ t) - u^0(0,\ t) = \int_0^t [u^e(0,\ s) - u^0(0,\ s)]\mathfrak{N}(0,\ t,\ s)\,ds, \qquad t \geq 0. \qquad (A.3)$$

This can be viewed as a homogeneous integral equation of the second kind for $u^e(0, t) - u^0(0, t)$ with $\mathfrak{N}(0, t, s)$ as the kernel. If we choose a t such that $|u^e(0, s) - u^0(0, s)| \leq |u^e(0, t) - u^0(0, t)|$ for $0 \leq s \leq t$, then (A.3) yields

$$|u^e(0,\ t) - u^0(0,\ t)| \leq |u^e(0,\ t) - u^0(0,\ t)| \int_0^t \mathfrak{N}(0,\ t,\ s)\,ds. \qquad (A.4)$$

For t sufficiently small, say $0 \leq t \leq \epsilon$, the integral in (A.4) is less than unity, which implies that $u^e(0, t) = u^0(0, t)$ for $t \leq \epsilon$. Using this fact in (A.3), we can show that $u^e(0, t) = u^0(0, t)$ in a larger interval. This procedure can be repeated to show that $u^e(0, t) = u^0(0, t)$ for all $t \geq 0$. Then (A.1) shows that $u^e(x, t) = u^0(x, t)$ for all $x \geq 0$, $t \geq 0$. Thus there is a common limit $u(x, t)$, so

$$u(x,\ t) = u^e(x,\ t) = u^0(x,\ t), \qquad x \geq 0, \quad t \geq 0. \qquad (A.5)$$

It follows from the definition (3.1) of u_j and from (3.2) that u_j and u_{j-1} satisfy

$$u_j(x,\ t) = u_\rho(x,\ t) + \int_0^t \{\rho(s) - \alpha[u_{j-1}(0,\ s)]^{n-1}\}u_j(0,\ s)G_\rho(x,\ t,\ s)\,ds. \qquad (A.6)$$

Then since $u_j \to u$ and $u_{j-1} \to u$, it is clear from (A.6) that u satisfies (2.1).

To show that the nonnegative solution constructed above is unique, we assume that there are two solutions T_1 and T_2. By subtracting (2.9) for T_2 from (2.9) for T_1 we obtain

$$T_1(0, t) - T_2(0, t) = \pi^{-1/2}\alpha \int_0^t [T_1(0, s) - T_2(0, s)]\left\{\frac{T_1^n(0, s) - T_2^n(0, s)}{T_1(0, s) - T_2(0, s)} (t - s)^{-1/2}\right\} ds. \quad \text{(A.7)}$$

Now by the same arguments used above to show that $u^\epsilon(0, t) = u^0(0, t)$, it follows that $T_1(0, t) = T_2(0, t)$. Then from (2.1) it follows that $T_1(x, t) = T_2(x, t)$.

Appendix B. Asymptotic behavior of $u_{\rho^*}(0, t)$. Tor establish the asymptotic property (3.7) for $u_{\rho^*}(0, t)$, we consider the initial boundary value problem (2.2)–(2.5) for u_{ρ^*} with $\rho(t) = \rho^*(t) = \alpha M t^{-1}$ and with $\delta(t - s)$ replaced by $f(t)$. Applying the Laplace transform to this problem yields

$$\hat{u}_{\rho^* xx}(x, p) + p\hat{u}_{\rho^*}(x, p) = 0, \qquad x > 0, \quad \text{(B.1)}$$

$$\hat{u}_{\rho^* x}(0, p) = \alpha M \int_0^\infty e^{-pt} t^{-1} u_{\rho^*}(0, t)\, dt - \hat{f}(p), \quad \text{(B.2)}$$

$$\hat{u}_{\rho^*}(x, p) \to 0, \qquad x \to \infty. \quad \text{(B.3)}$$

Here $\hat{u}_{\rho^*}(x, p)$ and $\hat{f}(p)$ are defined by

$$\hat{u}_{\rho^*}(x, p) = \int_0^\infty e^{-pt} u_{\rho^*}(x, t)\, dt, \qquad \hat{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt. \quad \text{(B.4)}$$

The solution of (B.1) satisfying (B.3) is

$$\hat{u}_{\rho^*}(x, p) = A(p) e^{-p^{1/2} x}. \quad \text{(B.5)}$$

Here $A(p) = \hat{u}_{\rho^*}(0, p)$ must be determined from the boundary condition (B.2). Upon substitution of (B.5) into (B.2) we obtain

$$-p^{1/2} A(p) = \alpha M \int_0^\infty e^{-pt} t^{-1} u_{\rho^*}(0, t)\, dt - \hat{f}(p). \quad \text{(B.6)}$$

Differentiation of (B.6) with respect to p yields

$$-\frac{d}{dp}[p^{1/2} A(p)] = -\alpha M A(p) - \frac{d}{dp}\hat{f}(p). \quad \text{(B.7)}$$

The solution of (B.7) which satisfies (B.6) is

$$A(p) = -p^{-1/2} \exp[2\alpha M p^{1/2}] \int_p^\infty \exp[-2\alpha M \xi^{1/2}] \hat{f}'(\xi)\, d\xi. \quad \text{(B.8)}$$

As $p \to 0$, (B.8) implies that

$$A(p) \sim p^{-1/2} \int_0^\infty \exp[-2\alpha M \xi^{1/2}] \int_0^\infty t f(t) e^{-\xi t}\, dt\, d\xi \quad \text{as} \quad p \to 0. \quad \text{(B.9)}$$

Then a classical asymptotic result on Laplace transforms shows that

$$u_{\rho^*}(0, t) \sim C t^{-1/2} \quad \text{as} \quad t \to \infty, \qquad C > 0. \quad \text{(B.10)}$$

Appendix C. Estimation of an integral. To estimate the integral in (3.12) we consider (2.2)–(2.5) with $\rho(t) = \mu =$ constant. Upon integrating the differential equation (2.2) we obtain

$$\int_{0-}^{t+} \int_0^\infty G_{\mu,t}(x, t, s)\, ds\, dx = \int_{0-}^{t+} \int_0^\infty G_{\mu,xx}(x, t, s)\, ds\, dx = -\int_{0-}^{t+} G_{\mu,x}(0, t, s)\, ds \qquad \text{(C.1)}$$

By virtue of the boundary condition (2.3) we then have

$$\int_{0-}^{t+} \int_0^\infty G_{\mu,t}(x, t, s)\, ds\, dx = 1 - \mu \int_0^t G_\mu(0, t, s)\, ds. \qquad \text{(C.2)}$$

Since $G_\mu(x, t, s)$ depends on the difference $t - s$, $G_{\mu,t} = -G_{\mu,s}$ and (C.2) becomes

$$0 \leq \int_0^\infty G_\mu(x, t, 0)\, dx = 1 - \mu \int_0^t G_\mu(0, t, s)\, ds, \qquad t > 0. \qquad \text{(C.3)}$$

This gives the desired inequality

$$\int_0^t G_\mu(0, t, s)\, ds \leq \mu^{-1}. \qquad \text{(C.4)}$$

References

[1] W. R. Mann and F. Wolf, *Heat transfer between solids and gases under nonlinear boundary conditions*, Quart. Appl. Math. 9, 163–184 (1951)

[2] J. H. Roberts and W. R. Mann, *A certain nonlinear integral equation of the Volterra type*, Pacific J. Math. 1, 431–445 (1951)

[3] K. Padmavally, *On a nonlinear integral equation*, J. Math. Mech. 7, 533–555 (1958)

[4] A. Friedman, *Generalized heat transfer between solids and gases under nonlinear boundary conditions*, J. Math. Mech. 8, 161–183 (1959)