

## REFLECTIONS OF A TEACHER OF APPLIED MATHEMATICS

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In my talk today I shall make some observations on the teaching of applied mathematics, both to non-mathematicians and to mathematicians, as they grew out of my experiences in teaching the subject, both in this country and in Switzerland. Hardly anything of what I shall say is especially new or startling, and I am sure that many experienced teachers will agree with most of my points. But to pass on even ordinary fruits of experience may be of some benefit to a younger generation.

Let me begin with some statistics. In the academic year 1970–71, the number of first year students at the ETH (Eidgenössische Technische Hochschule at Zürich) who took some kind of instruction in mathematics was 1308 [7]. The distribution of these students among the various fields was as follows:

Architects	197
Engineers (civil, mechanical, electrical)	521
Chemists, pharmacists, agricultural and forest engineers, biologists	424
Mathematicians and physicists	166

Of the 166 mathematicians and physicists, less than one half will ultimately study mathematics—the exact figure is not yet available, because the same mathematics courses are taken by mathematicians and physicists in the first year. I shall have to say something more about the proportion of mathematicians and physicists, and also about the different specializations chosen by mathematics students, but for the moment I simply wish to stress that only about 6% of all students who take mathematics in their first year will ultimately become mathematicians. (At my school the figures are of course slanted towards engineering, but I would be hard pressed to name any university where more than one-third of the students who take mathematics are future mathematicians.)

At the ETH, the remaining 94% do not take mathematics for pleasure. They are not interested (except as a private pastime) in mathematics as such. They study mathematics because mathematics is a tool—*one of many tools*—for the understanding of their own chosen subject. In one word, they study applied mathematics. As one who has taught both the 6% and the 94%, I now wish to call attention to three simple facts which seem obvious but are sometimes overlooked:

1. *The 94% will later provide an important link from mathematics to the outside world.* If in 25 years from now a mathematically educated person sits in a committee making decisions about or disbursing money for mathematicians, he will in all likelihood be one of the 94%, judging by numbers alone and even disregarding political ability. His will be the image which is shared by the public-at-large about mathematics.

2. *The 94% are not stupid.* They fail to study mathematics not because they hate it, but because they like something else better. I knew some mathematically highly-gifted civil engineering students who studied engineering because there was more money in it. At the ETH we have some mathematically very knowledgeable professors among the members of our chemistry faculty. (One of those men also knows a considerable amount of electronics and of quantum mechanics.) For the best classical analysis which I was taught as a student I am indebted to Wolfgang Pauli, a theoretical physicist.

3. It is commonly known that the modern engineer, chemist, and scientist in general require more mathematics than ever. All engineers and chemists want vector analysis, differential equations and linear algebra. The electrical engineers want, in addition, complex analysis, including Laplace transforms; even the architects want some linear programming as do, of course, many specialized engineers. Yet *the 94% do not have much time for mathematics.* There are many other things closer to their own chosen subject which they must learn.

Let me now report briefly on the conclusions which we have drawn at the ETH from the above in a reform of the mathematics curriculum for the 94% in 1967. Much of the credit for this reform is due to my friend and colleague Alfred Huber, an eminent analyst and one of the best teachers of mathematics at the ETH since Georg Polya.

1. The first conclusion is that the teaching of future non-mathematicians is probably the most important task of the professional mathematician, important both from a broad sociological point of view and from the "trade union" point of view. It is more important than much mathematical research, and certainly more important than the teaching of future research mathematicians. (In the typical small European pre-war university, there were very few courses for future mathematicians. These people largely taught themselves. If they fell by the wayside, this was considered a useful natural attrition, since the market for research mathematicians was very limited.) Judging from contacts which I have had with mathematicians in this country, one could almost form the impression that the teaching of future mathematicians was considered the only relevant teaching activity of a mathematician, and even this mostly at the research level.

2. The typical member of the 94% is exposed to many teachers who present subjects that, as a rule, are closer to his heart than mathematics. To compete successfully for his attention it is necessary to make mathematics relevant to him. At the ETH we do this by illustrating the theoretical material profusely by examples chosen from the natural intellectual habitat of the various students. Naturally, if an instructor presents examples, he must understand them. It so happens that Alfred Huber has a master's degree in civil engineering, and I have one in electrical engineering. It is one of the sad ironies of the present mathematical job crisis in this country that the typical Ph.D. product is well equipped, at least in theory, to become a professor of mathematics at Princeton University, but that he is unable to teach a good course in calculus for engineers because he has never been exposed to the rudiments of mechanics or circuit theory. At the ETH every student of mathematics has to take three semesters each of experimental and of theoretical physics.

To know applications is no sufficient qualification to be a successful mathematics teacher for scientists. To be a good teacher of anything requires an inner commitment to the subject which is sufficiently deep so as to be felt by the student. To walk into the classroom prepared only to recite from some well-worn textbook ("as if I was going out for lunch", as I have heard one American colleague describe it) is not enough. A good

way to acquire a commitment to the subject of calculus, for instance, is to invent new problems, and new and ever more simple ways to present the subject matter. One comment which one often hears is that the mathematics taught to the 94% is “trivial”. It may be so, but let nobody think that it is not important. The vast majority of the applications of mathematics in bread-and-butter engineering are simple applications of calculus, mainly differential equations. The circuitry of a TV set, or the mathematics involved in the trajectories of moon shots are of this kind.

3. How do we solve the dilemma of teaching to the 94% the vast amount of subject matter in a very limited amount of time? To teach all these things to the professional student of mathematics takes some two years of essentially full-time study of the subject. By that time he will have memorized the proofs, say, of a general form of Stokes’ theorem, or of an existence theorem for ordinary differential equations, still without knowing much of the physics behind it. At the ETH we use mainly two devices to teach, e.g., the engineer all the required mathematics within the first three semesters in competition with many other subjects:

(a) Intensive, supervised problem-solving sessions (“Laboratory”, “Uebungen”). In groups of at most 20, each supervised by a teaching assistant, the students spend 5 hours per week solving carefully selected problems. In course evaluation questionnaires a majority of the students stated that the learning process took place mainly in these problem-solving sessions. Here they also have an opportunity to ask questions; most of them are too shy to do so in the big lectures which frequently comprise 300 students. “Carefully selected problems” does not simply mean problems picked from a book. Those will never quite fit the special deficiency of which the treatment is required most at the moment, and they have been solved many times before. We are not fond of drill problems. The best problems are those that involve applications, and/or aid the intuitive understanding of the subject. Incidentally, the text of a problem is one of the few mathematics texts which our students actually read (in Switzerland the students are less well trained to read than they are here); thus it makes sense to devote great care to the formulation of the problem.

(b) The second device to gain time is a ruthless elimination of all formal rigor. We almost never present formal proofs, but instead rely on that invaluable asset which most of our students still bring along from high school and which our students of mathematics are trained to suppress: geometric intuition. It is amazing how many things in analysis become obvious and natural if one just draws the appropriate figure. Our definition of a limit is mathematically correct, but we never state it in delta-epsilon terms. Indeed, these letters do not occur in our course. Instead, close to one-half of the pages of our lecture notes [3, 4] (which are for sale to the students at no profit to ourselves) are covered with pictures. Our 94% are not taught how to prove things, because that is not what they need. Not only do we not prove anything that is intuitively obvious, but we also avoid mentioning things that could damage the student’s carefully nurtured faith in intuition. Continuous functions that are nowhere differentiable—their existence is certainly interesting, but from the point of view of applications this is one of those totally irrelevant results by which mathematics loses credibility. One of the difficult things a teacher must learn is not to tell everything he knows.

I suppose a true mathematician will now object: can such a course still be considered mathematics? If there is no rigor, how can there be any intellectual tension and excitement? Is such a course not simply a collection of cookbook recipes?

In replying, let me begin by asking whether the often-praised concept of mathematical rigor is as absolute as some of us seem to believe. The rigor that prevails in our analysis course for engineers would still be forbidding enough to send a chill down the spine, say, of a student of law. On the other hand, the rigor used in the introduction of complex numbers in a generally accepted mathematical text such as that by Ahlfors [1] leaves much to be desired from the point of view of the algebraist. And some of the things done in algebra do not hold water before the critical eye of a specialist in foundations. Anybody who views the history of mathematics and is not so immodest as to believe that today of all times is the finest hour of mathematics will have to admit that the standards of rigor are subject to strong variations in time and space. Perhaps it is true, to a higher degree than we like to admit, that "rigor" really means a concentration on issues that happen to be fashionable at the moment. If mathematics were done on some remote planets, and if such a mathematician visited an American symposium on analysis, he would perhaps be appalled by the vagueness with which the concept of existence is handled, and how little attention is paid to questions of constructivity.

In a similar spirit, I believe that stressing logical foundations in a mathematical service course would focus the attention at the wrong place. It goes without saying that also the consumer of mathematics should be taught to think logically. However, the foundations of analysis with their many quantifiers and logically complicated statements (think of the definition of non-uniform convergence!) are not a good place for first logical exercises. (There is a consensus at the ETH that even in the teaching of pure mathematicians linear algebra is a better vehicle for teaching rigor than analysis.) There are many interesting arguments in analysis that have a simpler logical structure and can be carried through completely. It suffices to recall the simple fact of vector analysis that a vector field has a potential if and only if the work along every closed path is zero. Often the idea of a proof emerges all the more clearly if the fine technical points are neglected. As an example, consider the divergence theorem. We start with the concept of flux, defined as the amount of fluid which passes through the surface per unit time. Then we prove the theorem for a cube. Then we approximate the given spatial domain by cubes. It is true that the surface is approximated only with regard to position; the normal direction is not approximated, but it is "intuitively clear" that the flux is not sensitive with regard to corrugations of the surface.

To say that our course is not rigorous is not to say that it is not modern in spirit. For example, we define all functions as mappings and write

$$f: x \rightarrow \sin x,$$

as in Bourbaki. The mapping concept is close to intuition, and therefore desirable. The above notation is also very useful when dealing with functions of several variables and their restrictions.

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In the second part of my talk I shall say something about teaching applied mathematics to mathematicians. Let me first come back to the 6% of our first-semester students who plan to become professional mathematicians. What will they do? It is impossible to tell, but I can show you some statistics which indicate what the students who just received their diplomas will do. In the academic year 1970-71, 113 students at the department of mathematics and physics of the ETH passed their final oral diploma

examinations. Among these were

Experimental physicists	60
Theoretical physicists	12
Mathematicians	41

Every mathematician has to pass examinations in analysis, algebra and geometry, and theoretical physics. In addition, he has to choose three electives, and the choice of these gives some indication as to what he plans to do and/or is interested in. In the year mentioned, the choice of electives broke down as follows [2]:

Elementary mathematics (required for gymnasium, i.e. high-school, teaching)	18
Applied mathematics (including one or several of numerical analysis, operations research, methods of mathematical physics, computer science)	25
Statistics or actuarial mathematics	9
No applied elective	5

(The total adds up to more than 41 because of overlaps.) The interesting fact is that only 5 out of 41 mathematical graduates show no interest whatsoever in applications; and this in spite of the fact that even after the era of Plancherel, Weyl, and Heinz Hopf our core mathematics faculty consists of men with international reputations. (Happily, our faculty is very open-minded and does not make it difficult for the student to choose applied electives by the gerrymandering of regulations.) The conclusion may be drawn that most of our students are interested in applications, even if this involves some loss of sheer mathematico-intellectual pleasure.

There is a cry for relevance echoing through our college halls. We can answer it in mathematics by getting our fingers dirty. Statisticians can talk about the statistics of poverty and of electrocardiograms, numerical analysts about the mathematics of reactors and of moon shots, operations research people about economics, computer scientists about linguistics. This is all obvious. But what can the pure mathematician do? His position is not easy. He has trained himself to observe standards of understanding that are exceedingly high. He will profess not to understand a subject until he has mastered it to such an extent that he can present it axiomatically. Unfortunately, few applied subjects are understood that well, or if they are, they cease to be interesting. Real life matter is often fragmentary, there are obscure odds and ends, and the understanding is partly intuitive rather than completely logical. Thus the pure mathematician often is temperamentally unfit to present applications. But there is one thing which he can do to increase the relevance of his subject: He can apply mathematics to mathematics.

Let me explain what I mean by this. In his remarkable article, "On the Ph.D. in mathematics" [5], Professor Herstein points out, among many other things, that there is a lack of graduate courses that give some broad view of the subject. According to Herstein, there is a tendency for the student to regard each course as an entity unto itself, with little interplay between the various courses. As an example he mentions the student who had memorized the spectral theorem for normal operators on a Hilbert space but did not know that a Hermitian matrix could be diagonalized. I, too, have made studies of the graduate offerings in various large mathematics departments and in at least one case have found that except for the basic courses the professors mostly taught highly specialized seminars, mainly about their own last paper. This, in my view, is a serious violation of the spirit of mathematics which to a significant extent owes its great fascination to its unifying power. One of my great experiences as a student of

electrical engineering was the discovery that the distribution of electrostatic potential, the flow of an ideal fluid, and the stationary distribution of temperature all were governed by one and the same differential equation. (But, of course, the student who never had physics will not be able to share this experience. To him, the theory of Laplace's equation is just another subject to be learned.)

Let me show by some examples what I mean by applying mathematics to mathematics. I have had occasion to teach such applications in courses on applied complex analysis which I have offered repeatedly in Zürich. (These courses are intended primarily for students of mathematics.) The following examples are taken from these courses.

1. Every student of complex analysis knows Rouché's theorem and the Lemma of Schwarz, and those who take numerical analysis know that fixed points (i.e. solutions of equations) can be obtained by iteration. But few students realize—or are made to realize—that for analytic functions the existence of a unique fixed point follows under very natural conditions from Rouché's theorem, and that under the same conditions the convergence of the iteration sequence, including error estimates, follows from the lemma of Schwarz. The resulting theory of fixed points is both simpler and more general than the real theory usually taught in numerical analysis, and it is equally applicable, since the functions whose fixed points are sought are usually analytic.

2. In complex analysis the decomposition of a rational function in partial fractions is a very simple corollary of the theory of isolated singularities and of Liouville's theorem, and there the matter seems to rest for most teachers of complex analysis. After all, the subject ceased to be of research interest some two hundred years ago. Admittedly, partial fractions are a simple matter, but this does not imply that they cannot be a source of excitement and wonder. They enable us to find formulas for the coefficients in the Taylor series representing a rational function, and from these formulas we can extract all kinds of information about their asymptotic behavior—which is exciting, for instance, if these coefficients have some combinatorial interpretation. We can also develop Bernoulli's method for finding zeros of polynomials and its modern extensions such as the QD algorithm. Similar remarks hold for infinite products. The student usually experiences these merely as one station on the way to the general theorem of Weierstrass on the existence of entire functions with prescribed zeros—one of those general theorems which, unless one is given the proper background, is neither expected nor unexpected. He is not aware that infinite products often have fascinating combinatorial interpretations, which even occupied the minds of an Euler, a Jacobi, a Ramanujan.

3. The calculus of residues is duly and regularly applied to the evaluation of integrals, hereby providing one of the relative maxima of interest in the complex analysis course. Again, that seems to be the end of residues. I have had great fun with my students summing systematically all kinds of series using residues, beginning with that old stand-by, the sum of the squares of the reciprocal integers. (I learned this from Hille's *Analytic function theory* [6].) The special series mentioned can also be summed in many other ways; I think it is instructive for the student to see the same problem attacked by several methods, and vice versa.

4. I have briefly mentioned potential problems. It may be stretching the concept of applying mathematics to mathematics, but when talking about conformal mapping it almost seems a crime not to mention the solution of plane potential problems by conformal transplantation. And where is the student who would remain blasé if shown that the lift exerted on an airfoil can be calculated by evaluating a certain residue at infinity?

5. Linear (or bilinear, or Moebius) transformations are discussed in all complex analysis courses, and some point out the connection between the composition of such transformations and the multiplication of  $2 \times 2$  matrices. The correlation with other mathematical subjects can be increased if we point out that unitary  $2 \times 2$  matrices correspond to rigid motions of the Riemann number sphere and thereby discover the representation of the group of rotations in 3-space by unitary matrices. This was the mathematics used by Wolfgang Pauli in his theory of spin.

6. It is generally known that some important branches of topology, notably homology theory and homotopy theory, have their historical roots in certain questions of analytic function theory. But are the students of topology made aware of this fact? I have recently seen a text on homotopy theory where that theory was not related to anything. I have pity for the student who has to study it, and somehow also for the single-purpose mind who designed it.

Another topological question arises in complex analysis in connection with the Jordan curve theorem. The use of this theorem greatly facilitates the statement and enhances the intuitive content of a significant part of complex integration theory. Yet many writers of complex analysis texts seem to have a curious hang-up about this theorem. They either do not prove it and use it anyway, which for them, being rigorous mathematicians, is very hard. Or they claim that the theorem "is never used in complex analysis" and that a proof "would lead too far astray from the main issues of the book". In an almost cynical way, they then proceed to define their way out of the difficulty. Now while the theorem is hard to prove in its most general form, the proof is not at all that difficult for piecewise regular Jordan curves, especially if the theory of the index of a point with respect to a curve is already available. I feel that some version of the theorem should be proved. It is wonderful to use a concept with so much intuitive appeal as the interior of a Jordan curve, and yet to have a completely clear mathematical conscience.

Some complex analysis texts—not very many—mention the Lagrange–Bürmann formula for the power series of the inverse function. It is seldom taught that this formula has a purely algebraic content, and that it can be proved algebraically, using concepts from the theory of formal power series and again a certain matrix isomorphism. In doing so we again relate complex analysis to algebra. The Lagrange series can also be related to numerical analysis, in several ways. It provides a simple method for constructing Schroeder's iteration functions of arbitrary order. In the special case where we apply it to the solution of polynomial equations, we can obtain Cardano's formulas for the general solution of the cubic equation as a corollary of the Lagrange series.

Several of my examples bear on computation. Most mathematicians have begun to realize that computation, indeed, can be a fascinating source of mathematical problems. Those who attended the International Congress of Mathematicians at Nice could sample the flavor of the modern theory of computation in Knuth's lecture. Knuth and the workers in his field are concerned with the difficult problems of *optimal* computation in simple arithmetical problems such as the multiplication of two  $n$ -digit numbers. But even disregarding problems of optimality, there are interesting problems. Some of these relate to complex analysis. I have mentioned partial fraction decompositions. Everybody knows that they exist, and that they can be found by some comparing of coefficients. But is this really the best way? It is not; there are other, shorter algorithms, better suited to the needs of the engineer. But where are the authors of calculus texts

who devote some attention to this problem, important not, as they think, for the evaluation of integrals, but in circuit theory? Here is a place where they can show ingenuity in a way which really impresses the customer.

On a yet simpler plane there is the problem of determining zeros of polynomials. Everybody knows the fundamental theorem of algebra. It is usually proved in function theory as one of the by-products of Liouville's theorem. That proof does not very much impress the student, because it offers no method for actually calculating the zeros. How many authors of differential equation books who glibly write "Let  $z_1, \dots, z_n$  be the zeros of  $p$ " have ever stopped to think how to compute the zeros? This problem, if mentioned at all, is quickly referred to "the machine people" which populate, it seems, a different world. Yet the very problem of determining zeros of polynomials offers fine opportunities for applying elementary but important theorems of complex analysis. I have already mentioned the lemma of Schwarz. We can also use the principle of the maximum, Moebius transformations, the argument principle, all of these in a number of ways. Naturally I am not thinking here of the old "theory of equations" material, where most polynomials seemed to have real roots only and the main point, anyway, seemed to *avoid* computing zeros. I am thinking of algorithms for obtaining all zeros of a complex polynomial to arbitrary accuracy. These things are elementary enough to be taught, in undergraduate courses even, yet some of them are very close to modern research in numerical analysis. I have pointed out elsewhere that attractive feature of numerical analysis (and of computer science in general): the distance from the subjects which we teach in the classroom to the frontiers of knowledge can be very short, especially in comparison with traditional mathematics.

You will have noticed that the mathematics which is used in most of the above applications is rather elementary. Naturally there are many opportunities to apply mathematics to mathematics on a more advanced level. In complex analysis, we may think of ordinary and partial differential equations, or of analytic number theory. But to deepen the understanding even of elementary mathematics by applying it should not be considered a waste of time. I think we generally make the mistake of trying to teach too much. What use is it to the student if he is exposed to a very general form of Cauchy's theorem as long as he does not see how to apply even a simple version of it in a specific situation?

Could it be that in mathematics, too, we need a new Consciousness? A Consciousness concentrating less narrowly on research progress in little isolated areas, on status symbols such as the publication of articles in reputable periodicals, on jealously guarded areas of teaching responsibility where no other man may enter; a new Consciousness stressing instead the exchange, communication, and experience of mathematical information, a Consciousness where mathematics is told in human words rather than in a maze of symbols, intelligible only to the initiated; a Consciousness where mathematics is experienced as an enlightening intellectual activity rather than as an almost fully automated logical robot, ardently performing simultaneously a large number of seemingly unrelated tasks?

Let us take to heart Charles Reich's message [8] not only in our daily lives, but also in our professional activities, to make sure that mathematics will not remain an isolated gray spot on an otherwise greening map of science.



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