QUARTERLY OF APPLIED MATHEMATICS

Vol. XXX APRIL 1972 No. 1

SPECIAL ISSUE: SYMPOSIUM ON
"THE FUTURE OF APPLIED MATHEMATICS"

INTRODUCTORY REMARKS

BY

WILLIAM PRAGER

Brown University

I realize that this symposium is concerned with the future of applied mathematics, but I believe that it will be useful briefly to recall from where we have come before we start looking ahead.

In the early thirties, American applied mathematics could, without much exaggeration, be described as that part of mathematics whose active development was in the hands of physicists and engineers rather than professional mathematicians. This is not to imply that there were no professional mathematicians genuinely interested in the applications, but their number was extremely small. Moreover, with a few notable exceptions, they were not held in high professional esteem by their colleagues in pure mathematics, because there was a widespread belief that you turned to applied mathematics if you found the going too hard in pure mathematics. As a distinguished evaluation committee appointed by President Wriston in 1941 put it: "in our enthusiasm for pure mathematics, we have foolishly assumed that applied mathematics is something less attractive and less worthy".

To keep informed about developments in applied mathematics, one had to attend meetings not only of the mathematical societies but also of several scientific and engineering societies, each meeting providing only a few items of interest.

It was not at all easy to get a paper in applied mathematics published in a journal where it would come to the attention of readers really concerned with the subject. Editors of mathematical journals usually insisted that the discussion of the technical background of the problem be drastically shortened, because only very few readers would care about this. On the other hand, editors of engineering journals demanded that the mathematical discussion be shortened or relegated to an appendix, because most readers would not be interested in this.

A personal experience along this line may amuse you. Shortly after coming to this country, I submitted a paper in continuum mechanics to an engineering journal. It was promptly returned by the editor with the remark that it could only be considered if I changed Cartesian tensor notation to conventional component notation, that is, denoted the components of the displacement vector by u, v, w rather than u_i, and so on. The reason given for this request was that only a few readers would be able to follow the mathenatical argument if the conciser notation were used. When I mentioned this incident to Professor Courant, he replied that his associates had experienced similar difficulties, and that something ought to be done to remedy the situation. This conversation started me thinking about the need for a journal devoted to applied mathematics and thus led to the foundation of the Quarterly of Applied Mathematics.

You will agree that the environment of applied mathematicians has greatly improved since these times. Brown University's Division of Applied Mathematics and its forerunner, the Program of Advanced Instruction and Research in Applied Mechanics, contributed substantially to this improvement, as did similar organizations at a few other universities, for instance, the Courant Institute of the Mathematical Sciences at New York University. Brown's contribution owes much to the foresight of two men to whom we should pay tribute today.

Dean Roland G. D. Richardson recognized early that, as a result of this country's unavoidable entry into World War II, there would be a sharply increased demand for applied mathematicians, and that an emergency training program in applied mathematics was therefore bound to find financial support. It was thanks to his initiative and drive that the Program of Advanced Instruction and Research in Applied Mechanics could be started in the summer of 1941.

At the end of the war, the funds that had so far supported this Program disappeared almost overnight, and President Henry M. Wriston had to decide whether to wind up the venture or continue it in some form with little prospect of outside support. It is thanks to his courage in committing University funds that Brown has a Division of Applied Mathematics today.

The growth of the Division, however, would have been considerably less vigorous without the support of much of its research by the Office of Naval Information and its successor, the Office of Naval Research, in the critical period before the creation of similar research offices by the other services and the establishment of the National Science Foundation. It was fortunate, indeed, that at that time, ONR was still free to construe its mission in very broad terms.

Beyond acknowledging these signal contributions, I shall not dwell on the history of applied mathematics at Brown, because you will have read about this in the brochure you received on registering for the symposium. When I accepted the assignment to talk at this opening session, I realized that the audience would not exclusively consist of applied mathematicians and that some general remarks on applied mathematics and applied mathematicians would therefore be more appropriate than a more technical talk.

Precisely to define applied mathematics is next to impossible. It cannot be done in terms of subject matter: the borderline between theory and application is highly subjective and shifts with time. Nor can it be done in terms of motivation: to study a mathematical problem *for its own sake* is surely not the exclusive privilege of pure mathematicians. Perhaps the best I can do within the framework of this talk is to describe applied mathematics as the bridge connecting pure mathematics with science and technology. I have deliberately described this bridge as *connecting* two areas of activity rather than *leading* from one to the other, because the bridge carries two-way traffic. Its importance to science and technology is obvious, but it is not less important to pure mathematics, which would be poorer without the stimuli coming from the applications.

With the pure mathematician, the applied mathematician shares the interest in developing new mathematics, and with the scientist or engineer, the interest in applying mathematics to the improvement of our understanding and control of natural or manmade environments. As an intermediary between these groups, the applied mathematician should appreciate, though not necessarily emulate, the pure mathematician's insistence on rigor as well as the willingness of scientists and engineers to accept heuristic reasoning. He must be able to construct, not only a rigorous proof of a mathematical proposition, but also a workable mathematical model of the phenomenon he plans to investigate.

Very often, the applied mathematician's skill in the construction of suitable models will contribute as much to the success of an investigation as his knowledge of the analytical or numerical techniques that will be needed to treat the mathematical relations governing the behavior of the adopted model. His familiarity with these techniques enables him to foresee difficulties that may result from the inclusion of certain effects in the model. He will then question his clients about the need for including these effects, warn of the mathematical consequences of this inclusion, and stress the fact that a coarse model that is readily manipulated mathematically may yield a better insight into a natural phenomenon or technical process than a more refined but mathematically unwieldy model.

Modern automatic computers have to some extent freed the applied mathematician from the restriction to highly idealized models. This has not always been a blessing, because the unrestrained inclusion of a medley of conflicting effects may lead to a spurious agreement between computation and experiment that is mistakenly interpreted as a confirmation of the model until the application of the model to slightly different situations reveals its inadequacy. In teaching, however, the removal of the restriction to the simplest models is highly beneficial, because it gives students the opportunity of developing skill in *model building* where, in the past, they had to be content to acquire skill in the *manipulation* of a few standard models.

This is but one instance of the impact of automatic computers on education in applied mathematics. The increasing availability of interactive computing and computer graphics to students in applied mathematics will profoundly change the traditional pattern of teaching this subject. I believe that the general trend of these changes will be to emphasize the *creative* and de-emphasize the *manipulative* aspects, and that this will make the study of applied mathematics more exciting.

Since these remarks may strike you as rather cryptic, let me illustrate them by an example. Numerous management problems, for instance the scheduling of production, storage, and distribution of a product, as well as numerous engineering problems, for instance plastic analysis and design of structures, may be formulated as problems in linear optimization. In these problems, a linear form of a finite number of nonnegative variables is to be maximized or minimized subject to some linear inequality constraints. Several methods are available for the solution of problems of this kind. Conceptually the simplest of them is the *simplex method* which, however, is not the most efficient one. Whereas the *basic ideas* of the simplex method are readily explained in less than an hour, considerable time will be required for the student to master the use of this method and familiarize himself with the theory and application of its more efficient variants. If we insist on this, not enough time will be available for the discussion of a truly representative array of problems from economics, mechanics, and other fields in which linear optimization is used. It seems to me that, in an introductory course at least, it would be sufficient to explain the basic ideas of the simplex method and show the students how the data of a linear optimization problem must be presented to the local computer and how the computer output is to be interpreted. The algorithmic aspects of linear optimization may be discussed much later as part of a numerical analysis sequence. I am convinced that this kind of approach will relieve the frustration many freshmen now experience because much of their academic work appears to be only a preparation for later work and not immediately useful.

The time and dollar costs of performing a given computational task continue to decrease rapidly. This often means that the use of a comparatively inefficient method that has already been programmed for the computer may be preferable to the use of a more efficient method for which a program has still to be written. The fear has been expressed that, in the long run, this situation is bound to devalue the services of applied mathematicians. This bleak forecast, however, would only be justified if these services consisted primarily in the selection or construction of efficient algorithms. While the algorithmic skills of the applied mathematician may indeed become less important, other skills will become ever more important. Foremost among these are, in my opinion, the abilities to construct realistic and workable mathematical models, to recognize the identical mathematical character of problems that are presented in widely different physical terms, and to synthesize general laws from the output of computer runs. Of these three important activities, I have already briefly mentioned model building; the other two, I should like to illustrate by an example. In view of my limited time, this example will have to be rather simple.

Let us imagine an applied mathematician tackling the problem of the brachystochrone at the beginning of the eighteenth century; that is, well in advance of the publication of Euler's "Methodus inveniendi lineas curvas maxima mimimave proprietates gaudentes", the first exposition of the calculus of variations. The brachystochrone indicates the shape of a wire connecting given points A and B in such a manner that a heavy bead released without initial velocity at the higher point A and sliding without friction along the wire will reach B in the shortest possible time (Fig. 1).

It is readily seen that the brachystochrone must lie in the vertical plane containing the segment AB. Furthermore, in view of the absence of friction, there is no dissipation of energy, and the kinetic energy gained during the descent of the bead must equal the potential energy lost. Accordingly, the speed v of the bead at an arbitrary point of its

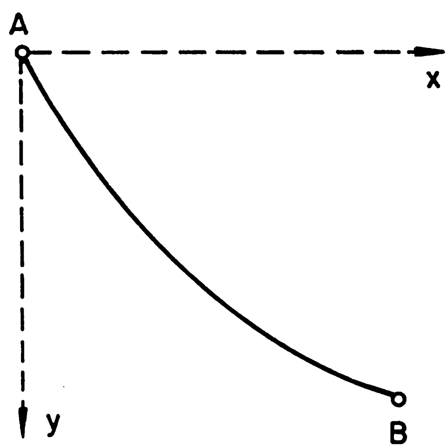

FIG. 1. Brachystochrone problem.

trajectory and the difference y between its initial and instantaneous levels are connected by the relation

$$v = (2gy)^{1/2}, \tag{1}$$

in which g denotes the acceleration of free fall.

Regardless of the shape of the wire, the speed thus depends only on the level the bead has reached. This suggests that we *discretize* the problem by regarding the levels of A and B as the median levels of the first and last of a number of horizontal layers of equal thickness and stipulating that, throughout any one of these layers, the bead must move with the speed corresponding to the median level of this layer. Considering two adjacent layers (Fig. 2), and assuming that the entrance P into the upper layer and the exit R from the lower layer are given, we must then locate the point Q at the interface of the two layers to minimize the total travel time along the segments PQ and QR, when the speeds v_1 and v_2 in the two layers have given values. At this stage, we recognize that, from the mathematical point of view, our problem is identical with that of determining the path of a light ray through layers in which the speed of light has given values v_1 and v_2. If the angles which PQ and QR form with the vertical, are denoted by θ_1 and θ_2, Snellius' law tells us that

$$v_1/\sin\theta_1 = v_2/\sin\theta_2\,. \tag{2}$$

The use of the mathematical analogy with an already solved optical problem considerably simplifies the treatment of our mechanical problem.

If you should be inclined to accuse my imaginary eighteenth-century mathematician of adopting a twentieth-century *finite-element* approach, let me assure you that the foregoing argument is due to Jean Bernoulli. Choosing A as the origin of rectangular coordinates x, y with the y axis vertical and directed downwards, he wrote (2) in the form

$$v/\sin\theta = \text{const}, \tag{3}$$

and used (1) and the relation $\sin\theta = (1 + y'^2)^{-1/2}$, where $y' = dy/dx$, to obtain the

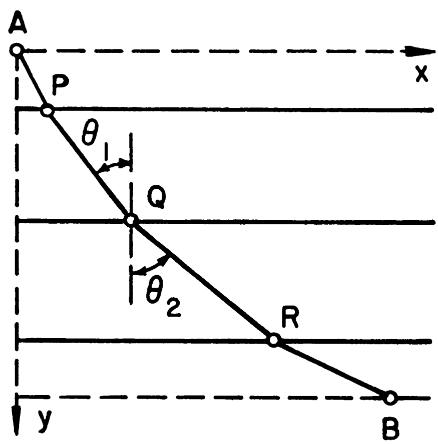

Fig. 2. Discretized problem.

differential equation of the brachystochrone as

$$y(1 + y'^2) = \text{const.} \tag{4}$$

It was, of course, fortunate that he could integrate this differential equation in closed form. In general, the more realistic we attempt to make our models, the greater will be their complexity, and the smaller the chance that the resulting differential equations can be integrated in closed form. It may then not even be worthwhile to establish these differential equations.

In the case of the brachystochrone, the finite-element approach may be pursued as follows. It is not hard to show that, to minimize travel time, the bead should start by gaining speed as quickly as is possible. This means that the tangent of the brachystochrone at A should be vertical. In addition to $v = 0$, we thus have $\theta = 0$ at A, but this, unfortunately, does not enable us to determine the value of the constant in (2). On the other hand, while the speed at B is determined by the level of this point, we do not know the angle θ there. To overcome this difficulty, let us continue the brachystochrone beyond B until its tangent becomes horizontal at some point C, the ordinate of which will be denoted by Y (Fig. 3). Since $\sin \theta = 1$ at C, it follows from (1) and (2) that

$$\sin \theta = (y/Y)^{1/2}. \tag{5}$$

This equation shows that the brachystochrone AC corresponding to an arbitrary value of the variable Y is obtained from the brachystochrone $A'C'$, for which this variable equals unity, by a similarity transformation with the center A and the ratio Y (Fig. 3). It will therefore suffice to determine the brachystochrone for $Y = 1$.

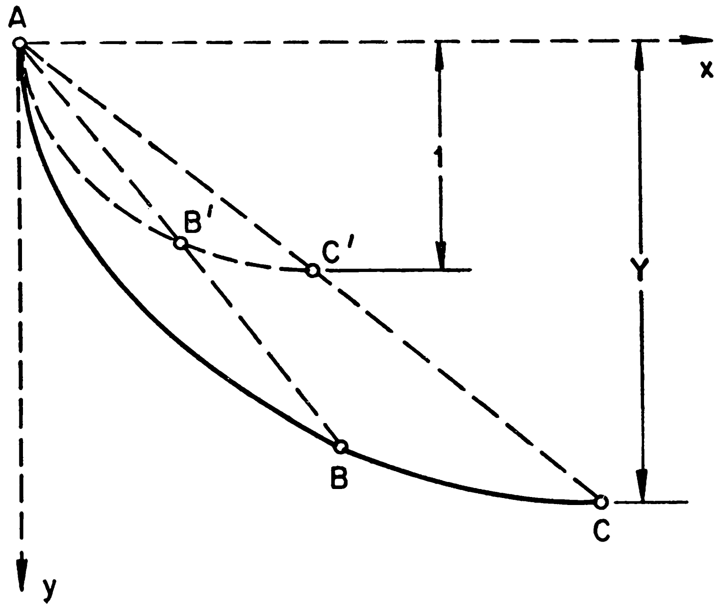

FIG. 3. Extended problem.

To this end, we discretize as above, using $n - 1$ layers of the thickness $1/n$ and half layers at top and bottom of thickness $1/(2n)$ each, and writing Eq. (5) in the form

$$\sin \theta_k = (k/n)^{1/2}, \qquad (k = 0, 1, \cdots, n) \tag{6}$$

where θ_k is the value of θ for $y = kY/n$. Since (6) furnishes $\theta_0 = 0$, we begin with a vertical segment from A to the level $y = 1/(2n)$ at the bottom of the first layer. From the endpoint of this segment, we draw a segment under the angle arcsin $(1/n)^{1/2}$ to the bottom of the second layer at $y = 3/(2n)$. Continuing in this manner, we face a difficulty when we reach the level $y = (n - 1)/(2n)$, because we have $\theta_n = \pi/2$ for the remaining half-layer. If we attempted to draw the next segment under this angle, we would never reach the final level $y = 1$. To bypass this difficulty, we draw our polygon from A to the point P at the median level $y = (n - 1)/n$ of the next-to-last layer (Fig. 4) and continue from P with a circular arc that is tangent to the last side of the polygon at P and also tangent to the line $y = 1$ at some point Q. As n tends to infinity, this circular arc should tend towards an arc of the circle of curvature of the brachystochrone at the limiting position of Q.

Computing the abscissas of the vertices of the polygon and the abscissa of Q, and plotting the result, we obtain an approximation to the brachystochrone which will improve in quality as n increases. Even for $n = 10$ (Fig. 4), this approximation should be adequate for most practical purposes. If, however, our solution is to serve as the starting point for an investigation of the influence of secondary effects, such as air resistance, we may wish to characterize the curve more precisely than by giving the approximate locations of a number of its points. This is a typical situation. An approximate discrete representation of the desired solution can usually be obtained by computa-

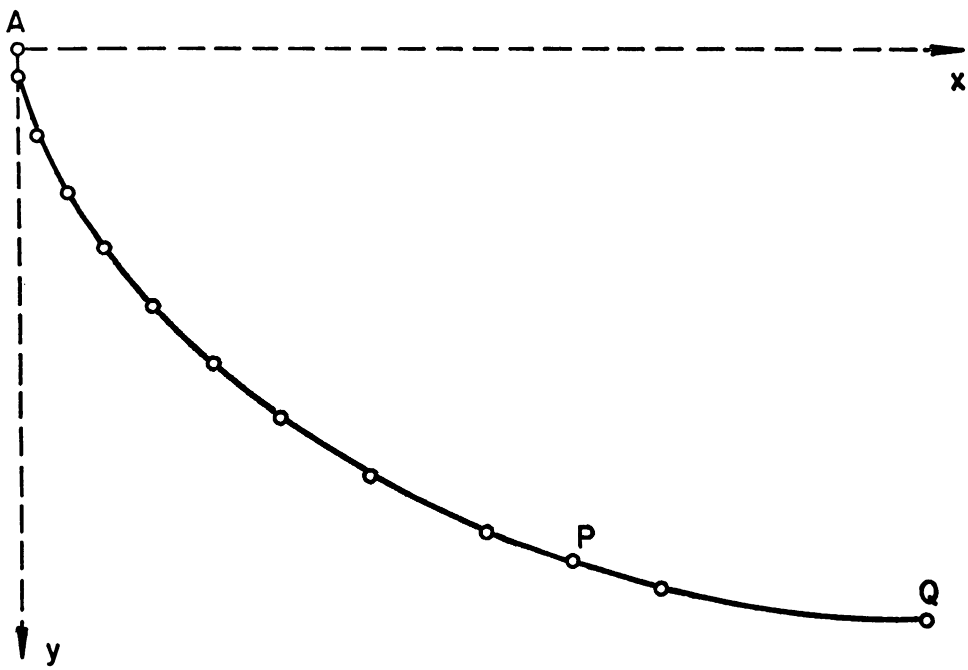

Fig. 4. Solution of discretized problem for $n = 10$.

tion, but for certain purposes this may not be sufficient. We may then try to synthesize an exact geometric or analytic description of the solution from observed features of the numerical approximation. For our brachystochrone problem, this synthesis might proceed along the following lines.

At first glance, Fig. 4 may suggest that the brachystochrone is a quadrant of an ellipse. This quadrant could, however, be continued beyond both its endpoints. The continuation beyond the lower endpoint would be mechanically meaningful because, thanks to its kinetic energy at this point, the bead could rise again along this continuation. The continuation beyond A, on the other hand, would not make sense because the bead, which has vanishing kinetic energy at A, cannot rise above this point. The brachystochrone thus cannot be continued beyond A into the upper half-plane bounded by the x-axis. This lets us surmise that A is a cusp of the brachystochrone, and this surmise appears to be confirmed by the fact that, assuming Q to lie on the other side of the y-axis, we obtain the mirror image of our curve on the y-axis.

The shape of the approximate brachystochrone in Fig. 4, and the fact that A is to be a cusp, now lets us suspect that the brachystochrone may be the common cycloid generated by the point A as point of the circle of radius $\frac{1}{2}$ and center 0, $\frac{1}{2}$ when this circle rolls without slipping on the x-axis. The fact that the point Q in Fig. 4 has the abscissa 1.572, which is a good approximation to $\pi/2 = 1.570796 \cdots$, further supports this suspicion, which may be confirmed as follows.

Consider an arbitrary position of the rolling circle with the point A in the position shown in Fig. 5. Since the instantaneous motion of the circle is a rotation about its contact point D with the x-axis, the line AD is the instantaneous normal to the cycloid

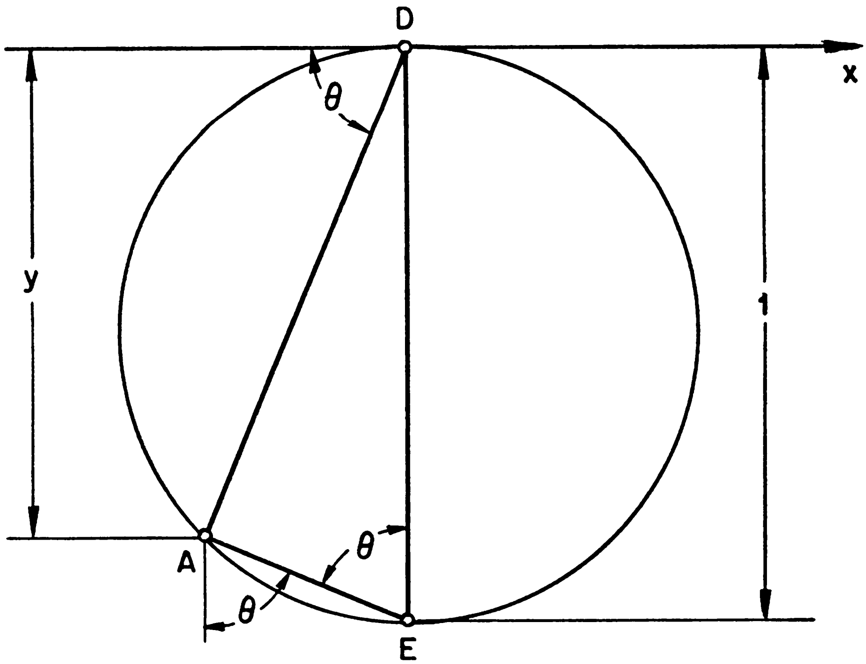

Fig. 5. Proof that brachystochrone is common cycloid.

described by A, and the line connecting A to the other endpoint E of the diameter through D is the instantaneous tangent, which forms the angle θ with the vertical. Moreover, the segment AD has the length $\sin\theta$, and we have

$$y = \sin^2\theta. \tag{8}$$

Since we have taken $Y = 1$, this equation agrees with (5) and thus confirms our guess that the brachystochrone is a common cycloid.

My example had to be extremely simple. I hope that it will, all the same, help me to make an important point. I believe that, in teaching applied mathematics, we should devote more time to the process of discovery than is customary today. This will not be easy on account of the tradition that requires us to present our results in an orderly and logical way and makes us reluctant to report the often erratic and illogical ways in which these results were obtained. We should overcome this reluctance because an account of the process of discovery will frequently be more useful to the student of applied mathematics than the particular results. Until the literature becomes richer in real case histories, we shall have to make up fictitious case histories as I have done today. I believe that, if we only knew them, many of the real case histories would have a strong geometric flavor. Accordingly, we should not hesitate to make ample use of computer graphics in our fictitious case histories.

With your indulgence, I should like to conclude on a personal note. In my student days, descriptive geometry, graphical statics, and kinematic geometry were still mandatory courses in applied mathematics, and differential geometry and projective geometry were not regarded as esoteric electives. I have regretted the gradual disappearance of these subjects from the curriculum and I am delighted at the prospect that, with the enormously powerful tool of computer graphics at our disposal, the geometric approach will become fashionable again, for I believe with Descartes that "nothing enters the mind as readily as geometric figures".