DIFFRACTION OF A PLANE SHEAR ELASTIC WAVE BY A CIRCULAR RIGID DISK AND A PENNY-SHAPED CRACK*

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1. Introduction. Diffraction of P-waves by disk-shaped obstacles and cracks embedded in a homogeneous and isotropic elastic medium has recently been given considerable attention. A bibliography of this subject is given in [1]. The corresponding problems of diffraction of S-waves are also of considerable interest in seismology and geophysics. Knowledge of the modification of S-waves in the vicinity of a crack is helpful in predicting the nature of flaws in the material. The dynamic stress intensity factors in the vicinity of a crack also help in predicting the fracture and failure of the material. Furthermore, the observation of diffraction pattern of S-waves generated by earthquakes or explosions reveal the presence of inhomogeneities and structural discontinuities in the medium. However, this problem has not received much attention. The main difficulty is that diffraction of S-waves leads to an asymmetric boundary value problem which is rather difficult to solve.

We present here the solution of the problem of diffraction of a plane S-wave for two cases. In the first case we discuss diffraction of S-waves by a rigid circular disk which is embedded in an infinite homogeneous and isotropic elastic medium. In the second case we analyze the corresponding problem of diffraction of a plane S-wave by a penny-shaped crack. It is assumed that the two faces of the crack are separated by a small distance so that they do not come into contact during vibration. The incident S-wave is assumed to be time-harmonic and is polarized in planes perpendicular to those of the disk and the crack. Furthermore, it is assumed to propagate along their axes of symmetry.

The method of solution is based on an integral equation technique suitable for mixed boundary value problems [2]. This method rests on giving an integral equation formulation to these problems by the usual Green's function approach. This leads to various Fredholm integral equations of the first kind which are subsequently transformed into Fredholm integral equations of the second kind suitable for iteration at low frequencies. The boundary conditions are prescribed in the interior and exterior of the disk and crack. These conditions are mixed with respect to the components of stress tensor and displacement vector. It emerges from the present analysis that the edge conditions play the same crucial role here as they do for the corresponding problems in electromagnetic theory.

We give the method of determining approximate values of the displacement and stress fields in the planes of the crack and the disk for these problems. In the case of

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the crack problem we also give the values of the dynamic intensity factors. The corresponding static intensity factors, when the loading is a uniform shear field at infinity, can be derived from the analysis of Westmann [3]. Second-order correction terms to these factors in the dynamic case were presented by Mal [4]. However, due to a conceptual error in one set of his relations and a slight algebraic error in another, his correction terms are, unfortunately, incorrect. Although we have given the values of these factors to the fourth order of the wave number, one can calculate these values to a higher-order accuracy also from our analysis. We have further derived the formulas for the far-field amplitudes and scattering cross-section for both of these problems. Finally, we establish that the crack faces would never make contact during vibration if a constant static pressure, however small, is applied over the crack.

The interesting feature of the present analysis is that when we take the elastostatic limit of the formulas for the various physical quantities as mentioned above, we find that even some of these limits appear to be new.

I: Diffraction By A Rigid Circular Disk

2. Formulation of the problem. We normalize all lengths with respect to the radius a of the disk and choose a cylindrical polar coordinate system (ρ, φ, z) in such a way that the disk occupies the region $z = 0, 0 \leq \rho \leq 1$, for all φ. The time dependence of all the relevant quantities is taken to be $e^{-i\omega t}$, where ω is circular frequency, and this factor is suppressed in the sequel. Let the incident wave be a shear wave propagating from $-\infty$ in the positive z-direction. The time-independent parts of the incident and the diffracted displacement fields can be written as

$$\mathbf{u}_0(r) = \mathbf{u}_0(\rho, \varphi, z) = \mathbf{e}_1 \exp(im_2 z), \tag{1}$$

$$\mathbf{u}(r) = \mathbf{u}(\rho, \varphi, z) = \{u_\rho(\mathbf{r}), u_\varphi(\mathbf{r}), u_z(\mathbf{r})\} = \{U_\rho(\rho, z) \cos\varphi, -U_\varphi(\rho, z) \sin\varphi, U_z(\rho, z) \cos\varphi\}, \tag{2}$$

where $\mathbf{r}$ is the position vector of a field point and $\mathbf{e}_1$ is the unit vector in the direction of x-axis. The function U_ρ, U_φ and U_z are given in terms of three scalar functions Φ_1, Φ_2, and Φ_3 as

$$U_\rho(\rho, z) = (\partial\Phi_1/\partial\rho) + (\partial^2 \Phi_2/\partial\rho\, \partial z) + (1/\rho)\Phi_3, \tag{3}$$

$$U_\varphi(\rho, z) = (1/\rho)\Phi_1 + (1/\rho)(\partial\Phi_2/\partial z) + (\partial\Phi_3/\partial\rho), \tag{4}$$

$$U_z(\rho, z) = (\partial\Phi_1/\partial z) + (\partial^2\Phi_2/\partial z^2) + m_2^2\Phi_2, \tag{5}$$

where the functions $\Phi_j(\rho, z)$, $j = 1, 2, 3$, satisfy the Helmholtz equation

$$(\nabla^2 + m_j^2)\Phi_j(\rho, z) \cos\varphi = 0, \tag{6}$$

and

$$m_1^2 = \omega^2\rho_0 a^2/(\lambda + 2\mu), \qquad m_2^2 = m_3^2 = \omega^2\rho_0 a^2/\mu, \tag{7}$$

while λ and μ are the Lamé's constants of the elastic medium.

The stress components $\tau_{\rho z}$, $\tau_{\varphi z}$, and τ_{zz} due to the diffracted field are

$$\tau_{\rho z}(\mathbf{r}) = \mu\left\{2\frac{\partial^2 \Phi_1}{\partial\rho\, \partial z} + \frac{\partial}{\partial\rho}\left(2\frac{\partial^2}{\partial z^2} + m_2^2\right)\Phi_2 + \frac{1}{\rho}\frac{\partial\Phi_3}{\partial z}\right\} \cos\varphi, \tag{8}$$

$$\tau_{\varphi z}(\mathbf{r}) = -\mu\left\{\frac{2}{\rho}\frac{\partial \Phi_1}{\partial z} + \frac{1}{\rho}\left(2\frac{\partial^2}{\partial z^2} + m_2^2\right)\Phi_2 + \frac{\partial^2 \Phi_3}{\partial \rho\, \partial z}\right\}\sin\varphi, \tag{9}$$

$$\tau_{zz}(\mathbf{r}) = \mu\left\{\left(2m_1^2 - m_2^2 + 2\frac{\partial^2}{\partial z^2}\right)\Phi_1 + 2\left(\frac{\partial^2}{\partial z^2} + m_2^2\right)\frac{\partial \Phi_2}{\partial z}\right\}\cos\varphi. \tag{10}$$

The boundary condition

$$\mathbf{u}(\mathbf{r}) + \mathbf{u}_0(\mathbf{r}) = 0, \qquad z = 0, \quad 0 \le \rho \le 1, \quad \text{all } \varphi,$$

is equivalent to

$$U_\rho(\rho, 0) = 1, \qquad U_\varphi(\rho, 0) = -1, \quad U_z(\rho, 0) = 0 \tag{11}$$

for $0 \le \rho \le 1$. Furthermore,

$$u_\rho,\ u_\varphi,\ u_z,\ \tau_{\rho z},\ \tau_{\varphi z},\ \tau_{zz} \text{ are continuous across the surface } z = 0,\ \rho > 1, \tag{12}$$

and

$$\Phi_i \text{ satisfy the radiation condition at infinity.} \tag{13}$$

It follows from the relations (3)–(10) and the boundary conditions (11)–(13) that Φ_i and $\partial\Phi_i/\partial z$ are continuous across $z = 0$, $\rho > 1$.

The integral representation formulas for the scalar functions Φ_i can be obtained from Eq. (6) by the Green's function method. These formulas are [5]

$$\Phi_i(\rho, z) = \frac{1}{2}\int_0^1 \int_0^\infty p\left[\frac{f_i(t)}{\gamma_i} \mp g_i(t)\right] J_1(pt) J_1(p\rho) \exp(-\gamma_i\, |z|\, t\, dp\, dt, \qquad z \gtrless 0, \tag{14}$$

where

$$\begin{aligned} \gamma_i &= (p^2 - m_i^2)^{1/2}, \qquad p \ge m_i, \\ &= -i(m_i^2 - p^2)^{1/2}, \qquad m_i \ge p, \end{aligned} \tag{15}$$

and the source densities $f_i(t)$ and $g_i(t)$ are defined as

$$f_i(t) = [(\partial/\partial z_1)(\Phi_i(t, z_1)]_{z_1=0-} - [(\partial/\partial z_1)(\Phi_i(t, z_1)]_{z_1=0+}, \tag{16}$$

$$g_i(t) = [\Phi_i(t, z_1)]_{z_1=0-} - [\Phi_i(t, z_1)]_{z_1=0+}. \tag{17}$$

The result of substituting these values of Φ_i in relations (3)–(5) and (8)–(10) and using the boundary conditions (11)–(12) is

$$g_1(t) = f_2(t) = g_3(t) = 0, \tag{18}$$

$$\int_0^1 t f_1(t) J_1(pt)\, dt = -p^2 \int_0^1 t g_2(t) J_1(pt)\, dt, \tag{19}$$

$$\int_0^1 \int_0^\infty p^2\left[\left(\gamma_2 - \frac{p^2}{\gamma_1}\right) g_2(t) + \frac{f_3(t)}{\gamma_3}\right] J_1(pt) J_0(pt) t\, dp\, dt = -4, \qquad 0 \le \rho \le 1, \tag{20}$$

$$\int_0^1 \int_0^\infty p^2\left[\left(\gamma_2 - \frac{p^2}{\gamma_1}\right) g_2(t) - \frac{f_3(t)}{\gamma_3}\right] J_1(pt) J_2(p\rho) t\, dp\, dt = 0, \qquad 0 \le \rho \le 1. \tag{21}$$

In order to apply the present integral equation technique [2] it is necessary that we decouple the integral equations (20) and (21) into two integral equations, one containing the unknown function $g_2(t)$ only and the other containing the function $f_3(t)$ only. This is readily achieved if we use the relations

$$J_0(p\rho) = \frac{1}{p\rho}\frac{d}{d\rho}[\rho J_1(p\rho)], \qquad J_2(p\rho) = -\frac{\rho}{p}\frac{d}{d\rho}[\rho^{-1}J_1(p\rho)]. \tag{22}$$

Indeed, when we substitute them in (20)–(21), integrate both sides of the resulting equations with respect to ρ, add and subtract them, we obtain

$$\int_0^1 \int_0^\infty p\left[\left(\frac{p^2}{\gamma_1} - \gamma_2\right)g_2(t)\right]J_1(pt)J_1(p\rho)t\,dp\,dt = \frac{C+4}{2}\rho, \qquad 0 \leq \rho \leq 1, \tag{23}$$

$$\int_0^1 \int_0^\infty \frac{pf_3(t)}{\gamma_2} J_1(pt)J_1(p\rho)t\,dp\,dt = \frac{C}{2}\rho, \qquad 0 \leq \rho \leq 1, \tag{24}$$

where C is an unknown constant of integration and will eventually be evaluated with the help of appropriate edge conditions. It follows from (23) and (24) that

$$\int_0^1 \frac{f_3(t)}{\gamma_3} J_1(pt)t\,dt = \frac{C}{C+4}\int_0^1 \left(\frac{p^2}{\gamma_1} - \gamma_2\right)g_2(t)J_1(pt)t\,dt. \tag{25}$$

The next step is to substitute the relation (25) in (20) and (21), use the formula (22), (24), as well as the relation

$$J_2(p\rho) = 2J_1(p\rho)/p\rho - J_0(p\rho),$$

and integrate by parts with respect to t. The result is

$$\int_0^1 \int_0^\infty p\left[\left(\frac{p^2}{\gamma_1} - \gamma_2\right)\right]I_2(t)J_0(pt)J_0(p\rho)\,dp\,dt = C + 4, \qquad 0 \leq \rho \leq 1, \tag{26}$$

and

$$\int_0^1 \int_0^\infty \frac{p}{\gamma_2} I_3(t)J_0(pt)J_0(p\rho)\,dp\,dt = C, \qquad 0 \leq \rho \leq 1, \tag{27}$$

where

$$I_2(t) = (d/dt)(tg_2(t)), \qquad I_3(t) = (t/dt)(tf_3(t)), \tag{28}$$

and

$$g_2(1) = f_3(1) = 0. \tag{29}$$

Furthermore, it is assumed that $\lim_{t\to 0}[tg_2(t)]$ and $\lim_{t\to 0}[tf_3(t)]$ are finite.

The integral equations (26) and (27) are Fredholm integral equations of the first kind. They can be converted to Fredholm integral equations of the second kind by using the technique explained in [2]. This results in the following two equations:

$$S_2(\rho) = \frac{2(C+4)}{m_1^2 + m_2^2} + \int_0^1 L_2(v, \rho)S_2(v)\,dv, \qquad 0 \leq \rho \leq 1, \tag{30}$$

$$S_3(\rho) = C + \int_0^1 L_3(v, \rho)S_3(v)\,dv, \qquad 0 \leq \rho \leq 1, \tag{31}$$

where

$$S_j(\rho) = \int_\rho^1 \frac{I_j(t)\, dt}{(t^2 - \rho^2)^{1/2}}, \qquad j = 2, 3, \tag{32}$$

$$L_2(v, \rho) = (v\rho)^{1/2} \int_0^\infty p\left\{1 - \frac{2p}{m_1^2 + m_2^2}\left(\frac{p^2}{\gamma_1} - \gamma_2\right)\right\} J_{-1/2}(pv) J_{-1/2}(p\rho)\, dp, \tag{33}$$

$$L_3(v, \rho) = (v\rho)^{1/2} \int_0^\infty p\left(1 - \frac{p}{\gamma_2}\right) J_{-1/2}(pv) J_{-1/2}(p\rho)\, dp. \tag{34}$$

Let us now assume that $m_1 \ll 1$, $m_2 \ll 1$ and $m_1 = O(m_2)$. The kernels L_2 and L_3 can then be expanded in ascending powers of m_1 and m_2 by using Noble's technique [6]. These expressions are

$$L_2(v, \rho) = -\frac{2}{(\tau^2 + 1)}\left[\frac{2i}{3\pi}(2\tau^3 + 1)m_2 - \frac{v}{8}(3\tau^4 + 1)m_2^2 - \frac{2i}{15\pi}(v^2 + \rho^2)(4\tau^5 + 1)m_2^3 + O(m_2^4)\right], \qquad v \geq \rho, \tag{35}$$

$$L_3(v, \rho) = -(2im_2/\pi) + (v/2)m_2^2 + (2i/3\pi)(\rho^2 + v^2)m_2^3 + O(m_2^4), \qquad v \geq \rho, \tag{36}$$

where $\tau = m_1/m_2$. The corresponding values of these kernels for the case $\rho \geq v$ are obtained by interchanging v and ρ in the above relations. The expansions (35) and (36) enable us to solve the Fredholm integral equations (30) and (31) by iteration and we obtain

$$\begin{aligned} S_2(\rho) = \frac{2(C + 4)}{(m_1^2 + m_2^2)}\Bigg[1 &- \frac{4i(2\tau^3 + 1)}{3\pi(\tau^2 + 1)} m_2 + \left\{-\frac{16}{9\pi^2}\frac{(2\tau^3 + 1)^2}{(\tau^2 + 1)^2} + \frac{1}{8}\frac{(3\tau^4 + 1)}{(\tau^2 + 1)}(1 + \rho^2)\right\}m_2^2 + \frac{i}{4\pi}\left\{\frac{4}{15}\frac{(4\tau^5 + 1)}{(\tau^2 + 1)}\left(\frac{1}{3} + \rho^2\right)\right. \\ &\left. - \frac{1}{6}\frac{(2\tau^3 + 1)(3\tau^4 + 1)}{(\tau^2 + 1)^2}\left(\frac{7}{3} + \rho^2\right) + \frac{64}{27\pi^2}\frac{(2\tau^3 + 1)^3}{(\tau^2 + 1)^3}\right\}m_2^3 + O(m_2^4)\Bigg], \end{aligned} \tag{37}$$

$$S_3(\rho) = C\left[1 - \frac{2i}{\pi} m_2 - \frac{4}{\pi^2} m_2^2 + \frac{m_2^2}{4}(1 + \rho^2) + \frac{i}{\pi}\left\{-\frac{17}{18} + \frac{8}{\pi^2} + \frac{\rho^2}{6}\right\}m_2^3 + O(m_2^4)\right]. \tag{38}$$

The unknown constant C occurring in the above relations is determined by applying the edge condition

$$[\tau_{yz}]_{z=0\pm} \text{ are continuous and finite at the edge of the disk.} \tag{39}$$

For this purpose we appeal to relations (8), (14), (18) and (19) and find that

$$\begin{aligned} [\tau_{\rho z}]_{z=0\pm} &= \pm\frac{\mu}{2}\cos\varphi\left[-m_2^2\frac{dg_2(\rho)}{d\rho} + \frac{1}{\rho}f_3(\rho)\right], \qquad 0 \leq \rho \leq 1, \\ &= 0, \qquad \rho > 1. \end{aligned} \tag{40}$$

Next, we invert the Volterra integral equations (32) and use relations (28) and (29). Thereby the functions $g_2(t)$ and $f_3(t)$ are evaluated in terms of S_2 and S_3. These values are then substituted in (40). Hence

$$[\tau_{\rho z}]_{z=0\pm} = \mp\frac{\mu}{\rho}\cos\varphi\left[\frac{m_2^2}{\rho}\frac{d}{d\rho}\int_\rho^1 \frac{uS_2(u)\,du}{(u^2-\rho^2)^{1/2}} - \frac{1}{\rho^2}\int_\rho^1 \frac{u[m_2^2S_2(u)+S_3(u)]\,du}{(u^2-\rho^2)^{1/2}}\right], \quad 0\le\rho\le 1, \tag{41}$$

$$= 0, \quad \rho > 1.$$

Similarly,

$$[\tau_{\varphi z}]_{z=0\pm} = \mp\frac{\mu}{\pi}\sin\varphi\left[\frac{1}{\rho}\frac{d}{d\rho}\int_\rho^1 \frac{uS_3(u)\,du}{(u^2-\rho^2)^{1/2}} - \frac{1}{\rho^2}\int_\rho^1 \frac{u(m_2^2S_2(u)+S_3(u))\,du}{(u^2-\rho^2)^{1/2}}\right], \quad 0\le\rho\le 1, \tag{42}$$

$$= 0, \quad \rho > 1.$$

With the help of relations (41) and (42) we can evaluate $[\tau_{yz}]_{z=0\pm}$. From this it emerges that the edge condition (39) is satisfied if

$$m^2S_2(1) + S_3(1) = 0. \tag{43}$$

The result of substituting the values of $S_2(1)$ and $S_3(1)$ from (37) and (38) in (43) is

$$C = -(8/(\tau^2+3))[1 + c_1m_2 + c_2m_2^2 + c_3m_2^3 + O(m_2^4)], \tag{44}$$

where

$$c_1 = [2i(1+3\tau^2-4\tau^3)]/3\pi(3+\tau^2),$$

$$c_2 = \frac{(3\tau^4-2\tau^2-1)}{4(3+\tau^2)} + \frac{16(2+\tau^3)(1+3\tau^2-4\tau^3)}{9\pi^2(3+\tau^2)^2},$$

$$c_3 = \frac{i}{\pi}\left\{-\frac{(2\tau^3+1)(3\tau^4+1)(3+5\tau^2)}{9(1+\tau^2)(3+\tau^2)^2} - \frac{(9\tau^4+8\tau^3+7)}{3(3+\tau^2)^2} + \frac{2(8(4\tau^5+1)-5(1+\tau^2))}{45(3+\tau^2)} + \frac{(1+\tau^2)(9\tau^4+8\tau^3-6\tau^2+13)}{6(3+\tau^2)^2} + \frac{128(2+\tau^3)^2(4\tau^3-3\tau^2-1)}{27\pi^2(3+\tau^2)^3}\right\}.$$

Substituting the value of C from (44) into (37)–(38), we have

$$S_2(\rho) = (8m_2^{-2}/(\tau^2+3))[d_0 + d_2\,\rho^2 + O(m_2^4)], \tag{45}$$

where

$$d_0 = 1 - \frac{8i(2+\tau^3)}{3\pi(3+\tau^2)}m_2 + \left\{\frac{(3\tau^4+1)(\tau^2-1)}{8(1+\tau^2)(3+\tau^2)} + \frac{1}{3+\tau^2} - \frac{64(2+\tau^3)^2}{9\pi^2(3+\tau^2)^2}\right\}m_2^2 + \frac{i}{\pi}\left\{\frac{4(\tau^2-5)(4\tau^5+1)}{45(3+\tau^2)(1+\tau^2)} + \frac{(1+2\tau^3)(1+3\tau^4)(3-7\tau^2)}{18(1+\tau^2)(3+\tau^2)^2} - \frac{(3\tau^4+1)(1+3\tau^2)}{2(3+\tau^2)^2(1+\tau^2)} - \frac{8}{3}\frac{(1+2\tau^3)}{(3+\tau^2)^2} + \frac{2(11\tau^2-3)}{9(3+\tau^2)^2} + \frac{512(2+\tau^3)^3}{27\pi^2(3+\tau^2)^3}\right\}m_2^3,$$

$$d_2 = \frac{(3\tau^4 + 1)}{8(1 + \tau^2)} m_2^2 + \frac{i}{3\pi(1 + \tau^2)} \left\{\frac{4}{5}(4\tau^5 + 1) - \frac{(2 + \tau^3)(1 + 3\tau^4)}{(3 + \tau^2)}\right\} m_2^3 .$$

and

$$S_3(\rho) = -(8/(\tau^2 + 3))\{e_0 + e_2\rho^2 + O(m_2^4)\}, \tag{46}$$

where

$$\begin{aligned} e_0 = 1 &- \frac{8i(2 + \tau^3)}{3\pi(3 + \tau^2)} m_2 + \left\{\frac{(3\tau^4 - \tau^2 + 2)}{4(3 + \tau^2)} - \frac{64(2 + \tau^3)^2}{9\pi^2(3 + \tau^2)^2}\right\} m_2^2 \\ &+ \frac{i}{\pi}\left\{\frac{16(4\tau^5 + 1)}{45(3 + \tau^2)} - \frac{(2\tau^3 + 1)(1 + 3\tau^4)(3 + 5\tau^2)}{9(1 + \tau^2)(3 + \tau^2)^2}\right. \\ &+ \frac{1}{6(3 + \tau^2)^2}[-12(1 + \tau^2)^2 + 3(3\tau^4 + 1)(\tau^2 - 1) + 4(2\tau^3 + 1)(\tau^2 - 1)] \\ &+ \frac{1}{18(3 + \tau^2)}[32(1 + \tau^2) - 9(3\tau^4 + 1) - 8(\tau^2 + 3) - 12(\tau^3 + 2)] \\ &\left. + \frac{512(2 + \tau^3)^3}{27\pi^2(3 + \tau^2)^3}\right\} m_2^3 , \end{aligned}$$

$$e_2 = \tfrac{1}{4}m_2^2 + (2im_2^3(1 + \tau^2 - \tau^3)/3\pi(3 + \tau^2)).$$

3. Far-field amplitudes and scattering cross-section. The far-field amplitudes are obtained from the integral representation formulas for Φ_j obtained from (6). Indeed, by using the formulas (18), (19), (28), (29) and (32) and introducing the spherical polar coordinates (r, θ, φ), we obtain [5]

$$\Phi_j(r, \theta) = \Phi_j(\rho, z) \sim (\exp(im_j r)/r)A_j(\theta) \qquad \text{as } r \to \infty, \quad j = 1, 2, 3, \tag{47}$$

where

$$A_1(\theta) = i(m_1 \sin\theta)^2 A(\theta, m_1), \quad A_2(\theta) = m_2 \cos\theta A(\theta, m_2), \quad A_3(\theta) = -iB(\theta, m_2), \tag{48}$$

and

$$A(\theta, m_\alpha) = (2\pi m_\alpha \sin\theta)^{-1/2} \int_0^1 v^{1/2} J_{-1/2}(m_\alpha v \sin\theta) S_2(v)\, dv, \qquad \alpha = 1, 2, \tag{49}$$

$$B(\theta, m_2) = (2\pi m_2 \sin\theta)^{-1/2} \int_0^1 v^{1/2} J_{-1/2}(m_2 v \sin\theta) S_3(v)\, dv. \tag{50}$$

Substitution of relations (47) in (3)–(5) and the resulting expressions in (2) yield

$$\begin{aligned} u_r(r, \theta, \varphi) &= -m_1^3 \sin^2\theta A(\theta, m_1) \cos\varphi(\exp(im_1 r)/r) + O(r^{-2}), \\ u_\theta(r, \theta, \varphi) &= -m_2^3 \sin\theta \cos\theta A(\theta, m_2) \cos\varphi(\exp(im_2 r)/r) + O(r^{-2}), \\ u_\varphi(r, \theta, \varphi) &= -m_2 \sin\theta B(\theta, m_2) \sin\varphi(\exp(im_2 r)/r) + O(r^{-2}), \end{aligned} \tag{51}$$

as $r \to \infty$.

The next step is to put the values of the functions S_2 and S_3 from (45)–(46) into (49)–(50) and derive the values of $A(\theta, m_\alpha)$ and $B(\theta, m_2)$ as

$$A(\theta, m_\alpha) = \frac{8m_2^{-2}}{\pi m_\alpha \sin\theta(\tau^2 + 3)} \cdot \left[\left(d_0 + \frac{d_2}{3}\right) - \frac{m_\alpha^2}{6}\left\{1 - \frac{8i(\tau^3 + 2)m_2}{3\pi(\tau^2 + 3)}\right\}\sin^2\theta + O(m_2^4)\right], \qquad \alpha = 1, 2, \tag{52}$$

$$B(\theta, m_2) = -\frac{8}{\pi m_2 \sin\theta(\tau^2 + 3)} \cdot \left[\left(e_0 + \frac{e_2}{3}\right) - \frac{m_2^2}{6}\left\{1 - \frac{8i(\tau^3 + 2)m_2}{3\pi(\tau^2 + 3)}\right\}\sin^2\theta + O(m_2^4)\right]. \tag{53}$$

Thus, the far-field behavior of the displacement field $u(r)$ is completely known from (51)–(53).

Finally, we use the formula for the scattering cross-section Σ_s as given in [7] and find its value (in physical units) as

$$\Sigma_s = (4\pi a^2/m_2)[\mathbf{e}_1 \cdot \mathbf{h}(\theta, \varphi)]_{\theta=\varphi=0}, \tag{54}$$

where

$$\mathbf{h}(\theta, \varphi) = -m_2 \sin\theta[m_2^2 \cos\theta A(\theta, m_2) \cos\varphi \hat{\boldsymbol{\theta}} + B(\theta, m_2) \sin\varphi \hat{\boldsymbol{\varphi}}], \tag{55}$$

while $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ and $\mathbf{e}_1$ are unit vectors in the θ, φ and x directions.

When we substitute the values of $A(\theta, m_2)$ and $B(\theta, m_2)$ from (52) and (53) in (55) and the resulting expression in (54), we obtain

$$\Sigma_s = \frac{32a^2}{3\pi(3 + \tau^2)^2}\left\{8(2 + \tau^3) + \left[\frac{(1 + 3\tau^4)(9 + 19\tau^2 + 8\tau^5)}{3(1 + \tau^2)(3 + \tau^2)} + \frac{2(15 - 11\tau^2 + 24\tau^3)}{3(3 + \tau^2)} + \frac{8(1 + 4\tau^5)(1 - \tau^2)}{15(1 + \tau^2)} - \frac{512(2 + \tau^3)^3}{9\pi^2(3 + \tau^2)^2}\right]m_2^2 + O(m_2^4)\right\}. \tag{56}$$

The first term agrees with the result derived by the authors [1] as a limiting case of an ellipsoid by a different method. The second term is new. The interesting feature of the present method is that we can obtain the higher-order terms without solving any extra integral equations as is necessary in using the perturbation technique ([1]; see also [5, Chapter 11], for explanation). However, the advantage of taking the limit from the result for an ellipsoid is that no constant of integration occurs in the analysis.

As explained in [1], by setting $\tau = 1$, the present problem reduces to that of diffraction of an acoustic plane wave by a perfectly soft disk and this serves as another check on formula (56). For finding the displacement and stress fields in the plane $z = 0$, we first have to evaluate the unknown functions $g_2(t)$ and $f_3(t)$ by inverting the Volterra integral equations (32) and by using relations (28)–(29) as well as relations (45)–(46). These values are

$$g_2(t) = \frac{-16(1 - t)^{1/2}}{(m_1^2 + 3m_2^2)t}\left[d_0 + \frac{d_2}{3}(2t^2 + 1) + O(m_2^4)\right], \tag{57}$$

$$f_3(t) = \frac{16(1 - t^2)^{1/2}}{(3 + \tau^2)t}\left[e_0 + \frac{e_2}{3}(2t^2 + 1) + O(m_2^4)\right]. \tag{58}$$

The required results can then be easily derived by substituting the above values in (3)–(5), (8)–(10) and (14).

In the limit when $\omega \to 0$, i.e. $m_1 \to 0$ and $m_2 \to 0$, we get the solution for the corresponding elastostatic problem when the rigid circular disk is given a uniform displacement $-\mathbf{e}_1$ in an infinite elastic solid. As far as the authors are aware even these limiting solutions are new. However, we have verified that the formula for the horizontal force acting on the plate in this limiting case agrees with the known result [8].

II: Diffraction By A Penny-Shaped Crack

4. Formulation of the problem. The mathematical formulation of this problem is similar to the one given for the disk problem in Sec. 2. However, it is not exactly a dual problem as we are not given the Neumann conditions in this case.

Let the crack occupy the region $0 \leq \rho \leq 1$, $z = 0$, for all φ. Eqs. (2)–(10) hold for the present case as well. The difference starts with the boundary condition which, for the present case, must be given in terms of the stress components. For this purpose, let us denote the stress components due to the incident field $\mathbf{u}_0(r) = iA_0\mathbf{e}_1 \exp(im_2z)$ as $\tau^0_{\rho z}$, $\tau^0_{\varphi z}$, τ^0_{zz}. Then

$$\tau^0_{\rho z} = -A_0\mu m_2 \cos\varphi \exp(im_2z), \qquad \tau^0_{\varphi z} = A_0\mu m_2 \sin\varphi \exp(im_2z), \qquad \tau^0_{zz} = 0. \tag{59}$$

The boundary conditions for the present case are therefore

$$(\tau_{\rho z} + \tau^0_{\rho z}) = 0, \quad (\tau_{\varphi z} + \tau^0_{\varphi z}) = 0, \quad \tau_{zz} = 0, z = 0, 0 \leq \rho \leq 1, \quad \text{for all } \varphi, \tag{60}$$

and (12) and (13). The integral representation formulas for Φ_i are the same as (14).

Application of the boundary conditions (12) and (60) yields in this case

$$f_1(t) = g_2(t) = f_3(t) = 0, \tag{61}$$

$$m_2^2 \int_0^1 tg_1(t)J_1(pt)\,dt = 2p^2 \int_0^1 t[g_1(t) + f_2(t)]J_1(pt)\,dt, \tag{62}$$

$$\int_0^1 \int_0^\infty \left[\frac{1}{2\gamma_2}\{4p^2\gamma_1\gamma_2 - (2p^2 - m_2^2)^2\}g_1(t) + p^2\gamma_2 g_3(t)\right]J_1(pt)J_0(p\rho)t\,dp\,dt = 4A_0m_2, \quad 0 \leq \rho \leq 1, \tag{63}$$

$$\int_0^1 \int_0^\infty \left[\frac{1}{2\gamma_2}\{4p^2\gamma_1\gamma_2 - (2p^2 - m_2^2)^2\}g_1(t) - p^2\gamma_2 g_3(t)\right]J_1(pt)J_2(p\rho)t\,dp\,dt = 0, \quad 0 \leq \rho \leq 1. \tag{64}$$

Eqs. (63) and (64) are to be processed as were the corresponding equations (20) and (21) and this leads to the equations corresponding to (23) and (24):

$$\int_0^1 \int_0^\infty \left[\frac{t}{2p\gamma_2}\{4p^2\gamma_1\gamma_2 - (2p^2 - m_2^2)^2\}\right]g_1(t)J_1(pt)J_1(p\rho)\,dp\,dt = A_0m_2(1 + D)\rho, \quad 0 \leq \rho \leq 1, \tag{65}$$

$$\int_0^1 \int_0^\infty tp\gamma_2 g_3(t)J_1(pt)J_1(p\rho)\,dp\,dt = A_0m_2(1 - D)\rho, \qquad 0 \leq \rho \leq 1, \tag{66}$$

where the constant of integration D will be found with the help of appropriate edge conditions.

The integral equations (65)–(66) are amenable to the present integral equation technique. When we apply this technique we can convert (65) to Fredholm integral equation of the second kind:

$$S_1(\rho) = \frac{2A_0 m_2(1 + D)\rho}{(m_2^2 - m_1^2)} + \int_0^1 L_1(v, \rho) S_1(v)\, dv, \qquad 0 \leq \rho \leq 1, \tag{67}$$

where

$$S_1(\rho) = \rho \int_\rho^1 \frac{g_1(t)\, dt}{(t^2 - \rho^2)^{1/2}} \tag{68}$$

and the kernel $L_1(v, \rho)$ is

$$L_1(v, \rho) = (v\rho)^{1/2} \int_0^\infty \left[p - \frac{1}{2(m_2^2 - m_1^2)\gamma_2}\{4p^2\gamma_1\gamma_2 - (2p^2 - m_2^2)^2\}\right] J_{1/2}(pv) J_{1/2}(p\rho)\, dp. \tag{69}$$

As in Part I, we assume that $m_1 \ll 1$, $m_2 \ll 1$, $m_1 = O(m_2)$. Then by Noble's technique we find that

$$L_1(v, \rho) = \frac{1}{(m_2^2 - m_1^2)}\left[\frac{v}{4}(m_1^4 + m_2^4) + \frac{iv\rho}{15\pi}(8m_1^5 + 7m_2^5) - \frac{(v^3 + 3v\rho^2)}{48}(m_1^6 + m_2^6) + O(m_2^7)\right], \qquad \rho \geq v. \tag{70}$$

The value for $v \geq \rho$ follows by interchanging v and ρ in (70). The integral equation (67) can now be readily solved by straightforward iteration. An approximate solution is

$$S_1(\rho) = \frac{2A_0 m_2(1 + D)}{(m_2^2 - m_1^2)}\left\{\rho + \frac{(1 + \tau^4)}{24(1 - \tau^2)}(3 - \rho^2)\rho m_2^2 + \frac{i\rho}{45\pi}\frac{(8\tau^5 + 7)}{(1 - \tau^2)} m_2^3 + \frac{(1 + \tau^6)}{960(1 - \tau^2)}(35\rho - 30\rho^3 + 3\rho^5)m_2^4 + O(m_2^5)\right\}. \tag{71}$$

To solve Eq. (66) we introduce a new function $g_4(t)$ such that

$$p^2 \int_0^1 t g_3(t) J_1(pt)\, dt = \int_0^1 t g_4(t) J_1(pt)\, dt. \tag{72}$$

Then from (66) and (72) we have

$$\int_0^1 \int_0^\infty \frac{t\gamma_2}{p} g_4(t) J_1(pt) J_1(p\rho)\, dp\, dt = A_0 m_2(1 - D)\rho, \qquad 0 \leq \rho \leq 1. \tag{73}$$

This equation reduces to the following Fredholm integral equation of the second kind:

$$S_4(\rho) = 2A_0 m_2(1 - D)\rho + \int_0^1 L_4(v, \rho) S_4(v)\, dv, \qquad 0 \leq \rho \leq 1, \tag{74}$$

where

$$S_4(\rho) = \rho \int_\rho^1 \frac{g_4(t)\, dt}{(t^2 - \rho^2)^{1/2}}, \tag{75}$$

$$L_4(v, \rho) = (v\rho)^{1/2} \int_0^\infty p\left(1 - \frac{\gamma_2}{p}\right) J_{1/2}(pv) J_{1/2}(p\rho)\, dp. \tag{76}$$

The expansion for the kernel L_4 is

$$L_4(v, \rho) = \frac{m_2^2 v}{2} + \frac{2i\rho v}{3\pi} m_2^3 - \frac{(v^3 + 3\rho^2 v)}{48} m_2^4 + O(m_2^5), \qquad \rho \geq v, \tag{77}$$

where for $v \geq \rho$ we interchange v and ρ in (77). This enables us to solve Eq. (74) by iteration and get

$$S_4(\rho) = 2A_0 m_2(1 - D)\left[\rho + \frac{(3\rho - \rho^3)}{12} m_2^2 + \frac{2i\rho}{9\pi} m_2^3 + \frac{(35\rho - 30\rho^3 + 3\rho^5)}{960} m_2^4 + O(m_2^5)\right]. \tag{78}$$

When we substitute the values of S_1 and S_4 from (71) and (78) into (68) and (75) and invert these Volterra integral equations, we derive the values of $g_1(t)$ and $g_4(t)$ as

$$g_1(t) = \frac{4A_0 m_2(1 + D)t}{(m_2^2 - m_1^2)\pi(1 - t^2)^{1/2}}\left[1 + \frac{(1 + \tau^4)}{12(1 - \tau^2)}(2 - \tau^2)m_2^2 + \frac{i(8\tau^5 + 7)m_2^3}{45\pi(1 - \tau^2)} m_2^3 + \frac{(1 + \tau^6)}{120(1 - \tau^2)}(t^4 - 8t^2 + 8)m_2^4 + O(m_2^5)\right] \tag{79}$$

and

$$g_4(t) = \frac{4A_0 m_2(1 - D)t}{\pi(1 - t^2)^{1/2}}\left[1 + \frac{(2 - t^2)}{6} m_2^2 + \frac{2i}{9\pi} m_2^3 + \frac{(t^4 - 8t^2 + 8)}{120} m_2^4 + O(m_2^5)\right]. \tag{80}$$

Let us now evaluate the constant D with the help of the edge conditions:

$[u_\rho(\rho, \varphi, z)]_{z=0\pm}$ and $[u_\varphi(\rho, \varphi, z)]_{z=0\pm}$ are finite and continuous at the edge of the crack, (81)

which as in the previous problem can be put as

$$\int_0^1 t^2\left\{g_4(t) - \frac{m_2^2}{2} g_1(t)\right\} dt = 0. \tag{82}$$

When the edge condition (82) is satisfied, the displacement vector components u_ρ, u_φ are continuous in the plane $z = 0$, at the edge of the crack and are of order $\cos \varphi\{O(1 - \rho^2)^{1/2}\}$ and $\sin \varphi\{O(1 - \rho^2)^{1/2}\}$ respectively, as $\rho \to 1$.

Substituting the values of the functions g_1 and g_4 from (79) and (80) in (82), we obtain the value of the constant D as

$$D = \frac{(1 - 2\tau^2)}{(3 - 2\tau^2)}\left[1 - \frac{2m_2^2(1 - \tau^2)\beta_0}{5(3 - 2\tau^2)(1 - 2\tau^2)} - \frac{4im_2^3\beta_1(1 - \tau^2)}{9\pi(1 - 2\tau^2)(3 - 2\tau^2)} + \frac{m_2^4(1 - \tau^2)(\beta_0^2 - 100\beta_2(3 - 2\tau^2))}{25(3 - 2\tau^2)^2(1 - 2\tau^2)} + O(m_2^5)\right], \tag{83}$$

where

$$\beta_0 = \frac{1 + \tau^4}{1 - \tau^2} - 2, \qquad \beta_1 = \frac{8\tau^5 + 7}{5(1 - \tau^2)} - 2, \qquad \beta_2 = \frac{2\tau^2(1 + \tau^4)}{105(1 - \tau^2)} - \frac{1}{50}\beta_0 .$$

Now we put this value of D in (71) and (78) and obtain

$$S_1(\rho) = \frac{8A_0}{m_2(3 - 2\tau^2)} \left\{\rho\left[1 - \frac{\beta_0}{10(3 - 2\tau^2)} m_2^2 - \frac{i\beta_1}{9\pi(3 - 2\tau^2)} m_2^3 + \frac{\beta_0^2 - 100\beta_2(3 - 2\tau^2)}{100(3 - 2\tau^2)^2} m_2^4\right] + \frac{(1 + \tau^4)}{24(1 - \tau^2)} (3\rho - \rho^3)m_2^2\left[1 - \frac{\beta_0}{10(3 - 2\tau^2)} m_2^2\right] + \frac{i\rho(8\tau^5 + 7)}{45\pi(1 - \tau^2)} m_2^3 + \frac{(1 + \tau^6)}{960(1 - \tau^2)} (35\rho - 30\rho^3 + 3\rho^5)m_2^4 + O(m_2^5)\right\}, \tag{84}$$

and

$$S_4(\rho) = \frac{4A_0 m_2}{(3 - 2\tau^2)} \left\{\rho\left[1 + \frac{(1 - \tau^2)\beta_0}{5(3 - 2\tau^2)} m_2^2 + \frac{2i\beta_1(1 - \tau^2)}{9\pi(3 - 2\tau^2)} m_2^3 - \frac{(1 - \tau^2)(\beta_0^2 - 100\beta_2(3 - 2\tau^2))}{50(3 - 2\tau^2)^2} m_2^4\right] + \frac{(3\rho - \rho^3)m_2^2}{12}\left[1 + \frac{(1 - \tau^2)\beta_0}{5(3 - 2\tau^2)} m_2^2\right] + \frac{2i\rho}{9\pi} m_2^3 + \frac{(35\rho - 30\rho^3 + 3\rho^5)}{960} m_2^4 + O(m_2^5)\right\}. \tag{85}$$

5. Dynamic stress intensity factors. The dynamic stress intensity factors N_ρ and N_φ are defined as (in physical units)

$$N_\rho = \lim_{\rho \to 1+0} (a)^{-1/2}(\rho - 1)^{1/2}[\tau_{\rho z}]_{z=0,\rho>1}, \tag{86}$$

and

$$N_\varphi = \lim_{\rho \to 1+0} (a)^{-1/2}(\rho - 1)^{1/2}[\tau_{\varphi z}]_{z=0,\rho>1}. \tag{87}$$

From (8), (14), (61), (62), (68), (72) and (75) it follows that

$$[\tau_{\rho z}(\rho, \varphi, z)]_{z=0} = -\mu \cos\varphi\left\{(m_1^2 - m_2^2)\frac{d}{d\rho} \cdot\left[\frac{1}{\rho}\int_0^\rho \frac{w}{(\rho^2 - w^2)^{1/2}}\int_0^1 L_1(v, w)S_1(v)\,dv\,dw - \frac{1}{\rho}\int_0^1 \frac{wS_1(w)\,dw}{(\rho^2 - w^2)^{1/2}}\right] + \frac{1}{\rho^2}\left[\int_0^\rho \frac{w}{(\rho^2 - w^2)^{1/2}}\int_0^1 L_4(v, w)S_4(v)\,dv\,dw - \int_0^1 \frac{wS_4(w)\,dw}{(\rho^2 - w^2)^{1/2}}\right]\right\}. \tag{88}$$

Similarly,

$$[\tau_{\varphi z}(\rho, \varphi, z)]_{z=0} = \frac{\mu \sin\varphi}{\pi}\left\{\frac{(m_1^2 - m_2^2)}{\rho^2} \cdot\left[\int_0^\rho \frac{w}{(\rho^2 - w^2)^{1/2}}\int_0^1 L_1(v, w)S_1(v)\,dv\,dw - \int_0^1 \frac{wS_1(w)\,dw}{(\rho^2 - w^2)^{1/2}}\right] + \frac{d}{d\rho}\left[\frac{1}{\rho}\int_0^\rho \frac{w}{(\rho^2 - w^2)^{1/2}}\int_0^1 L_4(v, w)S_4(v)\,dv\,dw - \frac{1}{\rho}\int_0^1 \frac{wS_4(w)\,dw}{(\rho^2 - w^2)^{1/2}}\right]\right\}. \tag{89}$$

Consequently, the formulas (86) and (87) become

$$N_\rho = \frac{-8A_0 m_2 \mu(1 - \tau^2)\cos\varphi}{\pi(2a)^{1/2}(3 - 2\tau^2)}\left\{1 + \left[\frac{(1 + \tau^4)}{12(1 - \tau^2)} - \frac{\beta_0}{10(3 - 2\tau^2)}\right]m_2^2 + \frac{i}{45\pi}\left[\frac{(8\tau^5 + 7)}{(1 - \tau^2)} - \frac{5\beta_1}{(3 - 2\tau^2)}\right]m_2^3 + \left[\frac{\beta_0^2 - 100(3 - 2\tau^2)\beta_2}{100(3 - 2\tau^2)} - \frac{\beta_0}{120(3 - 2\tau^2)} + \frac{1 + \tau^6}{120(1 - \tau^2)}\right]m_2^4 + O(m_2^5)\right\}, \tag{90}$$

$$N_\varphi = \frac{8A_0 m_2 \mu \sin\varphi}{\pi(2a)^{1/2}(3 - 2\tau^2)}\left\{1 + \left[\frac{1}{6} + \frac{(1 - \tau^2)\beta_0}{5(3 - 2\tau^2)}\right]m_2^2 + \frac{2i}{9\pi}\left[1 + \frac{(1 - \tau^2)\beta_1}{(3 - 2\tau^2)}\right]m_2^3 + \left[\frac{1}{120} - \frac{(1 - \tau^2)(\beta_0^2 - 100(3 - 2\tau^2)\beta_2)}{50(3 - 2\tau^2)^2} + \frac{(1 - \tau^2)\beta_0}{30(3 - 2\tau^2)}\right]m_2^4 + O(m_2^5)\right\}. \tag{91}$$

When $\omega \to 0$, that is $m_1 \to 0$ and $m_2 \to 0$, we recover the static intensity factors. These static factors can also be obtained from the analysis of Westmann [3], and given in [9]. Mal [4] has derived the terms of $O(m_2^2)$. However, due to a conceptual error in his Eqs. (28)–(29) and an algebraic error in his Eqs. (40), (42), (45), (48a) and (48b), his results are incorrect.

6. Far-field amplitudes and scattering cross-section. Let us use the same notation as in Sec. 3. Then the far-field amplitudes are defined by the formulas (47) where the values of A_1, A_2 and A_3 in the present case are

$$A_1(\theta) = m_1^2 \sin\theta\cos\theta A(\theta, m_1), \tag{92}$$

$$A_2(\theta) = -(i/2)m_2(\operatorname{cosec}\theta - 2\sin\theta)A(\theta, m_2), \tag{93}$$

$$A_3(\theta) = \cos\theta\operatorname{cosec}\theta B(\theta, m_2), \tag{94}$$

with

$$A(\theta, m_\alpha) = (\tfrac{1}{2}\pi m_\alpha \sin\theta)^{1/2}\int_0^1 v^{1/2}J_{1/2}(m_\alpha v\sin\theta)S_1(v)\,dv, \qquad \alpha = 1, 2, \tag{95}$$

and

$$B(\theta, m_2) = (\tfrac{1}{2}\pi m_2 \sin\theta)^{1/2}\int_0^1 v^{1/2}J_{1/2}(m_2 v\sin\theta)S_4(v)\,dv. \tag{96}$$

Thus, from relations (2), (3)–(5), (47) and (92)–(94) we find that the displacement field $\mathbf{u}$ has the following far-field behavior (in spherical polar coordinates):

$$u_r(r, \theta, \varphi) = im_1 A_1(\theta)\cos\varphi\,\frac{\exp(im_1 r)}{r} = im_1^3 \sin\theta\cos\theta\cos\varphi A(\theta, m_1)\,\frac{\exp(im_1 r)}{r} + O(r^{-2}),$$

$$u_\theta(r, \theta, \varphi) = -m_2^2 A_2(\theta)\sin\theta\cos\varphi\,\frac{\exp(im_2 r)}{r} = \frac{i}{2}m_2^3(1 - 2\sin^2\theta)\cos\varphi A(\theta, m_2)\,\frac{\exp(im_2 r)}{r} + O(r^{-2}), \tag{98}$$

$$u_\varphi(r, \theta, \varphi) = -im_2 A_3(\theta) \sin\theta \sin\varphi \frac{\exp(im_2 r)}{r}$$

$$= -im_2 \cos\theta \sin\varphi B(\theta, m_2) \frac{\exp(im_2 r)}{r} + O(r^{-2}) \quad (99)$$

as $r \to \infty$.

The next step is to put the values of the functions S_1 and S_4 from (84) and (85) into (95) and (96) and derive the values of $A(\theta, m_\alpha)$ and $B(\theta, m_2)$ as

$$A(\theta, m_\alpha) = \frac{8A_0}{3\pi m_2(3 - 2\tau^2)} \left\{1 - \frac{1}{10}\left[\frac{\beta_0}{(3 - 2\tau^2)} - \frac{1 + \tau^4}{1 + \tau^2}\right]m_2^2 + \frac{i}{45\pi}\left[\frac{(8\tau^5 + 7)}{(1 - \tau^2)} - \frac{5\beta_1}{(3 - 2\tau^2)}\right]m_2^3 - \frac{m_\alpha^2}{10}\sin^2\theta + O(m_2^4)\right\}, \quad (100)$$

$$B(\theta, m_2) = \frac{4A_0 m_2}{3\pi(3 - 2\tau^2)} \left\{1 + \frac{1}{5}\left[1 + \frac{(1 - \tau^2)\beta_0}{(3 - 2\tau^2)} - \frac{1}{2}\sin^2\theta\right]m_2^2 + \frac{2i}{9\pi}\left[1 + \frac{3(1 - \tau^2)\beta_1}{(3 - 2\tau^2)}\right]m_2^3 + O(m_2^4)\right\}. \quad (101)$$

Thereby, the far-field behavior of the displacement field is completely known.

The vector $\mathbf{h}(\theta, \varphi)$ corresponding to Eq. (55) for the present case is

$$\mathbf{h}(\theta, \varphi) = im_2[\tfrac{1}{2}m_2^2(1 - 2\sin^2\theta)\cos\varphi\, A(\theta, m_2)\hat{\boldsymbol{\theta}} - \cos\theta \sin\varphi B(\theta, m_2)\hat{\boldsymbol{\varphi}}]. \quad (102)$$

On substitution of the values of $A(\theta, m_2)$ and $B(\theta, m_2)$ from (100) and (101) in (102), we obtain (in physical units)

$$\Sigma_s = \frac{-4\pi a^2}{m_2 A_0} \operatorname{Re}[\mathbf{e}_1 \cdot \mathbf{h}]_{\theta=\varphi=0} = \frac{128 a^2 m_2^4}{135\pi}\left[\frac{(2\tau^5 + 3)}{(3 - 2\tau^2)^2} + O(m_2^2)\right]. \quad (103)$$

As far as the authors are aware, this formula has been derived for the first time.

Finally, we observe that we can find the values of the displacement and stress fields in the plane $z = 0$ from the above analysis, as explained in the solution of the previous problem. To ensure that the crack faces do not make contact during vibration we first find the value of $u_z(\rho, \varphi, z)$ on the crack faces. These values are

$$[u_z(\rho, \varphi, z)]_{z=0\pm} = -\frac{A_0 m_2 \tau^2 \rho \cos\varphi}{(3 - 2\tau^2)}\left\{\left(1 - \frac{\beta_0}{10(3 - 2\tau^2)} m_2^2 + \frac{(1 + \tau^4)}{8(1 - \tau^2)} m_2^2\right) - \frac{(1 + \tau^4)\rho^2}{32(1 - \tau^2)} m_2^2 + \frac{(1 + \tau^4)}{8\tau^2}(1 - \rho^2/4)m_2^2 + O(m_2^3)\right\}, \quad 0 \le \rho \le 1. \quad (104)$$

From this it follows that the crack faces would never make contact during vibration if we superpose on our solution the solution due to a static constant pressure $p^{(s)}$ applied on the crack, because this pressure gives rise to the following displacement at the crack faces [10]:

$$[u_z^{(s)}(\rho, \varphi, z)]_{z=0\pm} = \pm\frac{(1 - \rho^2)^{1/2} p^{(s)}}{\pi\mu(1 - \tau^2)}, \quad 0 \le \rho \le 1. \quad (105)$$

It is interesting to note that there is no minimum value of $p^{(s)}$ which ensures that the crack faces do not make contact during vibration; the only restriction on $p^{(s)}$ is that it is positive. Any constant static pressure, however small, will suffice to do what is needed.

From the above analysis we can also obtain the solution of the corresponding elastostatic problem when the prescribed stresses on the penny-shaped crack are

$$\tau_{\rho z} = S \cos \varphi, \quad \tau_{\varphi z} = -S \sin \varphi, \quad \tau_{zz} = 0, \quad z = 0, \quad 0 \leq \rho \leq 1, \tag{106}$$

and

$$u_\rho = u_\varphi = \tau_{zz} = 0, \qquad z = 0, \qquad \rho > 1, \tag{107}$$

where S is known. This follows when we let $m_2 \to 0$ and $A_0 \to \infty$ in such a manner that $A_0 m_2 \to S/\mu$; we have verified that the limiting formulas obtained in this way agree with those given by Westmann [3].

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