TWO-TIMING SOLUTION OF MATHIEU EQUATION TO SECOND ORDER*

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Introduction. We consider the Mathieu equation

$$v_{tt} + k^2 (1 + \epsilon \sin 2t) v = 0 \tag{1}$$

for small ϵ in the neighborhood of $k = 1$, which is the first positive instability region [1]. The solution in this region has been found to first order by Cole [2] by means of a two-timing perturbation. The method can be extended to higher orders for more accuracy, or as a test of the range of the method. In this paper the two-timing solution of the Mathieu equation is extended to second order and is found to differ in some interesting mathematical details from the first-order procedure.

First-order solution. Let $k^2 = 1 + \alpha\epsilon$, where α has any constant value. Then (1) becomes

$$v_{tt} + v + \epsilon(\alpha + \sin 2t)v + \epsilon^2(\alpha \sin 2t)v = 0. \tag{2}$$

Define the two-timing variables $\tau = \epsilon t$, $\gamma = \epsilon^2 t$, and let $v(t, \epsilon)$ be represented by a function of the slow and fast time scales $u(t, \tau, \gamma, \epsilon)$. Then the second derivative of v is replaced by

$$u_{tt} + 2\epsilon u_{t\tau} + \epsilon^2(u_{\tau\tau} + 2u_{t\gamma}) + 0(\epsilon^3).$$

Let u be expanded in a perturbation series in ϵ: $u(t, \tau, \gamma, \epsilon) = u_0(t, \tau, \gamma) + \epsilon u_1(t, \tau, \gamma) + \epsilon^2 u_2 + \cdots$. Substitute this series into the Mathieu equation and separate by powers of ϵ:

$$u_{0,tt} + u_0 = 0, \tag{3a}$$

$$u_{1,tt} + u_1 = -(2u_{0,t\tau} + \alpha u_0 + u_0 \sin 2t), \tag{3b}$$

$$u_{2,tt} + u_2 = -(2u_{1,t\tau} + \alpha u_1 + u_1 \sin 2t) - (2u_{0,t\gamma} + u_{0,\tau\tau} + \alpha u_0 \sin 2t), \tag{3c}$$

$$\cdots.$$

The zero-order solution found from (3a) is

$$u_0 = A_0(\tau, \gamma) \exp (it) + B_0(\tau, \gamma) \exp (-it). \tag{4}$$

We substitute this into the right-hand side of (3b) and suppress secular terms of the form $\exp (\pm it)$ in order to avoid growth of u_1 in t. To do this requires that the coefficients A_0 and B_0 satisfy

$$A_{0,\tau} - \tfrac{1}{4}B_0 - \tfrac{1}{2}i\alpha A_0 = 0, \qquad B_{0,\tau} - \tfrac{1}{4}A_0 + \tfrac{1}{2}i\alpha B_0 = 0. \tag{5}$$

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The solution of these two first-degree equations in τ is

$$\begin{aligned} A_0 &= a_1(\gamma) \exp(\lambda\tau) + a_2(\gamma) \exp(-\lambda\tau), \\ B_0 &= r_1 a_1(\gamma) \exp(\lambda\tau) + r_2 a_2(\gamma) \exp(-\lambda\tau) \end{aligned} \tag{6}$$

where $\lambda = \frac{1}{4}(1 - 4\alpha^2)^{1/2}$, $r_1 = 4(\lambda - \frac{1}{2}i\alpha)$, and $r_2 = -4(\lambda + \frac{1}{2}i\alpha)$. Thus as a function of t and τ (but not of γ) the complete first-order solution of Cole is given by (4) and (6). Using these in (3b) now determines the inhomogeneous part of u_1 :

$$u_1 = A_1(\tau, \gamma) \exp(it) + B_1(\tau, \gamma) \exp(-it) + \frac{1}{16i}(A_0 \exp(3it) - B_0 \exp(-3it)). \tag{7}$$

Second-order solution. The functions A_1 and B_1 must be determined from the next higher order equation. Proceeding in a similar manner as above, we suppress terms of the form $\exp(\pm it)$ in the right-hand side of (3c) by having the coefficients subject to the equations

$$\begin{aligned} A_{0,\gamma} - \tfrac{1}{2}i(\lambda^2 + \tfrac{1}{32})A_0 - \tfrac{1}{4}\alpha B_0 + \{A_{1,\tau} - \tfrac{1}{2}i\alpha A_1 - \tfrac{1}{4}B_1\} &= 0, \\ B_{0,\gamma} + \tfrac{1}{2}i(\lambda^2 + \tfrac{1}{32})B_0 - \tfrac{1}{4}\alpha A_0 + \{B_{1,\tau} + \tfrac{1}{2}i\alpha B_1 - \tfrac{1}{4}A_1\} &= 0. \end{aligned} \tag{8}$$

We have to determine both the dependence of A_0, B_0 on γ and the dependence of A_1, B_1 on τ from the two Eqs. (8). Assume that A_1 and B_1 have the same form of dependence on the slow variable as do A_0 and B_0, namely,

$$A_1 = c_1 \exp(\lambda\tau) + c_2 \exp(-\lambda\tau), \qquad B_1 = d_1 \exp(\lambda\tau) + d_2 \exp(-\lambda\tau). \tag{9}$$

The fact that Eqs. (8) are first order in γ indicates that all the coefficients that depend on γ have the same exponential form of dependence on γ, so we assume

$$\begin{aligned} a_1(\gamma) &= a_{11} \exp(\eta_1\gamma), \qquad & c_1(\gamma) &= c_{11} \exp(\eta_1\gamma), \\ a_2(\gamma) &= a_{22} \exp(\eta_2\gamma), \qquad & c_2(\gamma) &= c_{22} \exp(\eta_2\gamma). \end{aligned} \tag{10}$$

and similarly for d_1, d_2 where the η_i $(i = 1, 2)$ are yet to be determined.

It is clear that (8) really describes four conditions to satisfy because the four coefficients of $\exp(\eta_i\gamma \pm \lambda\tau)$ must be separately set equal to zero, which makes it possible to determine the four functions $A_1(\tau)$, $B_1(\tau)$, $A_0(\gamma)$ and $B_0(\gamma)$. Substituting the appropriate terms into (8) gives the equations

$$r_1 c_{11} - d_{11} = -4\{\eta_1 - \tfrac{1}{2}i(\lambda^2 + \tfrac{1}{32}) - \tfrac{1}{4}\alpha r_1\}a_{11}, \tag{11a}$$

$$c_{11} + r_2 d_{11} = 4\{\eta_1 r_1 + \tfrac{1}{2}i(\lambda^2 + \tfrac{1}{32})r_1 - \tfrac{1}{4}\alpha\}a_{11}, \tag{11b}$$

$$r_2 c_{22} - d_{22} = -4\{\eta_2 - \tfrac{1}{2}i(\lambda^2 + \tfrac{1}{32}) - \tfrac{1}{4}\alpha r_2\}a_{22}, \tag{11c}$$

$$c_{22} + r_1 d_{22} = 4\{\eta_2 r_2 + \tfrac{1}{2}i(\lambda^2 + \tfrac{1}{32})r_2 - \tfrac{1}{4}\alpha\}a_{22}. \tag{11d}$$

Note that we cannot simply calculate c_{11} and d_{11} as functions of a_{11} (or c_{22}, d_{22} as functions of a_{22}) because the determinant of the coefficients of the left-hand side is zero, since $r_1 r_2 = -1$. It is just this fact, however, which gives the values of the η_i. Eq. (11a) is a multiple of (11b), and Eq. (11c) is a multiple of (11d); by inspection,

$$-r_2(r_1 c_{11} - d_{11}) = (c_{11} + r_2 d_{11}), \qquad -r_1(r_2 c_{22} - d_{22}) = (c_{22} + r_1 d_{22}). \tag{12}$$

Solving for the η_i gives

$$\eta_1 = \frac{\alpha}{16\lambda}\left(\alpha^2 + \frac{5}{8}\right) = -\eta_2 \equiv \eta. \tag{13}$$

Now the terms d_{11} and d_{22} are determined as

$$d_{11} = r_1 c_{11} + r_3 a_{11}, \qquad d_{22} = r_2 c_{22} + r_4 a_{22}, \tag{14}$$

where

$$r_3 = 4[\eta - \tfrac{1}{2}i(\lambda^2 + \tfrac{1}{32}) - \tfrac{1}{4}\alpha r_1], \qquad r_4 = -4[\eta + \tfrac{1}{2}i(\lambda^2 + \tfrac{1}{32}) + \tfrac{1}{4}\alpha r_2].$$

This leaves two independent constants, a_{11} and a_{22}, to be determined by the two zero-order boundary conditions, and two more independent constants, c_{11} and c_{22}, to be determined by the first-order boundary conditions; all other terms have been found, and the total solution to second order in t is given by $u = u_0 + \epsilon u_1$.

Further remarks. It is clear that A_0, A_1 will occur only in the combination $A_0 + \epsilon A_1$, and similarly B_0, B_1 will occur only as $B_0 + \epsilon B_1$. Let $\phi = \lambda + \epsilon\eta$ and define the independent constants $P = a_{11} + \epsilon c_{11}$, $Q = a_{22} + \epsilon c_{22}$. Then the solution can be written in terms of Floquet theory as

$$v = P \exp(\epsilon\phi t)\left[\left(\exp(it) - \frac{i\epsilon}{16}\exp(3it)\right) + R_1\left(\exp(-it) + \frac{i\epsilon}{16}\exp(-3it)\right)\right]$$
$$+ Q \exp(-\epsilon\phi t)\left[\left(\exp(it) - \frac{i\epsilon}{16}\exp(3it)\right) + R_2\left(\exp(-it) + \frac{i\epsilon}{16}\exp(-3it)\right)\right] \tag{15}$$

where

$$R_1 = r_1 + \epsilon r_3 = 4(1 - \epsilon\alpha)(\phi - \tfrac{1}{2}i\alpha) - 2i\epsilon(\lambda^2 + \tfrac{1}{32}),$$
$$R_2 = r_2 + \epsilon r_4 = -4(1 - \epsilon\alpha)(\phi + \tfrac{1}{2}i\alpha) + 2i\epsilon(\lambda^2 + \tfrac{1}{32}).$$

Some terms of order ϵ^2 which u_2 may be expected to supply are included in (15) without altering the accuracy to second order.

The usual transition curve between the regions of stability and instability near $k^2 = 1$ is located wherever ϕ crosses from real to imaginary values. To first order this occurs when $\lambda = 0$, at $\alpha = \pm\frac{1}{2}$. To second order, however, we must determine the value of α that satisfies

$$\phi = \frac{1}{\lambda}\left[\lambda^2 + \frac{\epsilon\alpha}{16}\left(\alpha^2 + \frac{5}{8}\right)\right] = 0. \tag{16}$$

Assuming that $\alpha = \pm(\frac{1}{2} + \epsilon\delta)$, we find that $\delta = 7/64$, and the transition boundaries lie at

$$k^2 = 1 + \epsilon\alpha = 1 \pm \tfrac{1}{2}\epsilon\left(1 + \frac{7\epsilon}{32}\right) + \cdots.$$

References

[1] J. J. Stoker, *Nonlinear vibrations*, Interscience Publishers, Inc., N. Y., 1950
[2] J. D. Cole, *Perturbation methods in applied mathematics*, Blaisdell Publishing Co., 1968