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DYNAMIC BUCKLING OF AN IMPERFECT COLUMN ON NONLINEAR FOUNDATION*

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Introduction. In this paper a formal two-time perturbation expansion (see, e.g., [1, 2]) is used to study the dynamic response and buckling of a model imperfection-sensitive structure. We consider a finite imperfect column resting on nonlinear elastic foundation subject to step loading. A simple expression is obtained for the dynamic buckling load for small imperfections and small initial conditions. It is found that the effect of one Fourier coefficient in the expansion of the imperfection dominates in the asymptotic expression for the deflection and the dynamic buckling load. The static buckling of this structure has been studied extensively [3, 4, 5].

A number of related dynamic response and buckling analyses are noteworthy. Budiansky and Hutchinson [6, 7, 8] formulated a general approximate theory of dynamic buckling by an extension of Koiter's static theory of post-buckling behavior [9]. In these theories the imperfections are assumed to be in the shape of the classical buckling mode. With such a restriction this problem can be handled by the theory of Budiansky and Hutchinson. In this analysis this restriction on the imperfection need not be made. The formal two-time procedure *shows* that imperfections in the shape of the classical buckling mode produce the greatest degradation in the dynamic buckling strength of the structure.

The multi-time method had been used by Danielson [10]. This study, however, was for a two-degree-of-freedom system with application to spherical shells. In a more recent paper [11] asymptotic methods are used to study the time evolution of the response of an imperfection-insensitive structure.

Differential equation. The nondimensional form of the relevant equations for the lateral displacement $w(x, t)$ of a column supported laterally by continuous elastic foundations is

$$w_{tt} + w_{xxxx} + 2\lambda w_{xx} + w - \alpha w^3 = -2\lambda\epsilon\bar{w}_{xx}, \qquad 0 < x < \pi, \quad t > 0, \tag{1}$$

$$w = w_{xx} = 0 \quad \text{for} \quad x = 0, \pi, \qquad t \geq 0, \tag{2}$$

$$w(x, 0) = \epsilon f(x), \qquad w_t(x, 0) = \epsilon g(x), \qquad 0 < x < \pi, \tag{3}$$

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where $(\ )_x \equiv (\partial/\partial x)(\ )$, $(\ )_t \equiv (\partial/\partial t)(\ )$, $|\alpha| = 1$. The nondimensional axial coordinate x, additional lateral displacement w, axial load parameter λ, imperfection (stress-free initial displacement) $\bar{w}$, and time t are related to the corresponding physical quantities by

$$x = (k_1/EI)^{1/4}X, \qquad w = (k_3/k_1)^{1/2}W, \qquad \lambda = P/2(EIk_1)^{1/2},$$
$$\epsilon\bar{w} = (k_3/k_1)^{1/2}\bar{W}, \qquad t = (k_1/m)^{1/2}T.$$

We have neglected nonlinear geometric effects and axial inertia. ϵ is a small parameter. We choose to consider only nondimensional displacements and velocities of the same order as the imperfection; hence the form of Eq. (3). EI is the bending stiffness of the column. P is the magnitude of the axial step-loading applied at time $T = 0$. The lateral deflection W of the column is restrained by a continuous elastic foundation that produces a nonlinear restraining force per unit length of $k_1W - k_3W^3$, with $k_1 > 0$. $\alpha = 1$ $(k_3 > 0)$ corresponds to a softening foundation. m is the mass per unit length of the column. $\bar{w}$, f, and g are $O(1)$.

The lowest eigenvalue (classical buckling load) for the perfect ($\bar{w} \equiv 0$) time-independent problem is $\lambda = 1$ and the corresponding eigenfunction is $w = \sin x$. The length of the column is taken to be half the wavelength of the classical buckling mode.

The problem consists of determining the deflection of the column for small imperfections and $|1 - \lambda|$ small. We also seek the dynamic buckling load λ_D and its dependence on the imperfection and initial conditions.

Static theory. For the time-independent problem Eqs. (1) – (3) with $(\ )' \equiv d(\ )/dx$ become

$$w'''' + 2\lambda w'' + w - \alpha w^3 = -2\lambda\epsilon\bar{w}'', \quad 0 < x < \pi, \tag{4}$$

$$w = w'' = 0 \quad \text{at} \quad x = 0, \pi. \tag{5}$$

As shown in [4], a perturbation in a load parameter is appropriate for this problem. For convenience we reiterate the salient points. We let $\lambda = 1 - \delta^2/2$ and expand w and $\lambda\epsilon$ in power series in δ; thus

$$\begin{aligned} w(x) &= \sum_{n=1}^{\infty} \delta^n w_n(x), \\ \lambda\epsilon &= \sum_{n=1}^{\infty} \delta^n \mu_n . \end{aligned} \tag{6}$$

We note that although ϵ and δ are in general independent parameters they are, however, related on the stability boundaries (see, e.g., [12]) which we seek. The imperfection $\bar{w}(x)$ can be expanded in a Fourier sine series, namely

$$\bar{w}(x) = \sum_{n=1}^{\infty} \bar{a}_n \sin nx. \tag{7}$$

Substituting the expansions (6) into (4) and equating corresponding powers of δ gives the following sequence of equations:

$$\begin{aligned} Lw_1 &\equiv w_1'''' + 2w_1'' + w_1 = -2\mu_1\bar{w}'', \\ Lw_2 &= -2\mu_2\bar{w}'', \\ Lw_3 &= -2\mu_3\bar{w}'' + w_1'' + \alpha w_1^3 , \end{aligned}$$

etc. The boundary conditions (5) become $w_j = w_j'' = 0$ at $x = 0, \pi, j = 1, 2, \cdots$. Since for $\bar{w} \equiv 0$ the solution for w_1 is $w_1 = \sin x$, the existence of solution for w_1 requires that $\mu_1\bar{a}_1 = 0$. Now the imperfection in the shape of the classical buckling mode produces the greatest reduction in the strength of the column [4]. We retain this term in the Fourier expansion of $\bar{w}$. Thus $\bar{a}_1 \neq 0$; hence

$$\mu_1 = 0 \quad \text{and} \quad w_1 = a_1 \sin x.$$

Similarly $\mu_2 = 0$, $w_2 = a_2 \sin x$ and

$$Lw_3 \equiv 2\mu_3 \sum_{n=1}^{\infty} n^2\bar{a}_n \sin nx - a_1 \sin x + \tfrac{3}{4}\alpha a_1^3 \sin x - \tfrac{1}{4}\alpha a_1^3 \sin 3x.$$

For the existence of a solution for w_3,

$$a_1 - \tfrac{3}{4}\alpha a_1^3 = 2\mu_3\bar{a}_1 .$$

Substituting these results for μ_j in the expansion (6) gives

$$\lambda\epsilon\bar{a}_1 = 2^{1/2}(1 - \lambda)^{3/2}(a_1 - \tfrac{3}{4}\alpha a_1^3) + 0(\delta^4).$$

For $\alpha = -1$, the column is imperfection-insensitive and can support loads in excess of the classical buckling load $\lambda = 1$. For $\alpha = 1$ the column is imperfection-sensitive and an expression for the static buckling load λ_s is obtained by maximizing λ with respect to a_1. Thus

$$(1 - \lambda_s)^{3/2} = \frac{9}{4\sqrt{2}} \lambda_s |\epsilon\bar{a}_1| \tag{8}$$

where

$$\bar{a}_1 = \frac{2}{\pi} \int_0^{\pi} \bar{w}(x) \sin x \, dx. \tag{9}$$

For $\alpha > 0$, the right-hand side of Eq. (8) is multiplied by $\alpha^{1/2}$. The formula gives the same value, $\lambda_s = 0.81$, obtained numerically in [3] for $\alpha = 0.1$ and $\epsilon\bar{a}_n = 0.2$ for all n.

Dynamic theory. As for the static theory we use δ as the perturbation parameter. We assume that the dynamic problem depends on two time scales, t and $\tau \equiv \delta t$, which describe the short- and long-time behavior of the solution respectively.

We write $w(x, t) \equiv u(x, t, \tau; \delta)$ and note that $w_t = u_t + \delta u_\tau$. Now expand u and ϵ in power series in δ, namely

$$u(x, t, \tau; \delta) = \sum_{n=1}^{\infty} u^{(n)}(x, t, \tau)\, \delta^n, \qquad \lambda\epsilon = \sum_{n=1}^{\infty} \epsilon^{(n)}\, \delta^n. \tag{10}$$

Substituting these into Eqs. (1)–(3) and assuming termwise differentiation of the series gives the following equations:

$$Mu^{(1)} \equiv u_{tt}^{(1)} + u_{xxxx}^{(1)} + 2u_{xx}^{(1)} + u^{(1)} = -2\epsilon^{(1)}\bar{w}_{xx}, \tag{11}$$

$$Mu^{(2)} = -2\epsilon^{(2)}\bar{w}_{xx} - 2u_{t\tau}^{(1)}, \tag{12}$$

$$Mu^{(3)} = -2\epsilon^{(3)}\bar{w}_{xx} - 2u_{t\tau}^{(2)} - u_{\tau\tau}^{(1)} + u_{xx}^{(1)} + \alpha(u^{(1)})^3, \tag{13}$$

etc.;

$$
\begin{aligned}
&u^{(j)} = u^{(j)}_{xx} = 0 \quad \text{at} \quad x = 0, \pi, \\
&u^{(j)}(x, 0, 0) = b^{(j)} f(x), \\
&u^{(1)}_t(x, 0, 0) = b^{(1)} g(x), \\
&u^{(j+1)}_t(x, 0, 0) + u^{(j)}_\tau(x, 0, 0) = b^{(j+1)} g(x), \qquad j = 1, 2, \cdots
\end{aligned}
\tag{14}
$$

where

$$
b^{(j)} = \sum_{m=1}^{j} \epsilon^{(m)} [1 + (-1)^{j-m}] 2^{(m-j-2)/2}.
$$

We now seek necessary conditions for the existence of bounded solutions to Eqs. (10)–(14). Consider the following problem:

$$
\begin{aligned}
MV &= h(x, t), \qquad 0 < x < \pi, \quad t > 0, \\
V &= V_{xx} = 0 \quad \text{at} \quad x = 0, \pi, \qquad t > 0, \\
V &= V_t = 0 \quad \text{at} \quad t = 0, \qquad 0 < x < \pi.
\end{aligned}
$$

A solution exists in the form

$$
V(x, t) = \sum_{n=1}^{\infty} V_n(t) \sin nx
$$

if

$$
V_n'' + (n^2 - 1)^2 V_n = \frac{2}{\pi} \int_0^\pi h(x, t) \sin nx \, dx.
$$

For a bounded solution to this problem the inhomogeneous term must be orthogonal to the solutions of the corresponding homogeneous problem $h \equiv 0$. That is,

$$
\begin{aligned}
\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^\pi h(x, t) \sin nx \sin\, (n^2 - 1)t \, dx \, dt &= 0, \\
\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^\pi h(x, t) \sin nx \cos\, (n^2 - 1)t \, dx \, dt &= 0.
\end{aligned}
\tag{15}
$$

Thus the use of the Fourier series (7) and the conditions (15) in Eq. (11) gives the condition

$$
\epsilon^{(1)} \bar{a}_1 = 0.
$$

As noted before, we want $\bar{a}_1 \neq 0$; thus $\epsilon^{(1)} = 0$ and Eqs. (11) and (14) for $u^{(1)}$ become

$$
\begin{aligned}
Mu^{(1)} &= 0, \qquad 0 < x < \pi, \quad t > 0, \\
u^{(1)} &= u^{(1)}_{xx} = 0 \quad \text{at} \quad x = 0, \pi, \qquad t > 0, \\
u^{(1)}(x, 0, 0) &= u^{(1)}_t(x, 0, 0) = 0.
\end{aligned}
$$

Let

$$
u^{(1)}(x, t, \tau) = \sum_{n=1}^{\infty} u^{(1)}_n(t, \tau) \sin nx; \tag{16}
$$

then

$$u^{(1)}_{n,tt} + (n^2 - 1)^2 u^{(1)}_n = 0,$$
$$u^{(1)}_n(0, 0) = u^{(1)}_{n,t}(0, 0) = 0.$$

For $n = 1$, the bounded solution is

$$u^{(1)}_1 = B^{(1)}_1(\tau) \tag{17}$$

with

$$B^{(1)}_1(O) = 0. \tag{18}$$

For $n \neq 1$,

$$u^{(1)}_n = A^{(1)}_n(\tau) \sin (n^2 - 1)t + B^{(1)}_n(\tau) \cos (n^2 - 1)t$$

with $B^{(1)}_n(O) = 0$, $A^{(1)}_n(O) = 0$. Thus, by (16),

$$^{(1)}(x, t, \tau) = B^{(1)}_1(\tau) \sin x + \sum_{n=2}^{\infty} [A^{(1)}_n(\tau) \sin (n^2 - 1)t + B^{(1)}_n(\tau) \cos (n^2 - 1)t] \sin nx.$$

Substituting for $u^{(1)}$ in (12) and using (7) and the boundedness conditions leads to

$$\epsilon^{(2)} = 0, \qquad A^{(1)}_{n,\tau}(\tau) = 0, \qquad B^{(1)}_{n,\tau}(\tau) = 0, \qquad n \geq 2.$$

Use of the initial conditions for $A^{(1)}_n$ and $B^{(1)}_n$ gives

$$A^{(1)}_n = B^{(1)}_n \equiv 0, \qquad u^{(1)}(x, t, \tau) = B^{(1)}_1(\tau) \sin x. \tag{19}$$

The resulting equations for $u^{(2)}$ are

$$Mu^{(2)} = 0, \qquad 0 < x < \pi, \quad t > 0,$$
$$u^{(2)} = u^{(2)}_{xx} = 0 \quad \text{at} \quad x = 0, \pi, \qquad t > 0,$$
$$u^{(2)}_t(x, 0, 0) = 0,$$
$$u^{(2)}_t(x, 0, 0) + u^{(1)}_\tau(x, 0, 0) = 0.$$

Let

$$u^{(2)}(x, t, \tau) = \sum_{n=1}^{\infty} u^{(2)}_n(t, \tau) \sin nx; \tag{20}$$

then

$$u^{(2)}_{n,tt} + (n^2 - 1)^2 u^{(2)}_n = 0,$$
$$u^{(2)}_n(0, 0) = u^{(2)}_{n,t}(0, 0) = 0.$$

For $n = 1$, the bounded solution is

$$u^{(2)} = B^{(1)}_1(\tau).$$

The initial conditions reduce to

$$B^{(2)}_1(0) = 0, \quad B^{(1)}_{1,\tau}(0) = 0. \tag{21}$$

For $n \neq 1$,

$$u^{(2)}_n = A^{(2)}_n(\tau) \sin (n^2 - 1)t + B^{(2)}_n(\tau) \cos (n^2 - 1)t$$

with $A_n^{(2)}(0) = B_n^{(2)}(0) = 0$. Thus

$$u^{(2)}(x, t, \tau) = B_1^{(2)}(\tau) \sin x + \sum_{n=2}^{\infty} [A_n^{(2)}(\tau) \sin (n^2 - 1)t + B_n^{(2)}(\tau) \cos (n^2 - 1)t] \sin nx.$$

We substitute these results for $u^{(1)}$ and $u^{(2)}$ into (13) and use the boundedness condition to obtain

$$A_{n,\tau}^{(2)}(\tau) = 0, \quad B_{n,\tau}^{(2)}(\tau) = 0 \quad \text{for} \quad n \geq 2,$$
$$B_{1,\tau\tau}^{(1)} + B_1^{(1)} - \tfrac{3}{4}\alpha(B_1^{(1)})^3 = 2\epsilon^{(3)}\bar{a}_1 . \tag{22}$$

It is seen that $A_n^{(2)} = B_n^{(2)} \equiv 0$. This procedure may be continued to obtain $u^{(3)}$. Our primary interest is in the analysis of the amplitude equation (22) to determine the dynamic buckling load. For convenience we let

$$B(\tau) \equiv B_1^{(1)}(\tau), \quad B' = dB/d\tau, \quad c \equiv 2\epsilon^{(3)}\bar{a}_1 .$$

The amplitude equation (22) becomes

$$B'' + B - \tfrac{3}{4}\alpha B^3 = c, \quad \tau > 0, \tag{23}$$

and from Eqs. (18) and (21)

$$B(0) = B'(0) = 0. \tag{24}$$

Exact solutions to Eqs. (23) and (24) can be obtained in terms of elliptic functions (see [13]). It is, however, more convenient and adequate for this investigation to seek qualitative features of the solution by phase plane analysis.

We multiply Eq. (23) by B' and integrate to obtain the equation for the integral curve:

$$(B')^2 + B^2 - \tfrac{3}{8}\alpha B^4 = 2cB. \tag{25}$$

We note from Eq. (10) and the fact that $\epsilon^{(1)} = \epsilon^{(2)} = 0$ that $\lambda\epsilon = \epsilon^{(3)}\delta^3 + 0(\delta^4)$; thus

$$c \sim [2(1 - \lambda)]^{-3/2} 2\lambda\epsilon\bar{a}_1 \quad \text{for} \quad \lambda \to 1^-. \tag{26}$$

A study of the phase plane (see Figs. 1 and 2) reveals that for $\alpha = -1$ the solutions to (23) and (24) are bounded and periodic. For $\alpha = 1$ and $|c| \leq 2\sqrt{2}/9$ Eqs. (23) and (24) have bounded solutions for all τ, the solution being periodic for $|c| < 2\sqrt{2}/9$ and $(B, B') \to (2\sqrt{2}/3, 0)$ as $\tau \to \infty$ for $|c| = 2\sqrt{2}/9$. On the other hand, for $\alpha = 1$ and $|c| > 2\sqrt{2}/9$, the solutions are unbounded as $\tau \to \infty$. Since c is a monotone increasing function of λ for $0 < \lambda < 1$ there exists a value of λ, say λ_D, which possesses the following property. For $0 < \lambda < \lambda_D$ there exist bounded periodic solutions to (23) and (24) and hence to the nonlinear dynamic problem, while for $\lambda_D < \lambda < 1$ the deflection of the column is unbounded. We call λ_D the dynamic buckling load. This criterion of unboundedness which has to be modified if damping is introduced is equivalent to the definition and criterion of dynamic stability formulated and discussed extensively in [14] by Hsu.

The use of the critical value of c for $\alpha = 1$ in Eq. (26) gives a simple asymptotic expression for λ_D, namely

$$(1 - \lambda_D)^{3/2} = \tfrac{9}{4}\lambda_D |\epsilon\bar{a}_1|, \tag{27}$$

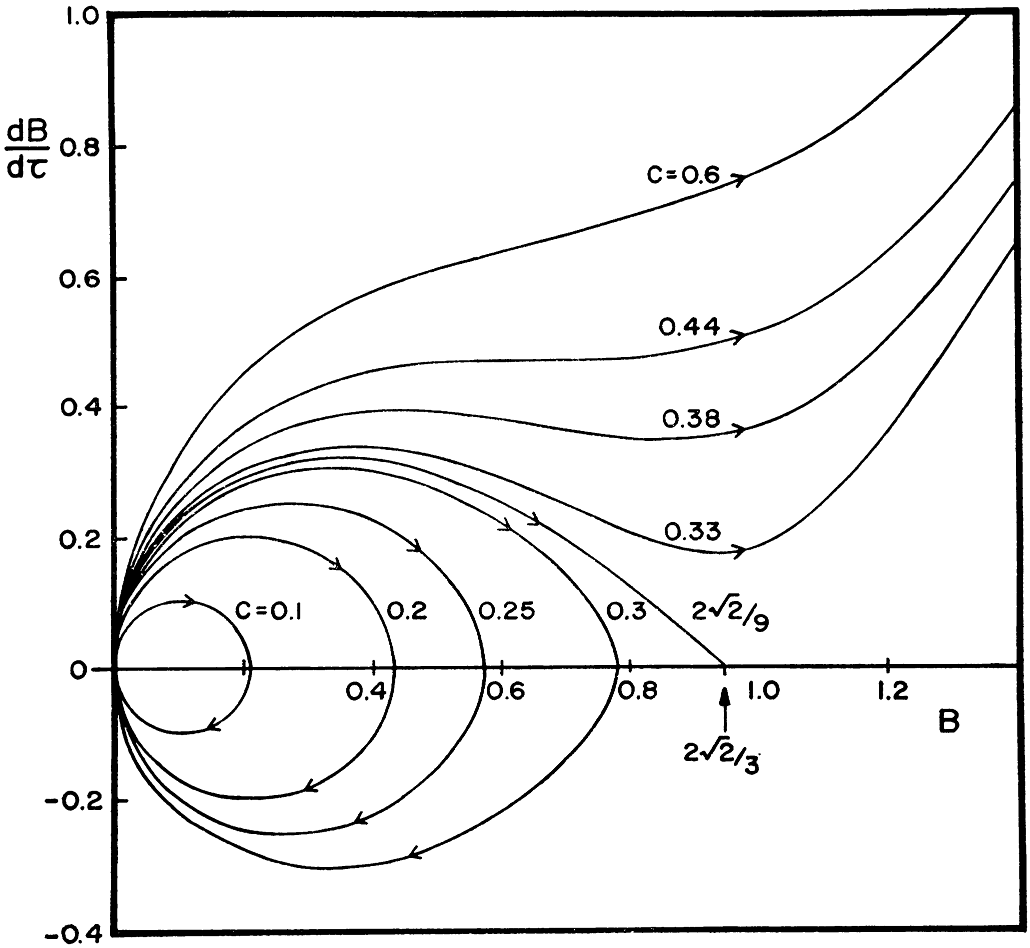

FIG. 1. Phase plane for imperfection-sensitive case, $\alpha = 1$ (Eq. (25)).

where $\bar{a}_1$ is defined in Eq. (9). As noted in [6–8], a simple relation between λ_D and λ_s may be obtained by eliminating $\epsilon\bar{a}_1$ from Eqs. (8) and (27), namely

$$((1 - \lambda_D)/(1 - \lambda_s))^{3/2} = \sqrt{2}\lambda_D/\lambda_s\,. \tag{28}$$

We note that Eq. (27) and hence (28) will have to be modified if nondimensional initial conditions of order lower than that of the imperfection are imposed.

For $\lambda = \lambda_D$, the amplitude equation can be solved to obtain

$$B(\tau) = 4\sqrt{2}/(3[3\coth^2(9\tau/4\sqrt{2}) - 1]).$$

The dominant term for the deflection as $\epsilon \to 0$, $(\lambda_D \to 1^-)$ is

$$w(x, t) = \frac{8(1 - \lambda_D)^{1/2}\sin x}{3[3\coth^2(9(1 - \lambda_D)^{1/2}t/4) - 1]} \tag{29}$$

We note that the initial displacement and velocity, if restricted to the same order of magnitude as the imperfection, do not affect the dominant term in the deflection

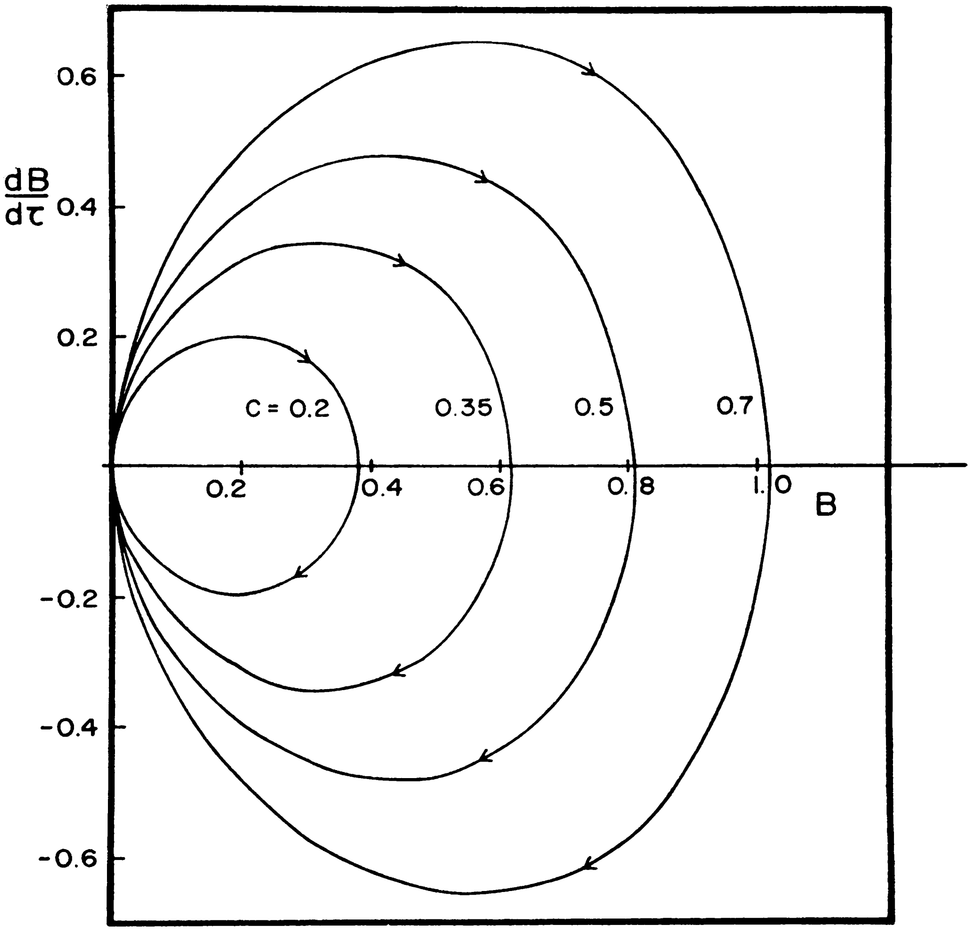

FIG. 2. Phase plane for imperfection-insensitive case, $\alpha = -1$.

or the first significant term in the asymptotic expansion of the dynamic buckling load λ_D . Asymptotically the dynamic buckling load and deflection depend only on one Fourier coefficient $\bar{a}_1$ of $\bar{w}(x)$ corresponding to imperfection in the shape of the classical buckling mode.

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