BOUNDARY CONDITIONS AND STABILITY OF INVISCID PLANE-PARALLEL FLOWS*

BY

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Abstract. The problem of describing the manner in which disturbances in an incompressible flow propagate downstream, when the infinite speed of signal transmission would make it theoretically possible for the disturbances to travel upstream also, and indeed would make upstream values dependent on boundary conditions imposed at arbitrarily great distances downstream, arises in numerous cases of physical interest. One such case is the growth of perturbations on plane-parallel flows in long channels, where the nature of the perturbation at any particular location is "quasi-steady" in time, although the perturbation grows with distance downstream. Another such case arises with the inevitable numerical disturbances in numerical "experiments" with computers, where an infelicitous choice of downstream boundary conditions may lead to results wholly at variance with observations. The purpose of this paper is to indicate how the choice of boundary conditions can drastically influence the phenomena observed, and to select those boundary conditions most consistent with reality. Our ideas are applied in some detail to the description of the linear theory of growth of perturbations on a two-dimensional inviscid plane-parallel basic flow. In particular, we show that the stability or instability of the flow depends on the nature of the boundary conditions imposed, and we find boundary conditions whose application leads to results in conformity with actual experimental situations.

1. Introduction. The linear theory of two-dimensional disturbances in a plane-parallel, two-dimensional flow of inviscid, incompressible fluid—to which we restrict ourselves in this paper—was first given a mathematical treatment by Rayleigh [1]. In that work the assumption was introduced that the disturbance had the form of a "wave", exp $(i\alpha(x - ct)$, where x is the spatial coordinate in the direction of the basic flow, t is the time, α is a real constant, and c is a complex constant. This assumption has characterized most subsequent work in the field. In this paper we shall treat the evolution of disturbances, in the linear approximation, in the context of an initial- and boundary-value problem. Our purpose in doing this is not to solve any specific problem, but, first, to show that the choice of the boundary conditions to be imposed has a profound influence on the growth of disturbances in space and time and, second, to exploit this dependency and find boundary conditions under the influence of which the growth of disturbances in time is limited, as appears to be the case in a number of experimental situations.

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Note that it suffices, within the framework of a linear treatment of the growth of disturbances, to consider an initial-value problem with homogeneous boundary conditions, since if the boundary conditions are not homogeneous we can always subtract from the perturbation a quantity which satisfies the inhomogeneous boundary conditions but not the governing partial differential equations, and the difference between the perturbation and that quantity satisfies the homogeneous boundary conditions and has a source term in the partial differential equations governing it. The contributions of this source term at a time t can be represented by the integral over t_0, from 0 to t, of the contributions from a set of "initial" disturbances at "initial" times t_0. In accordance with this fact, it is clear that the linear problem may be analyzed in terms of solutions satisfying many different sets of homogeneous boundary conditions. One may then ask, in view of this freedom, why we go to such trouble to find "suitable" boundary conditions. The reason is that it is important, particularly in a stability analysis where we want to examine regions of growth of disturbances in space-time, to choose boundary conditions such that deviations from them are uniformly small over the space-time region in question. For, should the deviations not be uniformly small, our disturbance would be expressed as a superposition of quantities which were not uniformly small, and little can be inferred about such a superposition.

At first, we shall be especially interested in two sets of initial and boundary conditions. The first set, similar to the initial and boundary conditions customarily used to study viscous flow, entails the specification of an initial divergenceless velocity field, the normal component of velocity on the boundary of the region, and the tangential component of velocity on that portion of the boundary where fluid enters the region. In the second set, originally proposed by Louis N. Howard, we prescribe the initial vorticity field, the normal component of velocity on the boundary of the region, and the vorticity on that portion of the boundary where fluid enters the region. One reason for considering these boundary conditions is that the Euler equations have been shown to have a unique solution when they are imposed [2]. We shall then consider variations on these two sets of initial and boundary conditions.

We study perturbations on a basic flow with velocity $U(y)\mathbf{i}$ in the rectangle R: $0 \leq x \leq L$ and $0 \leq y \leq D$. The limit $L \to \infty$ will be of special interest. We shall let $U_0 > 0$ be the minimum of $U(y)$ for $0 \leq y \leq D$ and U_1 be the maximum of $U(y)$ for $0 \leq y \leq D$. Derivatives of U with respect to y will be denoted by primes, and otherwise subscripts x, y, or t on quantities will denote derivatives with respect to those variables. We shall assume that U'' exists for $0 \leq y \leq D$. Let u and v denote the x- and y-components of the perturbation velocity, ω the perturbation of the vorticity, and ψ the perturbation of the stream function. p will be the pressure perturbation. Then the perturbation satisfies the kinematical relations

$$u = \psi_y , \qquad v = -\psi_x , \quad \text{and} \quad \omega = -\Delta\psi\left(\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{1.1}$$

and also the familiar dynamical relations

$$\begin{aligned} u_t + Uu_x + uu_x + vU' + vu_y &= -p_x , \\ v_t + Uv_x + uv_x + vv_y &= -p_y , \end{aligned} \tag{1.2}$$

or equivalently

$$\omega_t + U\omega_x + u\omega_x - vU'' + v\omega_y = 0. \tag{1.3}$$

It is customary to linearize these equations by keeping only first-order terms in the perturbation, so that (1.2) and (1.3) are replaced by

$$u_t + Uu_x + vU' = -p_x ,$$

$$v_t + Uv_x = -p_y , \tag{1.2'}$$

$$\omega_t + U\omega_x - vU'' = 0. \tag{1.3'}$$

Since we are particularly interested in flows with the normal component of velocity prescribed on the boundary, when we express ψ in terms of ω we will have need for the Green's function

$$G_0(x, y; x_0 , y_0) = -\frac{2}{D}\sum_{n=1}^{\infty}\frac{\sin(n\pi y/D)\sin(n\pi y_0/D)}{(n\pi/D)\sinh(n\pi L/D)}\sinh\frac{n\pi x_<}{D}\sinh\frac{n\pi(L - x_>)}{D}, \tag{1.4}$$

$$x_< = \min(x, x_0), \qquad x_> = \max(x, x_0). \tag{1.5}$$

The limiting form of the Green's function as $L \to \infty$ is

$$G_0^{(\infty)}(x, y; x_0 , y_0) = -\frac{2}{D}\sum_{n=1}^{\infty}\frac{\sin(n\pi y/D)\sin(n\pi y_0/D)}{(n\pi/D)}\sinh\frac{n\pi x_<}{L}\exp\left(-\frac{n\pi x_>}{D}\right). \tag{1.6}$$

(Actually, existence and uniqueness was proved [2] only for flows in bounded domains whose boundaries had Hölder-continuous curvature. However, the extension of such theorems to regions with corners, like that currently under consideration, poses no difficulty. And, because of the nature of G_0 , which behaves like $\exp(-(\pi/D)|x - x_0|)$ when $|x - x_0|$ is large compared to D, the constructions used in the proofs of existence and uniqueness can be shown to converge uniformly in L.)

The procedure adopted by Rayleigh to solve Eqs. (1.1) and (1.3′) was to assume a solution in the form of a "wave", $\psi = \psi_1 \exp(i(\alpha x - \beta t))$. Here ψ_1 is a function of y only. If this form is substituted into the equations for the perturbation, we get the familiar Rayleigh equation

$$\left(U - \frac{\beta}{\alpha}\right)\left(\frac{d^2\psi_1}{dy^2} - \alpha^2\psi_1\right) - U''\psi_1 = 0. \tag{1.7}$$

In Rayleigh's formulation, α is real but β may be complex. In fact, Eq. (1.7) determines β as a function of α through the eigenvalue equation obtained by appending the conditions

$$\psi_1(0) = \psi_1(D) = 0. \tag{1.8}$$

If the imaginary part of β is positive for some real α, one concludes that the flow is unstable; otherwise it is stable. So, if the basic flow is unstable, we deduce that, at least as long as the equations linear in the disturbance are valid, that disturbance will grow with the passage of time. This interpretation has been criticized by Watson [3] on the ground that in actual experiments the amplitude of the disturbance is "quasi-steady" in time, but increases exponentially with distance downstream (in the case of "instability"). Watson's solution to the problem was to assume a solution of the same form as that assumed by Rayleigh, but with the difference that β is real and α may be complex. Eqs. (1.7) and (1.8) still hold in this case, but they are now to be construed as determining α as a function of β. If, for some real β, the imaginary part of α is negative, the basic flow is interpreted as unstable; otherwise it is stable.

There is some merit to Watson's approach, especially when one considers actual experiments like that of Browand [4]. For it is true that there is no time limitation to such experiments, as would be the case if the disturbance increased indefinitely in time. In other words, no matter how long the experiment has run, we would expect to be able to make measurements near the source of the disturbance and find the perturbation there to be "small". On the other hand, the form of Watson's assumed disturbance has some consequences which may be considered even graver than those deriving from Rayleigh's assumption. This is especially true if we consider fluid motions taking place in a "long" rectangle (that is, L large compared to D), the most common experimental situation. For suppose the assumed disturbance increases with x faster than $\exp((\pi/D)x)$. If we still assume that the normal velocity of the flow at $x = L$ is fixed, we shall find, because G_0 falls off no faster than $\exp((-\pi/D)\,|x - x_0|)$, that the bulk of the perturbation velocity near $x = 0$ comes from perturbation vorticity near $x = L$. Not only that, but the disturbance velocity near $x = 0$ will be strongly dependent on our choice of L, that is, where we require the normal velocity to be fixed. This strongly contradicts our intuition of what a reasonable experimental situation should be, and it prevents our passage to the limit $L \to \infty$, as would be reasonable for most such experiments. Indeed, Watson's assumed form contradicts our most elementary notions of causality, which admit the possibility of describing the evolution in time of any mechanical system by means of an initial-value problem. Now, perhaps causality should be thrown out, because it is obvious that a major culprit here is the assumption of incompressibility, with its attendant infinite speed of signal propagation. Perhaps we can hope to eliminate the problem by considering a fluid with a finite, but large, sound speed. But in fact this will not work, for we are dealing with a situation where effects really can be propagated upstream (and downstream) with a velocity much greater than the flow velocity. In Browand's experiment, the source of the disturbance is actually a loudspeaker placed downstream from the region where the disturbance is studied. Thus, we seem to arrive at a dilemma with either Rayleigh's or Watson's assumed form—with Rayleigh's when $t \to \infty$ and with Watson's when $L \to \infty$.

Even if neither of the assumed forms entailed any difficulties, it is clear that there is generally a considerable difference between exponential growth in space and exponential growth in time, and it would certainly behoove us to study the evolution of the disturbance as an initial- and boundary-value problem, whose solution is uniquely determined, and thereby remove any ambiguity. Nevertheless, the assumed forms of Rayleigh and Watson lead to results which, when correctly interpreted, have ample experimental confirmation. Thus, there must be some element of truth underlying these assumptions. We hope in this paper to get at least a little better understanding of what that element of truth is. We also expect to indicate how to get a more realistic first-order perturbation, to provide us with a sounder base for a nonlinear treatment of the stability problem. In passing, let us make an observation about the disturbance, using either the Rayleigh or Watson form. Besides the expression for ψ, let us write

$$\omega = \omega_1(y)\exp(i(\alpha x - \beta t)), \qquad v = v_1(y)\exp(i(\alpha x - \beta t)). \tag{1.9}$$

We obtain immediately

$$v_1(y) = -i\alpha \int_0^D H(y, y_0)\omega_1(y_0)\,dy_0\,, \tag{1.10}$$

where

$$H(y, y_0) = \frac{\sinh \alpha y_< \sinh \alpha(D - y_>)}{\alpha \sinh \alpha D} \tag{1.11}$$

and $y_< = \min(y, y_0)$, $y_> = \max(y, y_0)$. Thus, it is apparent that the term v, which occurs in Eq. (1.3′) for ω, depends only on the local (in x) values of ω.

Let us now enumerate some of the properties which we should expect of any reasonable experiment designed to study the stability of parallel flow. Satisfaction of these requirements will aid us in our search for boundary conditions which are appropriate for theoretical analysis of such experiments. First of all, we should in fact get stability in the sense of a uniform bound in time for each given x. For generally the experiment will provide the disturbance, and we expect that, in a reasonable experiment, if we turn off the disturbance, then the flow will revert to the basic flow under study. Second, we want the effect of disturbances downstream to be damped out to such an extent that they do not dominate the perturbation far upstream. This is certainly reasonable if the experimental results are to be reproducible, for generally we may expect highly turbulent conditions far downstream. And the third requirement is closely related, that the behavior of the perturbation near $x = 0$ should not be strongly dependent on the detailed nature of the boundary conditions at $x = L$ when L is large, and that we should be able to make the passage $L \to \infty$.

As a final point, perhaps we can elucidate the purpose of this paper in a negative way by describing a case where the boundary conditions we seek are not at all applicable: the case of Couette flow. Boundary conditions of the sort used by Taylor [5] seem to be entirely adequate for a description of the experimental situation, where in the unstable case disturbances grow in time.

2. Heuristic treatment. In this section we shall tentatively give some boundary conditions under the influence of which the flow will have the properties enumerated above. An interpretation of these conditions is deferred to the next section.

There is a case when the homogeneous boundary conditions $\psi = 0$ at $x = 0$ and L and $\omega = 0$ at $x = 0$ give results of the sort required, and we shall discuss that case first. If we refer to equation (1.3′) and denote the substantial derivative $(\partial/\partial t) + U(\partial/\partial x)$ by D/Dt, we have

$$D\omega/Dt = vU''. \tag{2.1}$$

With the homogeneous boundary conditions on ψ,

$$v(x, y) = \iint_R \frac{\partial G_0(x, y; x_0, y_0)}{\partial x} \omega(x_0, y_0)\, dx_0\, dy_0 . \tag{2.2}$$

We can always find a positive function $\epsilon(x)$ which majorizes $\omega(x)$ ($|\omega(x, y, t)| \leq \epsilon(x, t)$ for $0 \leq x \leq L$ and $t \geq 0$) such that $U''v$ is majorized by $\Omega_0\epsilon(x)$ for some constant $\Omega_0 > 0$, independent of x and t, which may be thought of as a constant characteristic of the basic flow $U(y)$ (and also of the majorization $\epsilon(x)$). If $\epsilon(x, 0)$ is given at $t = 0$, it is clear that we can find a suitable $\epsilon(x, t)$ which satisfies the equation

$$D\epsilon/Dt \leq \Omega_0\epsilon. \tag{2.3}$$

For example, if $\epsilon(x, 0)$ is just a constant ϵ, we see that $\epsilon \exp(\Omega_0 t)$ will majorize ω at later times. (We have assumed that U'' exists, since this is not a physically unreasonable assumption, and since it makes the assertion of the existence of the constant Ω_0 transparent. However, it can be shown [2] that an exponential-type bound majorizes the growth of disturbances when U' is only required to be Hölder-continuous, and we may confidently expect the constant Ω_0 to exist for such flows.) So far we have obtained a bound which increases exponentially with time, when what we really want is a uniform bound in time. So let us sharpen our analysis of Eq. (2.3). Note that Eq. (2.3) gives a bound on the growth of a disturbance as we move with the flow, whereas what we want is a bound on the growth as we remain fixed. Thus, the disturbance at x and $t + \Delta t$ is bounded by the disturbance at t and at a point at least $U_0\ \Delta t$ to the "left" of x, augmented by no more than $\Omega_0\ \Delta t$ times the disturbance at x and t. But this bound may be less than that on the disturbance at x and t if the disturbance falls off sufficiently rapidly as x decreases (or rises sufficiently rapidly as x increases). Let us suppose that, at a given time, the disturbance can be majorized by $\epsilon \exp(ax)$ where ϵ and a are positive constants. Then at a small time Δt later the disturbance can be majorized by $\epsilon \exp(ax) + \Delta t(\Omega_0 - aU_0)\epsilon \exp(ax)$. Thus, we get a uniform bound in time if

$$aU_0 \geq \Omega_0 . \tag{2.4}$$

We have been careless, of course, about the dependence of Ω_0 on the form of the majorizing function. Since we are particularly interested in the case when L is large, let us consider the contributions to $\Omega_0\epsilon(x)$ from points x_0 with $|x - x_0|$ large. If we recall the asymptotic behavior of G_0 when $|x - x_0|$ is large, and the majorization $\epsilon \exp(ax)$ used, we see that the contributions to $\Omega_0\epsilon(x)$ are proportional to $\exp((a - (\pi/D))x_0)$. It is clear that we will be courting disaster if $a > \pi/D$, or perhaps even if a is comparable to π/D. But if a is small compared to π/D, the constant Ω_0 may be determined in a manner which is uniformly bounded in L. So, for those basic flows for which

$$\pi/D \text{ is large compared to } a \geq \Omega_0/U_0 , \tag{2.5}$$

we may expect that the imposition of the conditions $\psi = 0$ at $x = 0$ and $x = L$ will in no way violate our requirements for a meaningful experiment. Note that, in this case, the effect of the disturbances far downstream at a particular upstream point decreases with their increasing downstream distance, despite the fact that the disturbances themselves are increasing with this distance. Thus v in Eq. (1.3′) depends primarily on local values of ω.

Presently we shall give other boundary conditions which ensure that the constant Ω_0 can be chosen so that it is uniformly bounded in L, for any basic flow U. Assuming that we can do so, we may return to the discussion where we left it at Eq. (2.4). We get uniform boundedness in time when the spatial growth rate a is related to the temporal growth rate Ω_0 by that equation. (Of course, Ω_0 is not really the maximum rate of increase of disturbances in time, but is generally much larger, majorizing that maximum rate. But in the crude analysis of this section we shall assume that it is. And most of the statements of this section involving Ω_0 actually apply to the maximum growth rate.) The analysis that led to Eq. (2.4) indicates that if $aU_1 < \Omega_0$, we would definitely get instability. So we must have

$$aU_1 \geq \Omega_0 . \tag{2.6}$$

Note that Eq. (2.6) is necessary, Eq. (2.4) is sufficient. Suppose $aU_0 > \Omega_0$. Then the disturbance dies out exponentially in time. Now, in an actual experimental situation there will always be some production of disturbances, and we may expect that the disturbances, over a long period of time, will be "quasi-steady". So effectively the net rate of growth will vanish, a situation which corresponds to

$$aU_0 \leq \Omega_0 . \tag{2.7}$$

Combining Eq. (2.6) with Eq. (2.7), we see that the ratio Ω_0/a lies in the range of U. This result is strongly reminiscent of the result of Gaster [6], if use is made of the semicircle theorem of Howard [7].

We still have to show that constants Ω_0 and a can be chosen so that the inequality (2.4) is satisfied, and so that $\Omega_0\epsilon \exp(ax)$ majorizes $U''v$ whenever ω is majorized by $\epsilon \exp(ax)$, and that those choices can be made to be uniform in L, if appropriate boundary conditions are prescribed. We shall now introduce a sequence of boundary conditions and we shall show that for some members of this sequence all these choices can be made. When this is done there may appear to the reader to be considerable freedom regarding appropriate boundary conditions to be imposed. In fact, the criterion mentioned in the introduction, that deviations from the boundary conditions should be uniformly small, as well as the criterion of simplicity, will severely delimit the boundary conditions we use. These points will be discussed in more detail in the third section of this paper. For the present let us merely note that, although we introduce a sequence of boundary conditions whose members correspond to an integer n, the choice of n for a particular experimental situation is not at all capricious. Whatever value is chosen for a, let n be the integer such that

$$n\pi/D \leq a < (n+1)\pi/D. \tag{2.8}$$

(For example, a might be $(n + \frac{1}{2})\pi/D$.) Besides the Green's function G_0 we shall need the Green's functions $G_n(x, y; x_0, y_0)$ defined by

$$\begin{aligned} G_n(x, y; x_0, y_0) &= \frac{2}{D}\sum_{p=1}^{n} \frac{\sin(p\pi y/D)\sin(p\pi y_0/D)}{(p\pi/D)} \sinh\frac{p\pi(x-x_0)}{D} \qquad (x > x_0) \\ &= 0 \qquad (x < x_0) \\ &\quad - \frac{2}{D}\sum_{p=n+1}^{\infty} \frac{\sin(p\pi y/D)\sin(p\pi y_0/D)}{(p\pi/D)\sinh(p\pi L/D)} \sinh\frac{p\pi x_<}{D}\sinh\frac{p\pi(L-x_>)}{D}, \end{aligned} \tag{2.9}$$

or, equivalently,

$$G_n(x, y; x_0, y_0) = G_0 + \frac{2}{D}\sum_{p=1}^{n} \frac{\sin(p\pi y/D)\sin(p\pi y_0/D)}{(p\pi/D)\sinh(p\pi L/D)} \sinh\frac{p\pi x}{D}\sinh\frac{p\pi(L-x_0)}{D}. \tag{2.9'}$$

Now let homogeneous boundary conditions be prescribed so that Eq. (2.2) is replaced by

$$v(x, y) = \iint_R \frac{\partial G_n(x, y; x_0, y_0)}{\partial x}\,\omega(x_0, y_0)\,dx_0\,dy_0 . \tag{2.10}$$

We proceed as before and determine Ω_0 so that $\Omega_0\epsilon \exp(ax)$ majorizes $U''v$ with v given by this new expression. Again, the case when L is large is of special interest. When $x - x_0$ is large compared to D, G_n behaves like $\exp((n\pi/D)(x - x_0))$, and when $x_0 - x$ is large compared to D, G_n behaves like $\exp((-(n+1)\pi/D)(x_0 - x))$. The contributions to v from large distances are majorized by a multiple of

$$\int_0^x \epsilon e^{ax_0} \exp\left(\frac{n\pi}{D}(x - x_0)\right) dx_0 + \int_x^L \epsilon e^{ax_0} \exp\left(-\frac{(n+1)\pi}{D}(x_0 - x)\right) dx_0$$
$$= \frac{\epsilon}{a - (n\pi/D)} e^{ax}\left(1 - \exp\left[-\left(a - \frac{n\pi}{D}\right)x\right]\right)$$
$$+ \frac{\epsilon}{[(n+1)\pi/D] - a} e^{ax}\left(1 - \exp\left[-\left(\frac{(n+1)\pi}{D} - a\right)(L - x)\right]\right). \quad (2.11)$$

With a (and thus n) chosen sufficiently large, it is clear that we can find Ω_0 to satisfy all the conditions required of it. Once again, the effect of disturbances far removed from a given point on the velocity perturbation there diminishes with increasing distance from the point, although the disturbances themselves increase with the downstream distance. v in Eq. (1.3′) depends primarily on local values of ω. The value of ψ at $x = L$ that corresponds to the use of G_n is, of course, no longer 0 but

$$-\iint_R G_n(L, y; x_0, y_0)\omega(x_0, y_0)\, dx_0\, dy_0$$
$$= -2 \sum_{p=1}^{n} \frac{\sin(p\pi y/D)}{p\pi} \iint_R \sin\frac{p\pi y_0}{D} \sinh\frac{p\pi(L - x_0)}{D}\, \omega(x_0, y_0)\, dx_0\, dy_0\,. \quad (2.12)$$

G_n satisfies the following boundary conditions:

$$\int_0^D G_n(x, y; 0, y_0) \sin\frac{p\pi y_0}{D}\, dy_0 = 0 \qquad (p \geq n + 1)\,,$$
$$G_n(x, y; L, y_0) = 0,$$
$$\int_0^D \frac{\partial G_n(x, y; L, y_0)}{\partial x_0} \sin\frac{p\pi y_0}{D}\, dy_0 = 0 \qquad (1 \leq p \leq n). \quad (2.13)$$

ψ satisfies the boundary conditions

$$\psi(0, y) = 0,$$
$$\int_0^D \sin\frac{p\pi y}{D}\, \psi_x(0, y)\, dy = 0 \qquad (1 \leq p \leq n),$$
$$\int_0^D \sin\frac{p\pi y}{D}\, \psi(L, y)\, dy = 0 \qquad (p \geq n + 1). \quad (2.14)$$

If we were to take the limit as $n \to \infty$ we would get an improperly posed problem, the prescription of Cauchy data for Poisson's equation. But for any finite n the boundary-value problem is well-posed. (It is necessary to prove that there is a unique solution to the initial- and boundary-value problem with the boundary conditions (2.14). However, the proof is straightforward for large, but finite, L.) Under the boundary conditions corresponding to the use of G_n, disturbances proportional to $\sin(p\pi y/D)$ for $1 \leq p \leq n$ do not propagate upstream, and they are magnified when they propagate downstream. Disturbances proportional to $\sin(p\pi y/D)$ for $p \geq n + 1$ propagate in both directions, without amplification.

3. Refinement and interpretation of boundary conditions. In the last section we found that a parallel flow, under the influence of the boundary conditions (2.14), behaves as we have required it to behave. The purpose of this section is to make more precise some of the arguments which led to those boundary conditions, and to rephrase those boundary conditions and others like them in a way which is susceptible to a more direct physical interpretation.

First, note that the boundary conditions (2.14) are not the only boundary conditions for which the flow has the desired characteristics. If we change the values of $\psi(0, y)$ and $\int_0^D \sin(p\pi y/D)\psi_x(0, y)\, dy$ for $1 \leq p \leq n$ by amounts of the order of ϵD^2, and $\int_0^D \sin(p\pi y/D)\psi(L, y)\, dy$ for $p \geq n + 1$ by an amount of the order of $\epsilon \exp(aL)\, D^3$, we will not change the basic properties of the flow.

Suppose that, whatever homogeneous boundary conditions we decide to prescribe for ψ, we denote the corresponding Green's function by G^*, so that

$$\psi(x, y) = -\iint_R G^*(x, y; x_0, y_0)\omega(x_0, y_0)\, dx_0\, dy_0 . \tag{3.1}$$

If we let $G = \partial G^*/\partial x$, we get

$$v(x, y) = \iint_R G(x, y; x_0, y_0)\omega(x_0, y_0)\, dx_0\, dy_0 . \tag{3.2}$$

Define M as the Laplace transform of ω:

$$M(x, y, s) = \int_0^\infty \omega(x, y, t)e^{-st}\, dt. \tag{3.3}$$

Inserting Eq. (3.2) into Eq. (1.3′), taking the Laplace transform of the result, and considering only the homogeneous form of the equation thereby obtained, we find

$$\left(s + U\frac{\partial}{\partial x}\right)M = U'' \iint_R G(x, y; x_0, y_0)M(x_0, y_0, s)\, dx_0\, dy_0 . \tag{3.4}$$

The reason for considering the homogeneous equation (3.4) is that its nontrivial solutions (when an additional homogeneous boundary condition is prescribed at $x = 0$) determine the values of s for which the associated inhomogeneous equation has poles in the complex s-plane. The value of s with the greatest real part determines the growth of the perturbation in time. If we require that $\omega = 0$ at $x = 0$, Eq. (3.4) can be integrated to yield the integral equation

$$M(x, y, s) = \frac{U''(y)}{U(y)} \iint_R M(x_0, y_0, s) \cdot \int_0^x \exp\left(-\frac{s}{U(y)}(x - x_1)\right)G(x_1, y; x_0, y_0)\, dx_1\, dx_0\, dy_0 . \tag{3.5}$$

If a diferent boundary condition is set at $x = 0$, Eq. (3.5) must be replaced by

$$M(x, y, s) = M(0, y, s) \exp\left(-\frac{sx}{U(y)}\right) + \frac{U''(y)}{U(y)} \iint_R M(x_0, y_0, s) \cdot \int_0^x \exp\left(-\frac{s}{U(y)}(x - x_1)\right)G(x_1, y; x_0, y_0)\, dx_1\, dx_0\, dy_0 . \tag{3.5′}$$

It will prove convenient for us to simplify our notation somewhat. Define the function $\tilde{G}$ by

$$\tilde{G}(x, y; x_0, y_0) = \frac{U''(y)}{U(y)} \int_0^x \exp\left(-\frac{s}{U(y)}(x - x_1)\right)G(x_1, y; x_0, y_0)\, dx_1 . \tag{3.6}$$

Let us consider quantities like G and $\tilde{G}$ as continuous "matrices" and use the symbol $\circ$ to denote their multiplication. Then Eqs. (3.5) and (3.5′) may be written respectively as

$$M = \tilde{G} \circ M, \qquad (I - \tilde{G}) \circ M = \exp\left(-\frac{sx}{U(y)}\right) M(0, y, s), \tag{3.7}$$

where $I(x, y; x_0, y_0) = \delta(x - x_0)\,\delta(y - y_0)$. We may also define the operator

$$\Delta(x, y; x_0, y_0) = \delta(x_0)\,\delta(y - y_0).$$

When, instead of the condition $\omega = 0$ at $x = 0$, we require that $v = 0$ at $x = 0$, we obtain

$$\Delta \circ G \circ M = 0. \tag{3.8}$$

Before proceeding further we introduce the operator K defined by

$$K(y, y_0) = \int_0^L G(0, y; x_0, y_0) \exp(-sx_0/U(y_0))\, dx_0 . \tag{3.9}$$

K may operate on functions of y alone or it may multiply functions of x and y, in which case the multiplication is understood to involve an integration over the y variable alone. Multiplication by K will be denoted by the symbol $\times$. Consider a function $F(x, y)$. Multiplication of F by $\exp(sx/U(y))$, then by $\exp(-sx/U(y))$, then by G, and finally by Δ is, of course, equivalent to multiplication by G, then by Δ. Suppose it should happen that $\exp(sx/U(y))F(x, y)$ is independent of x. Then the composite operation just described is equivalent to multiplication of F by $\exp(sx/U(y))$, and then by K. According to Eq. (3.7), $(I - \tilde{G}) \circ M$ is such a function. Thus,

$$K \times \exp(sx/U(y))(I - \tilde{G}) \circ M = \Delta \circ G \circ (I - \tilde{G}) \circ M = -\Delta \circ G \circ \tilde{G} \circ M \tag{3.10}$$

by Eq. (3.8), if $v = 0$ at $x = 0$. If we write Eq. (3.10) out, we find that it is a homogeneous integral equation,

$$\begin{aligned} &\int_0^D M(x, y_0, s) \int_0^L \exp\left(\frac{s(x - x_0)}{U(y_0)}\right) G(0, y; x_0, y_0)\, dx_0\, dy_0 \\ &\quad = \iint_R M(x_0, y_0, s)\Bigg[\iint_R \left(\int_{x_1}^{x} \frac{U''(y_1)}{U(y_1)} \exp\left(-\frac{s(x_1 - x_2)}{U(y_1)}\right) G(x_2, y_1; x_0, y_0)\, dx_2\right) \\ &\quad\quad \cdot G(0, y; x_1, y_1)\, dx_1\, dy_1\Bigg] dx_0\, dy_0 , \end{aligned} \tag{3.11}$$

which should be contrasted with Eq. (3.5). (It should be observed that for some Green's functions G^*, the boundary condition $v = 0$ at $x = 0$ will entail some redundancy with conditions priorly established, such as the second set of Eqs. (2.14). In that case Eq. (3.11) will not have a unique solution unless other homogeneous boundary conditions are appended. This matter will be discussed more fully in two more paragraphs.) If Eq. (3.11) has a nontrivial solution for a value of $s = s_1 + is_2$ with $s_1 > 0$, then the flow with boundary conditions (3.1) and $v = 0$ at $x = 0$ is unstable. If Eq. (3.5) has a nontrivial solution for $s_1 > 0$, the flow with boundary conditions (3.1) and $\omega = 0$ at $x = 0$ is unstable. In the following, we shall see that, if we let $G^* = G_n$ and vary n, as we increase n we can make s_1 for all values of s at which we get nontrivial solutions of Eqs. (3.5) and (3.11) arbitrarily small.

We start with discussion of Eq. (3.5), which may also conveniently be written in the symbolic form of the first of Eqs. (3.7). Our method of proof is very simple—we establish that, for any given s_1, a value of n can be found such that an appropriate norm of $\tilde{G}$ is less than unity for any value of s with real part $\geq s_1$. It is common knowledge that for any value of n the norm of $\tilde{G}$ goes to 0 as the real part of $s \to +\infty$. We shall not reproduce the proof here, as it involves only a rephrasing of the familiar uniqueness theorems for the various types of boundary conditions. What we need here, of course, is a finer result. The norm which we use for $\tilde{G}$ is based on the following norm for functions $M(x, y, s)$:

$$||M|| = \sup_{(x,y)\in R} |M(x, y, s) \exp(-ax)|. \tag{3.12}$$

(It is apparent from Eq. (3.5) that, if $G^* = G_n$, generally $|M(x, y, s)|$ will increase with x at least as fast as $\exp(n\pi x/D)$.) A rough but tedious calculation, which we do not reproduce here, based on Eqs. (3.6) and (2.9), shows that, for a suitable value of a with $n\pi/D < a < (n + 1)\pi/D$, the norm of $\tilde{G}$ is bounded by

$$||\tilde{G}|| \leq \sup_{0\leq y\leq D} \left|\frac{D^2 U''(y)}{U(y)}\right| \cdot \left[\frac{A_1 + A_2 \ln(n+1)}{n} + \frac{A_3 + A_4 \ln\left(1 + n + \frac{D}{\pi}\frac{s_1}{U_0}\right) + A_5 \ln\left(1 + \frac{D}{\pi}\frac{|s_1|}{U_0}\right)}{n + (D/\pi)(s_1/U_0)}\right]. \tag{3.13}$$

The result stated in Eq. (3.13) applies only for values of s such that $n + (D/\pi)(s_1/U_0) > 0$. The numbers A_1 to A_5 are positive numerical constants independent of s and n, whose precise values do not concern us here. (In fact, we might expect a more careful calculation to show that $A_1 \to 0$ and $A_2 \to 0$ as $s_1 \to +\infty$, an expectation derived from the knowledge that the norm must go to 0 as $s_1 \to +\infty$, as mentioned above. However, that particular limit does not interest us here.) If Eq. (3.5) is to have a solution as n increases without limit, the norm of $\tilde{G}$ must be at least 1, and an examination of Eq. (3.13) for a given basic flow shows that, if this condition is to be met, $(n + (D/\pi)(s_1/U_0))/\ln n$ cannot increase without limit. In the limit as $n \to \infty$, the requirement $||\tilde{G}|| \geq 1$ becomes

$$\frac{B_1 \ln n}{n + \frac{D}{\pi}\frac{s_1}{U_0}} > B_2 . \tag{3.14}$$

(From Eq. (3.13) it is clear that asymptotically $B_1 = A_5$,

$$B_2 = \frac{1}{\sup_{0\leq y\leq D} \left|\frac{D^2 U''(y)}{U(y)}\right|}.)$$

So for n large, the values of s for which the homogeneous equation has a nontrivial solution satisfy

$$\frac{D}{\pi}\frac{s_1}{U_0} < \frac{B_1}{B_2} \ln n - n. \tag{3.15}$$

(Perhaps, with a more careful bound on $||\tilde{G}||$, we can get rid of the term ln n here. But its presence does not affect the validity of our argument.)

A similar result can be found for Eq. (3.11). However, it is easier to analyze that equation if we write it in a different form. If we write $M(0, y, s) = \phi(y)$, Eq. (3.7) is $(I - \tilde{G}) \circ M = \exp(-sx/U(y))\phi(y)$. Now, suppose n is large and s is such that $(n + (D/\pi)(s_1/U_0))/\ln n$ is also large. We aim to show that Eq. (3.11) has no nontrivial solutions for such values of s. With that established, it is an easy matter to infer that, for all values of s corresponding to nontrivial solutions of Eq. (3.11), we must have

$$\frac{D}{\pi}\frac{s_1}{U_0} < B_3 \ln n - n \tag{3.16}$$

for some constant B_3. To establish this result, note that, with n large and $(n + (D/\pi)(s_1/U_0))/\ln n$ large, $||\tilde{G}||$ can be made arbitrarily small. Thus $I - \tilde{G}$ is invertible and

$$M = (I - \tilde{G})^{-1} \circ \exp(-sx/U(y))\phi(y). \tag{3.17}$$

In fact, we may use the expansion $(I - \tilde{G})^{-1} = I + \tilde{G} + \tilde{G} \circ \tilde{G} + \cdots$. The boundary condition (3.8) is $\Delta \circ G \circ (I - \tilde{G})^{-1} \circ \exp(-sx/U(y))\phi(y) = 0$. In the limit as $||\tilde{G}|| \to 0$ we get, approximately,

$$\Delta \circ G \circ \exp(-sx/U(y))\phi(y) = 0. \tag{3.18}$$

Eq. (3.18) may be written more conveniently in terms of the operator K introduced in Eq. (3.9):

$$K \times \phi = 0. \tag{3.18$'$}$$

As we mentioned above, Eq. (3.18′) does not determine $\phi(y)$ uniquely, for when we set $G^* = G_n$, the first n Fourier components $\int_0^D \sin(p\pi y/D)v(0, y)\,dy$ $(1 \leq p \leq n)$ of v already vanish at $x = 0$. Thus, we may feel free to add n similar conditions at $x = 0$ which are not redundant. We shall add the following conditions:

$$\int_0^D \sin\frac{p\pi y}{D}\,\omega(0, y)\,dy = 0 \qquad (1 \leq p \leq n). \tag{3.19}$$

That is,

$$\int_0^D \sin\frac{p\pi y}{D}\,\phi(y)\,dy = 0 \qquad (1 \leq p \leq n). \tag{3.19$'$}$$

Eqs. (3.18′) and (3.19′) determine $\phi(y)$ uniquely (up to a constant). It is easy to see that

$$K(y, y_0) = \sum_{p=n+1}^{\infty} \sin\frac{p\pi y}{D}\sin\frac{p\pi y_0}{D}\,f_p(y_0) \tag{3.20}$$

where

$$f_p(y_0) = \frac{L}{D}\,\frac{\eta_p \exp(-\xi) - \eta_p \cosh \eta_p + \xi \sinh \eta_p}{(\sinh \eta_p)(\eta_p^2 - \xi^2)}$$

and $\eta_p = p\pi L/D$, $\xi(y_0) = sL/U(y_0)$. The easiest case to treat is when $U(y)$ is a constant U. Then ξ is a constant, and the value of s with the largest real part for which the Eqs. (3.18′) and (3.19′) have a nontrivial solution is a solution of the equation

$$\eta_{n+1} \exp(-\xi) - \eta_{n+1} \cosh \eta_{n+1} + \xi \sinh \eta_{n+1} = 0, \tag{3.21}$$

the corresponding $\phi(y)$ being $\sin((n+1)\pi y/D)$. Since the limit $L/D \to \infty$ is the one of interest, we give only the result in that limit. In that limit,

$$s_1 L/U \to -(n+1)\pi L/D$$

or

$$s_1 \to -(n+1)(\pi U/D) \tag{3.22}$$

for the value of s with the largest real part. Thus, the desired result is obtained for the very special case when $U(y) = \text{constant}$. We can obtain a similar result for an arbitrary velocity profile of the sort we have considered. For the value of s with the largest real part, in the limit $L/D \to \infty$, we get

$$-\frac{(n+1)\pi U_1}{D} \le s_1 \le -\frac{(n+1)\pi U_0}{D}. \tag{3.23}$$

The proof is given in the following paragraph.

Let ϕ be a nontrivial solution of Eqs. (3.18′) and (3.19′) for a given profile $U(y)$ and a given s, and let θ be a solution of the adjoint problem

$$\int_0^D K(y_0, y)\theta(y_0)\, dy_0 = 0,$$
$$\int_0^D \sin\frac{p\pi y}{D}\,\theta(y)\, dy = 0 \qquad (1 \le p \le n). \tag{3.24}$$

If the velocity profile U is perturbed by an amount δU, there will be induced a variation δs in the corresponding value of s for which a nontrivial solution exists. This is

$$\delta s = \frac{\displaystyle\int_0^D\int_0^D \phi(y_0)\theta(y) \sum_{p=n+1}^{\infty} \frac{\partial f_p(y_0)}{\partial\xi(y_0)}\frac{\xi(y_0)}{U(y_0)}\,\delta U(y_0)\sin\frac{p\pi y_0}{D}\sin\frac{p\pi y}{D}\,dy\,dy_0}{\displaystyle\int_0^D\int_0^D \phi(y_0)\theta(y) \sum_{p=n+1}^{\infty} \frac{\partial f_p(y_0)}{\partial\xi(y_0)}\frac{\xi(y_0)}{s}\sin\frac{p\pi y_0}{D}\sin\frac{p\pi y}{D}\,dy\,dy_0} \tag{3.25}$$

For the special case when the unperturbed velocity is a constant U this becomes (for that value of s with the largest real part)

$$\delta s = \frac{s}{U}\frac{2}{D}\int_0^D \delta U(y_0)\sin^2\frac{(n+1)\pi y_0}{D}\,dy_0\,. \tag{3.26}$$

Suppose we have the constant profile $U = U_1$ and then seek to vary U by an amount $\delta U(y_0) \le 0$ no more than Δ in magnitude ($\sup_{0\le y_0\le D} |\delta U(y_0)| \le \Delta$), where Δ is "small". We ask for the profile $U(y)$ with $U_1 - \Delta \le U(y) \le U_1$ for which the value of s has the largest real part. From Eq. (3.26) it is apparent that we should have $U(y) = U_1 - \Delta$. Similarly, of all profiles with $U_0 \le U(y) \le U_1$, the one for which the real part of s is largest is the constant profile $U(y) = U_0$. And the one with the smallest real part of s is the constant profile $U(y) = U_1$. This proves Eq. (3.23) and, by indirection, Eq. (3.16).

Thus, for any velocity profile we can make disturbances decay in time arbitrarily rapidly by a suitable choice of boundary conditions. As we stated in the discussion which preceded Eq. (2.7), in an actual experimental situation the production of disturbances and their decay will cancel one another out. Accordingly, the most appropriate sets of boundary conditions might be assumed to be those for which the maximum of s_1 is 0.

Now, we have discussed only a relatively restricted class of boundary conditions, corresponding to the choice $G^* = G_n$ $(n = 0, 1, \cdots)$. But clearly we can make many other choices for G^*, and ones such that the maximum of s_1 is 0. For example, suppose that the maximum of s_1 for $G^* = G_0$ is >0 and that that maximum for $G^* = G_n$ is <0. By continuity we may expect that the maximum of s_1 for $G^* = \sum_{i=0}^{n} \alpha_i G_i$ with $\alpha_i \geq 0$ and $\sum_{i=0}^{n} \alpha_i = 1$ lies between these extremes, and that there are many choices of the α's for which the maximum of s_1 is 0. With such a wealth of possible G^*'s, it is desirable to try to select from among them by other criteria. This is especially important in our present situation, when we have not given a physical explanation for any of the G^*'s chosen. Our intuition suggests that, if G^* is chosen as a linear combination of Green's functions G_i, we only include those up to and including the first one, say G_n, for which the maximum of s_1 is ≤ 0. The remainder of this section will deal with the search for a more specific set of boundary conditions, which can be interpreted in terms of physical requirements on the flow.

Perhaps the most obvious lesson to be derived from the foregoing analysis is that stability seems to be enhanced when more conditions are prescribed at $x = 0$ and fewer are given at $x = L$. We could not go all the way, even if we desired to, and give *all* the conditions at $x = 0$ and *none* at $x = L$, for we would run into trouble with the existence problem. So we will always have some propagation of disturbances, like sound waves, upstream. But we have freed ourselves from the conception that we must have $\psi = 0$ at $x = L$.

There is a relatively straightforward physical way to insure stability of the flow, although it alters the region of flow. In Eq. (2.5) we saw that we could get stability if somehow we could shrink D so that $\pi/D > \Omega_0/U_0$. So suppose we insert rigid vanes in the region of flow, parallel to the x-axis, at intervals along the y-axis such that the flow in each subregion will be stable. We shall see shortly that no such drastic measures are really required to insure stability, but the suggestion of the use of vanes, with the idea that more conditions can be required at $x = 0$, points the way to a resolution of the problem. After all, in experiments considerable care is generally taken to eliminate disturbances in the fluid entering the region under study, and this pertains not only to the x-component of velocity at $x = 0$ but to the y-component and vorticity there as well. We may imagine that vanes have been installed parallel to the x-axis for $x < 0$. Then $\psi = 0$ on these vanes, and ψ_{xx} and ψ_x vanish there. Now of course ψ will not be exactly 0 at $x = 0$ for $0 \leq y \leq D$, but in a good experiment it will be small there. So conceivably there will be a small jump in ψ_y at $x = 0$ in passing from one side of a vane to the other. And in that case $\omega = -\psi_{yy}$ there will behave like a δ-function. But of course a δ-function has significant higher Fourier components and, no matter how small the viscosity is, as long as it is not zero, it will cause the diffusion of such a disturbance. Thus, a short distance into the region from $x = 0$, we may with impunity prescribe conditions like $\psi = 0$, $\omega = 0$, and $v = 0$ at those points where there are vanes.

Let us set, at $x = 0$,

$$\psi(0, y) = \omega(0, y) = 0, \qquad \psi_x(0, y_i) = 0 \quad (1 \leq i \leq n) \tag{3.27}$$

where the y_i, $0 \leq y_i \leq D$, are n selected points at the end $x = 0$. The precise conditions at $x = L$ will not turn out to be crucial, for we shall see that the boundary conditions (3.27) cause the decay of disturbances propagating upstream, and we shall rather arbitrarily choose our boundary conditions at $x = L$ to minimize $\int_0^D (\nabla\psi)^2 \, dy$ there, subject to the conditions (3.27). In analogy with Eq. (2.9), we write

$$\psi(x, y) = -\iint_R G_n(x, y; x_0, y_0)\omega(x_0, y_0)\, dx_0\, dy_0 + \sum_{p=1}^{\infty} \gamma_p \sin\frac{p\pi y}{D} \sinh\frac{p\pi x}{D}. \tag{3.28}$$

The n conditions $\psi_x(0, y_i) = 0$ then take the form

$$\sum_{p=1}^{\infty} \frac{p\pi}{D} \sin\frac{p\pi y_i}{D} \gamma_p + \frac{2}{D} \sum_{p=n+1}^{\infty} \frac{\sin(p\pi y_i/D)}{\sinh(p\pi L/D)} \cdot \iint_R \sin\frac{p\pi y_0}{D} \sinh\frac{p\pi(L - x_0)}{D}\, \omega(x_0, y_0)\, dx_0\, dy_0 = 0,$$

or

$$\sum_{p=1}^{\infty} a_{ip}\gamma_p = b_i, \qquad a_{ip} = (p\pi/D)\sin(p\pi y_i/D),$$

$$b_i = -\frac{2}{D} \sum_{p=n+1}^{\infty} \frac{\sin(p\pi y_i/D)}{\sinh(p\pi L/D)} \iint_R \sin\frac{p\pi y_0}{D} \sinh\frac{p\pi(L - x_0)}{D}\, \omega(x_0, y_0)\, dx_0\, dy_0. \tag{3.29}$$

We are to minimize

$$\begin{aligned}
&\int_0^D (\nabla\psi(L, y))^2\, dy \\
&= \frac{D}{2} \sum_{p=1}^{n} \left(\frac{p\pi}{D} \cosh\frac{p\pi L}{D} \gamma_p - \frac{2}{D} \iint_R \sin\frac{p\pi y_0}{D} \cosh\frac{p\pi(L - x_0)}{D}\, \omega(x_0, y_0)\, dx_0\, dy_0\right)^2 \\
&\quad + \frac{D}{2} \sum_{p=n+1}^{\infty} \left(\frac{p\pi}{D} \cosh\frac{p\pi L}{D} \gamma_p - \frac{2}{D} \iint_R \frac{\sin(p\pi y_0/D)}{\sinh(p\pi L/D)} \sinh\frac{p\pi x_0}{D}\, \omega(x_0, y_0)\, dx_0\, dy_0\right)^2 \\
&\quad + \frac{D}{2} \sum_{p=1}^{n} \left(\frac{p\pi}{D} \sinh\frac{p\pi L}{D} \gamma_p - \frac{2}{D} \iint_R \sin\frac{p\pi y_0}{D} \sinh\frac{p\pi(L - x_0)}{D}\, \omega(x_0, y_0)\, dx_0\, dy_0\right)^2 \\
&\quad + \frac{D}{2} \sum_{p=n+1}^{\infty} \left(\frac{p\pi}{D} \sinh\frac{p\pi L}{D} \gamma_p\right)^2.
\end{aligned} \tag{3.30}$$

We can always write $\gamma_p = \gamma_p^{(0)} + \gamma_p^{(H)}$ for $1 \leq p \leq n$ so that

$$\sum_{p=1}^{n} a_{ip}\gamma_p^{(0)} = b_i, \qquad \sum_{p=1}^{n} a_{ip}\gamma_p^{(H)} + \sum_{p=n+1}^{\infty} a_{ip}\gamma_p = 0. \tag{3.31}$$

$\gamma_p^{(0)}$ is readily found as

$$\gamma_p^{(0)} = \frac{2}{p\pi} \int_0^D \sin\frac{p\pi y}{D} \sum_{i=1}^{n} b_i \frac{\sin(\pi y/D)}{\sin(\pi y_i/D)} \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{\cos(\pi y/D) - \cos(\pi y_j/D)}{\cos(\pi y_i/D) - \cos(\pi y_j/D)}. \tag{3.32}$$

We anticipate that, as a result of the minimization of the expression (3.30), the constants γ_p for $p \geq n + 1$ will all go to 0 as $L/D \to \infty$, that the quantities $\gamma_p^{(H)}$ will similarly go to zero, and that γ_p will be essentially given by $\gamma_p^{(0)}$. From the minimization we get

$$2C_p\gamma_p + E_p + \sum_{i=1}^{n} \lambda_i a_{ip} = 0, \tag{3.33}$$

where $C_p = (p\pi/D)^2 \cosh(2p\pi L/D)$,

$$E_p = -\frac{4p\pi}{D^2}\iint_R \sin\frac{p\pi y_0}{D}\cosh\frac{p\pi(2L - x_0)}{D}\,\omega(x_0, y_0)\,dx_0\,dy_0 \qquad (1 \le p \le n),$$

$$= -\frac{4p\pi}{D^2}\iint_R \frac{\sin(p\pi y_0/D)}{\sinh(p\pi L/D)}\sinh\frac{p\pi x_0}{D}\,\omega(x_0, y_0)\,dx_0\,dy_0 \qquad (p \ge n + 1)$$

and λ_i is given by

$$\sum_{j=1}^{n} D_{ij}\lambda_j + b_i + \sum_{p=1}^{\infty} a_{ip}\frac{E_p}{2C_p} = 0, \tag{3.34}$$

with

$$D_{ij} = \sum_{p=1}^{\infty}\frac{a_{ip}a_{jp}}{2C_p} = \sum_{p=1}^{\infty}\frac{\sin(p\pi y_i/D)\sin(p\pi y_j/D)}{2\cosh(2p\pi L/D)}. \tag{3.35}$$

As an example, we can give the solution when $n = 1$, keeping only the lowest powers in $\exp(-\pi L/D)$. Then

$$D_{11} \to \exp(-2\pi L/D)\sin^2\frac{\pi y_1}{D} \tag{3.36}$$

and

$$\lambda_1 \to \frac{2}{D\sin(\pi y_1/D)}\exp(2\pi L/D)\frac{1}{\cosh(2\pi L/D)} \cdot \iint_R \sin\frac{\pi y_0}{D}\cosh\frac{\pi(2L - x_0)}{D}\,\omega(x_0, y_0)\,dx_0\,dy_0 - \frac{\exp(2\pi L/D)b_1}{\sin^2(\pi y_1/D)}. \tag{3.37}$$

From Eq. (3.33) we get

$$\gamma_1 \to (b_1/(\pi/D)\sin(\pi y_1/D)),$$

$$\gamma_p \to \frac{-2\sin(p\pi y_1/D)}{p\pi\sin(\pi y_1/D)}\exp(-2(p - 1)\pi L/D)\iint_R \sin\frac{\pi y_0}{D}\exp(-\pi x_0/D)\omega(x_0, y_0)\,dx_0\,dy_0$$
$$+ \frac{\sin(p\pi y_1/D)}{(p\pi/D)\sin^2(\pi y_1/D)}\exp(-2(p - 1)\pi L/D)b_1 + \frac{4}{p\pi}\exp(-3p\pi L/D) \cdot \iint_R \sin\frac{p\pi y_0}{D}\exp(p\pi x_0/D)\omega(x_0, y_0)\,dx_0\,dy_0\,. \tag{3.38}$$

Eqs. (3.28) and (3.33), (3.34), (3.35) (or the simpler version given by Eq. (3.32)), serve as a rationalization for our use of the Green's function G_n. As a practical matter, G_n would generally be used for a calculation, and any resultant departures of ψ_x from 0 at selected points where $x = 0$ or of $\int_0^D (\nabla\psi(L, y))^2\,dy$ from a minimum would be accounted for by a time-dependent source of small perturbations. Now, again, some care should be taken in choosing n. For we have seen that there is some trade-off between the rates of increase in time and the increase in distance, and there is no point prescribing so many conditions at $x = 0$ that we achieve a rapid decay in time but have such a rapid rise with distance that the region where linear perturbation theory is valid is greatly restricted. So, also as a practical matter, it is reasonable to choose n just large enough to insure stability in time. This is closer to the actual physical situation anyway, since in fact we cannot set $\psi_x = 0$ exactly at selected points any more than we can set ψ or ω equal to zero exactly at those points, and the actual boundary conditions at $x = 0$ might be more like $\sup_{0 \leq y \leq D} |\omega(0, y)| \leq \epsilon$, $\sup_{0 \leq y \leq D} |\nabla\psi(0, y)| \leq \epsilon D$, where ϵ measures some tolerance in the system. But when we start talking about properties of the system, we are talking about nonlinear effects, which are beyond the scope of this discussion.

Nevertheless, it is clear that boundary conditions of the sort we have considered may also be prescribed when nonlinear effects of the perturbation are significant. There is nothing of special import for the linear theory, as opposed to the nonlinear theory, in setting $\psi_x = 0$ at certain points, or in minimizing the integral of $(\nabla\psi)^2$ over other points. In fact, the preceding results are readily carried over to the nonlinear case. The nonlinear case may be thought of simply as a succession of linear cases, each obtained from the preceding one by varying a parameter ζ describing the strength of the nonlinear perturbation by an amount $d\zeta$. So in the nonlinear case we are essentially dealing with linear perturbations about time-dependent flows. Thus, if Ψ is the time-dependent stream function describing the "basic" (actually, perturbed) flow and ψ is the stream function of the perturbation, ψ satisfies the linear dynamic equation

$$\Delta\psi_t + \Psi_y \, \Delta\psi_x - \Psi_x \, \Delta\psi_y + \psi_y \, \Delta\Psi_x - \psi_x \, \Delta\Psi_y = 0. \tag{3.39}$$

We may impose boundary conditions on ψ just as we have done in the basically linear theory. And all our conclusions above follow, as long as Ψ satisfies some rather gross conditions, like $\Psi_y > 0$ for all points in R.

Although the linear theory presented in this paper is consistent in itself, it is fitting nevertheless to make some mention of the nonlinear theory, since in practice Nature may provide us with small disturbances, but never infinitesimal ones. So suppose L is a length which is longer than any physical length of interest in the experiment, but finite still. Let ϵ be the size of a typical perturbation which is introduced by the nature of the apparatus itself (and by the inadequacies of the Euler equations), and assume that boundary conditions are prescribed consistent with a growth exp (ax) in linear theory. It may happen that ϵ exp (aL) is unacceptably large, indicating trouble with the linear theory and possibly also with the boundary conditions. The problem of linearity is resolved by proceeding as indicated above, replacing the perturbation by one of size $\epsilon\zeta$ $(0 \leq \zeta \leq 1)$ and varying ζ by $d\zeta$. As we increase ζ we will arrive at a point where nonlinear effects of the total perturbation (deviation from the constant basic flow) are important. Initially, of course, the total perturbation with $\zeta = 1$ can be treated by linear theory and will grow in time. (This does not contradict our requirement earlier that the disturbance should be quasi-steady, a requirement related to the asymptotic behavior in time.) If the experiment is capable of permitting observation of the growth of linear disturbances over an indefinite period of time, we may expect that, at some time and some point in space, when a nonlinear region is reached, the growth in time will start leveling off. This corresponds to a decrease, in the nonlinear region, of the quantity Ω_0 introduced in the second section. But we saw that Ω_0 induced a growth with distance if we were to have asymptotic boundedness in time, and that that growth was proportional to Ω_0 . So, for a satisfactory experiment, we may expect that the nonlinear effects will permit a majorization of the disturbance by ϵ exp $(\alpha(x))$, where $\alpha(x) \to$ constant as $x \to L$. Thus, suppose that for $0 \leq x \leq L_1$, $\alpha(x) = ax$ and the solution can be majorized by ϵ exp (ax). This is the linear region where, say, G_n is the appropriate Green's function. We expect a nonlinear region $L_1 \leq x \leq L_2$ where $\alpha(x) = aL_1 + b(x - L_1)$ with $b < a$. Perhaps in this region we can use G_{n-1}, using the results of the calculation in the linear region $0 \leq x \leq L_1$ to supply boundary data on the flow at $x = L_1$. Similarly, in a region $L_2 \leq x \leq L_3$ we may use G_{n-2}. Finally there will be a region where G_0 may be used. (Of course, $L_i < L$ for all i in the problem.) If, by going out far enough, we can use G_0 as the Green's function, then in fact we can use G_0 throughout a sufficiently large region $(0 \leq x \leq L$, say). The question naturally arises, then, as to why we cannot use G_0 from the outset, for the whole region, including the linear subregion. (After all, it is still possible to describe the flow using the Green's function G_0, even if downstream conditions at $x = L$ are not rigidly prescribed.) The answer to the question is that in principle we can use G_0, and that, according to the nonlinear theory, effects proportional to sin $(p\pi y/D)$, $1 \leq p \leq n$, from disturbances downstream would cancel one another out. But we would have to perform a nonlinear calculation to verify this, and we could not begin a linear treatment of the stability problem. Nor could we resolve the problem in the context of linear stability theory by using G_0 and picking ζ small enough to allow the use of linear theory out to $x = L$. For what we want is a uniform majorization for all time, and the linear theory would ultimately break down, no matter how small the value of ζ chosen. Now, it may happen that, by prescribing that u at $x = 0$ be within a certain tolerance of 0, and by prescribing that v at certain selected points at $x = 0$ be within a given tolerance of 0, the solution to the problem, for perturbations of a size that are unavoidable in the experiment, will still get unacceptably large or be physically unreasonable in other ways in the nonlinear region. In that case, we must regretfully conclude that the boundary conditions at $x = 0$ were not imposed correctly, that in fact the asymptotic behavior in time near $x = 0$ is nonlinear, and that the given experiment is unsuitable for the observation of the linear growth of disturbances over an indefinite period of time.

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