ON THE EXISTENCE OF SOLUTIONS TO OPTIMIZATION PROBLEMS WITH EIGENVALUE CONSTRAINTS*

BY

KOSLA VEPA

University of Waterloo

Abstract. The optimum tapering of Bernoulli–Euler beams, i.e. the shape for which a given total mass yields the highest possible value of the first fundamental frequency of harmonic transverse small oscillations, is determined. The question of the existence of a solution to the optimization problem is considered. It is shown that, irrespective of the relationship between the flexural rigidity and linear mass density of the cantilever beam, the necessary conditions for optimality lead to a contradiction. This result is in partial disagreement with that obtained by earlier investigators. By imposing additional constraints on the optimization variable, a numerical solution for the case of the cantilever beam is obtained, using the formulation of the maximum principle of Pontryagin.

Introduction. Optimization of elasto-mechanic systems in which the behavioral constraint is a deformation bound has become a standard design procedure. However, optimization with a frequency constraint is a more difficult problem. The object of optimization in such problems is to establish, from among all designs of given style and specified total mass, the one for which the lowest natural mode is a maximum. This design is at the same time the minimum-volume (or -mass) structure for a specified fundamental natural frequency.

The proper vehicle for modelling the above optimization problem is the calculus of variations. The theory for the extremum problem of Bolza, which is the classical mathematical tool, was developed by Bliss, McShane, Hestenes and others. On the non-classical side two principles have evolved and since become popular, namely the Maximum Principle of Pontryagin [1] which will be used here and the Principle of Optimality of Bellman. These represent the necessary extremum conditions obtainable by the use of first derivatives.

This paper is motivated by two publications, one by Brach [2] and the other by Karihaloo and Niordson [3]. Brach has raised the questions of the existence of the solution of the optimization problem for a cantilever beam (and free-free beams) for the restricted case where the flexural rigidity and the mass distribution along the span of the beam are linearly related. Although the distinction between the simply supported case and the cantilever/free-free cases is not very clearly brought out, it is clear that the existence

* Received April 19, 1972. This research was carried out with the support of the National Research Council of Canada, grant No. A3728. The author is also grateful to the NRC of Canada for a PIER fellowship during 1969–1971. The author acknowledges with pleasure helpful discussions with Prof. John Roorda of the Solid Mechanics Division, University of Waterloo, who guided his postgraduate research work.

of nontrivial solutions hinges crucially on the nature of the boundary conditions. Weisshaar [4] and Karihaloo and Niordson [3] have arrived at the same result with less rigorous reasoning. But the interesting and not entirely convincing result obtained by the latter authors is that an extremal fundamental frequency exists when the restriction of linearity between flexural rigidity and the linear mass density is dropped.

In the following, we discuss a few shortcomings of the above proofs, and present alternate proofs for existence and nonexistence with the help of the maximum principle.

1.0 Statement of the problem. The systems under consideration are Bernoulli–Euler beams of length l performing small harmonic transverse vibrations of magnitude $y(x, t)$. The following relationship between the flexural rigidity α and the linear mass density μ is assumed:

$$\alpha = c\mu^n, \qquad n > 0. \tag{1}$$

Without loss of generality, c may be assumed to be unity. The motion of the beam may then be characterized in any of the following ways:

a. By minimizing the integral

$$\int_{t_0}^{t_1} L\,dt = \int_{t_0}^{t_1} (T - V)\,dt = \int_{t_0}^{t_1}\int_0^l (-c\mu^n\bar{y}''^2 + \mu\dot{\bar{y}}^2)\,dx\,dt \tag{2}$$

with appropriate boundary conditions. L, T, and V denote the Lagrangian, kinetic energy and the potential energy respectively.

b. By obtaining the stationary value of the Rayleigh quotient

$$\omega^2 = \frac{\int_0^l \mu^n y''^2\,dx}{\int_0^l \mu y^2\,dx} \tag{3}$$

among all functions $y(x) \in C^2$ (twice differentiable) satisfying the geometric boundary conditions.

c. By solving the partial differential equation

$$(\mu^n\bar{y}'')'' + \mu\ddot{\bar{y}} = 0 \tag{4}$$

with appropriate boundary conditions given by the expressions

$$[(\mu^n\bar{y}'')\delta\bar{y}']_0^l = 0, \qquad [(\mu^n\bar{y}'')'\delta\bar{y}]_0^l = 0, \tag{5}$$

where the $\delta\bar{y}$ represent small variations in $\bar{y}$, and $\delta\bar{y} = \epsilon\eta$, as in the usual notation.

2. Formulation of the equations. The optimization problem may be formulated in each of three different ways corresponding to the above approaches. We are interested primarily in optimizing the natural modes of the system. The natural modes are solutions of the form

$$\bar{y} = f(x)g(t), \tag{6}$$

since for such solutions all elements of the system are in phase with each other.

From Eq. (4),

$$\frac{(\mu^n f'')''}{\mu f} + \frac{\ddot{g}}{g} = 0. \tag{7}$$

The first term is a function of x and the second term a function of t. Consequently,

$$\frac{(\mu^n f'')''}{\mu f} = \omega^2, \qquad \frac{\ddot{g}}{g} = -\omega^2. \tag{8}$$

The second of these equations yields

$$g(t) = Y \exp(i\omega t) \tag{9}$$

where Y is a constant depending on initial conditions at $t = 0$. Thus, the equation of a natural mode is given by

$$(\mu^n f'')'' - \mu\omega^2 f = 0, \tag{10}$$

with boundary conditions

$$[(\mu^n f'')\delta f']_0^l = 0 \tag{10a}$$

and

$$[(\mu^n f'')'\delta f]_0^l = 0. \tag{10b}$$

In what follows, we write y for f, noting that it represents the maximum amplitude of the motion.

To obtain the variational form for Eq. (10) we multiply throughout by y and integrate by parts where necessary in order to obtain the quadratic functional

$$H^* = \int_0^l [(\mu^n (y'')^2 - \mu\omega^2 y^2] \, dx. \tag{11}$$

a. Corresponding to the energy functional H^* we may formulate the optimization problem as an isoperimetric problem in the calculus of variations.

It is required to find the shape function $\mu(x)$ of the beam with a fixed total mass

$$M = \int_0^l \mu \, dx \tag{12}$$

while simultaneously performing the operation

$$\sup_{\mu \in \mathfrak{U}} \left\{ \inf_{y \in Q} \int_0^l [\mu^n (y'')^2 - \mu\omega^2 y^2] \, dx \right\}. \tag{13}$$

b. The use of the Rayleigh quotient by Niordson [5] and Prager [6] yields somewhat similar expressions. Thus,

$$\omega^2 = \sup_{\mu \in \mathfrak{U}} \left\{ \inf_{y \in Q} \frac{\int_0^l \mu^n y''^2 \, dx}{\int_0^l \mu y^2 \, dx} \right\} \tag{14}$$

subject to the constraint that

$$M = \int_0^l \mu \, dx \tag{12}$$

is fixed.

To obtain stationarity conditions, one can adjoin the additional constraint by a constant Lagrange multiplier Λ so that the problem is transformed to that of finding

$$J = \sup_{\mu \in \mathfrak{U}} \left\{ \inf_{y \in Q} \int_0^l (\mu\omega^2 y^2 - \mu^n y''^2 - \Lambda\mu)\, dx \right\} + \Lambda M$$
$$= \sup_{\mu \in \mathfrak{U}} \left\{ \inf_{y \in Q} \int_0^l \mathcal{G}\, dx \right\} + \Lambda M \tag{15}$$

where $\mathcal{G}$ is the integrand of the function in (15). For stationarity,

$$\frac{d^2}{dx^2}\left(\frac{\partial \mathcal{G}}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{G}}{\partial y'}\right) + \frac{\partial \mathcal{G}}{\partial y} = 0 \tag{16}$$

and

$$\partial \mathcal{G}/\partial \mu = 0. \tag{17}$$

Eq. (16) yields the equation of motion (10) obtained already. It is Eq. (17) which is mainly of interest in optimization studies. It yields the following additional equation which may be termed the optimality condition:

$$\omega^2 y^2 - n\mu^{n-1} y''^2 - \Lambda = 0 \tag{18}$$

which can be rewritten as

$$\frac{\omega^2 \mathfrak{I} - n\mathfrak{V}}{\mu} = \text{constant} \tag{19}$$

where $\mathfrak{I}$ = maximum amplitude of kinetic energy density = μy^2 and $\mathfrak{V}$ = maximum amplitude of strain energy density = $\mu^n y''^2$. This relationship has been obtained by Prager [6] using a different argument.

Corollary 1: The optimal structure is characterized by the property that a linear combination of the kinetic energy density and the strain energy density is proportional to the optimization or design variable.

c. The problem which has been most thoroughly investigated in the classical calculus of variations is the problem of Lagrange, namely the problem of minimizing a functional subject to differential constraints. It is in the justification of the Lagrange multiplier method when applied to the nonclassical problems of the calculus of variations that major advances have been made by Pontryagin and his associates. The objective is to minimize the mass

$$M = \int_0^l \mu\, dx$$

subject to

$$(\mu^n y'')'' - \mu\omega^2 y = 0, \tag{10}$$
$$[(\mu^n y'')\, \delta y']_0^l = 0,$$
$$[(\mu^n y'')' \delta y]_0^l = 0. \tag{10a}$$

Eq. (10) may be rewritten as a first-order system

$$\mathbf{q}' = \mathbf{A}\mathbf{q}, \tag{20}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\mu^n} & 0 \\ 0 & 0 & 0 & 1 \\ \mu\omega^2 & 0 & 0 & 0 \end{bmatrix}, \tag{20a}$$

forming the Lagrangian

$$\mathcal{J} = -p_0\mu + \mathbf{p}^T(\mathbf{q}' - \mathbf{A}\mathbf{q}) = \mathbf{p}^T\mathbf{q}' - p_0\mu - \mathbf{p}^T\mathbf{A}\mathbf{q}. \tag{21}$$

We define Π, the pre-Hamiltonian, as the Legendre transform with respect to the momenta $\mathbf{p}$:

$$\Pi(\mathbf{q}, \mathbf{p}, \mu) = \mathbf{p}^T\mathbf{q}' - \mathcal{J} = p_0\mu + \mathbf{p}^T\mathbf{A}\mathbf{q} = p_0\mu + \mathbf{q}^T\mathbf{A}^T\mathbf{p}, \tag{22}$$

since

$$\mathbf{p}^T\mathbf{A}\mathbf{q} = \mathbf{q}^T\mathbf{A}^T\mathbf{p},$$

$$\Pi(q, p, \mu) = p_0\mu + p_1q_2 + \frac{p_2q_3}{\mu^n} + p_3q_4 + p_4q_1\mu\omega^2. \tag{22a}$$

The equations governing the optimal solution are the canonical equations

$$\mathbf{q}' = \partial\Pi/\partial\mathbf{p} = \mathbf{A}\mathbf{q}, \tag{20}$$

$$\mathbf{p}' = -\partial\Pi/\partial\mathbf{q} = -\mathbf{A}^T\mathbf{p}. \tag{23}$$

p_0 is a constant Lagrange multiplier [1] and the optimality condition is

$$\partial\Pi/\partial\mu = 0. \tag{24}$$

In other words, the optimal μ can be shown to maximize the pre-Hamiltonian

$$\frac{\partial\Pi}{\partial\mu} = p_0 - \frac{np_2q_3}{\mu^{n+1}} + p_4q_1\omega^2 = 0, \tag{25}$$

or

$$\mu^{n+1} = \frac{np_2q_3}{p_0 + p_4q_1\omega^2}. \tag{26}$$

Introducing variables y^* adjoint to y, we can write the relationships between $\mathbf{q}$, y, $\mathbf{p}$ and y^*:

$$\begin{aligned} q_1 &= y, & p_4 &= y^*, \\ q_2 &= y', & p_3 &= -y^{*\prime}, \\ q_3 &= \mu^n y'', & p_2 &= \mu^n y^{*\prime\prime}, \\ q_4 &= (\mu^n y'')', & p_1 &= -(\mu^n y^{*\prime\prime})'. \end{aligned}$$

Eq. (23) may be rewritten in terms of y^* as

$$(\mu^n y^{*\prime\prime})'' - \mu\omega^2 y^* = 0 \tag{27}$$

with boundary conditions $[(\mu^n y^{*\prime\prime})\delta y^{*\prime}]_0^l = 0$, $[(\mu^n y^{*\prime\prime})\delta y^*]_0^l = 0$. At this point we note that the adjoint equation governing y^* is identical to the system described by (10). This is a consequence of the self-adjointness of the governing equations. Hence we may assume

$$y^* = y/c_1^2 \tag{28}$$

For the sake of definiteness, the proportionality constant has been assumed to be positive. The optimality condition (26) may be rewritten as

$$\mu^{n-1} = \frac{p_0 + yy^*\omega^2}{ny''y^{*\prime\prime}}$$

Using (28) we get an expression identical to (18):

$$n\mu^{n-1}y''^2 - \omega^2 y^2 = p_0 c_1^2 .$$

Dividing and multiplying throughout by μ, we get

$$\frac{\omega^2 \mathfrak{J} - n\mathfrak{V}}{\mu} = \text{constant},$$

which we have already obtained by other methods.

A similar expression has been obtained by Niordson, but the corresponding equation in Brach's paper is incorrect. The right-hand side is zero, which is equivalent to assuming that p_0, the multiplier associated with the total mass, is zero. Such an assumption violates the normality requirement in the Lagrange multiplier rule (see Bliss [7], for instance). Since all his conclusions on the existence of solutions are based on this incorrect equation, they are suspect.

3. Existence of optimal solutions. We consider the case of the cantilever beam noting that similar arguments hold (but not the same conclusions) for the free-free case. Inspection of (26) shows that as a result of the boundary conditions ($q_3 = 0$ at $x = l$) the mass density μ at the free end is 0, or

$$\mu = 0 \quad \text{at} \quad x = l. \tag{29}$$

To proceed further we need an additional property of the optimal solution, namely that

$$H(\mathbf{q}, \mathbf{p}) = \sup_{\mu \in \mathfrak{U}} \Pi(\mathbf{q}, \mathbf{p}, \mu) = \text{constant}.$$

If Π is not an explicit function of x,

$$\frac{d\Pi}{dx} = \frac{\partial \Pi}{\partial \mathbf{q}}\mathbf{q}' + \frac{\partial \Pi}{\partial \mathbf{p}}\mathbf{p}' + \frac{\partial \Pi}{\partial \mu}\mu'.$$

Noting that by definition

$$\mathbf{q}' = \partial\Pi/\partial\mathbf{p} \quad \text{and} \quad \mathbf{p}' = -\partial\Pi/\partial\mathbf{q},$$

and that for optimality $\partial\Pi/\partial\mu = 0$, it follows that $d\Pi/dx = 0$, or

$$H = \sup_{\mu \in \mathfrak{U}} \Pi = \text{constant in } 0 \le x \le l. \tag{30}$$

The multiplier p_0 can be chosen to be arbitrarily greater than 0 and μ at $x = 0$ is greater than 0 as a result of Eq. (26). The value of Π at $x = 0$ is

$$\Pi\,|_{x=0} = p_0\mu + \frac{p_2 q_3}{\mu^n} = p_0\mu + \frac{q_3^2}{c_1^2\mu^n} \tag{31}$$

Therefore $H|_{x=0} \neq 0$ at $x = 0$, and $H(x) = \text{constant} \neq 0$ in $0 \leq x \leq l$; at $x = l$

$$\Pi\,|_{x=l} = p_0\mu + \mu^2\omega q_1 p_4 = \mu(p_0 + \omega^2 q_1^2)/c_1^2$$

$$H\,|_{x=l} = 0 \quad \text{at} \quad x = l \tag{32}$$

since $\mu = 0$ at $x = l$ from (29). But this contradicts the result obtained in (31) and (30).

These considerations indicate that there exist no nontrivial solutions which satisfy the conditions of optimality for a cantilever beam. No restriction need be made with respect to the exponent n in (1).

Corollary 2: The problem of minimizing the volume or weight of a cantilever beam, keeping the first fundamental frequency in transverse vibration constant, does not possess a solution in the absence of geometric constraints on the design variable.

Contrary to the contention of the author of [2], one does not arrive at a similar conclusion in the case of free-free beams. It is clear that it is the nature of the boundary conditions which is significant, rather than the relationship between the flexural rigidity and the mass density, in the demonstration of existence. This result does not agree with that obtained by Niordson and his co-workers [3]. Karihaloo and Niordson have proved the non-existence of a solution, in an indirect manner, for the case where the flexural rigidity and mass density are linearly related, i.e. when the exponent n in (1) is equal to unity. These authors imply, however, that a solution to the problem exists when the restriction of linearity between $\alpha(x)$ and $\mu(x)$ is dropped. The question arises as to what conditions are necessary in order to ensure that an optimal solution exists. This may be done in either of two ways:

(1) by the introduction of non-structural mass, ensuring a non-zero density at all sections of the structure,

(2) by the introduction of inequalities in the optimization variable, with the result that $\partial\Pi/\partial\mu$ is not necessarily zero at all points in the beam. Consequently $H = \sup_{\mu\in\mathfrak{U}} \Pi$ is no longer constant along the beam.

4. Computational procedure (cantilever beam). The equations to be satisfied in order that optimality be attained, in the sense of minimum weight, are repeated for convenience:

$$\mathbf{q}' = \mathbf{A}\mathbf{q} \text{ with associated boundary conditions,} \tag{20}$$

$$\mathbf{p}' = -\mathbf{A}^T\mathbf{p} \text{ with associated boundary conditions,} \tag{23}$$

$$\mu^{n+1} = \frac{np_2q_3}{p_0 + p_4q_1\omega^2}. \tag{26}$$

If the lower bound is placed on the solution of μ, as indeed one must in order to obtain a solution, these equations must be supplemented by

$$\mu \geq \mu_b \tag{33}$$

where μ_b = lower bound on the linear mass density of the beam.

The numerical method used is the Min-H method described by Gottleib in [8]. Since the equations are self-adjoint, the systems of equations (20) and (23) are identical. The calculations have been carried out for the case of a circular cross-section, in which case

$$\alpha = E\frac{\pi r^4}{4} = \frac{E}{4\pi}(\pi r^2)^2 = \frac{E}{4\pi\gamma^2}\mu^2, \tag{34}$$

where γ = density of the material, E = Young's modulus, and r = radius of cross-section. Comparing with Eq. (3.1), therefore, we obtain

$$c = \frac{E}{4\pi\gamma^2} \quad \text{and} \quad n = 2.$$

As a result of the self-adjointness of the system equations, there are only two unknown initial conditions in the system, and as a consequence only two integrations are required to obtain the characteristic determinant. The numerical procedure is as follows:

Step 1: The design variable $\mu(x)$ is assumed to be of a certain shape.

Step 2: The resulting linear boundary value problem is solved, $\mathbf{q}' = \mathbf{Aq}$ with associated boundary conditions, and the fundamental frequency of the system is obtained.

Step 3: Since $p_4 = q_1/c_1^2$ and $p_2 = q_3/c_1^2$, $n = 2$, the design variable satisfying the optimality condition (26) can now be computed:

$$\mu_c^3 = \left(\frac{2q_3^2}{c_1^2 p_0 + q_1^2\omega^2}\right) \tag{35}$$

or $\mu = \mu_b$. If $(\mu^{(m+1)} - \mu^{(m)})/\mu^{(m)}$ is less than a small quantity (e.g., 10^{-4}), the computation is terminated (superscripts represent successive number of iteration). Before proceeding to the next iteration, one can choose to solve either of two dual problems: (a) keeping the mass constant, the natural frequency of the system can be optimized, or (b) keeping the natural frequency constant, the mass can be minimized. Due to the practical difficulty of solving the boundary-value problem with a fixed natural frequency, the former alternative is more convenient.

Step 4: To force the resulting mass to remain constant, the computed values of the design variable must be scaled by a factor c_2 such that

$$M = \int_0^{l_b} c_2\mu_c(x)\,dx + \int_{l_b}^{l} c_2\mu_b\,dx \tag{36}$$

where l_b = spanwise location in beam where $\mu = \mu_b$, and M = the constant total mass.

In order to avoid oscillatory instability in the iterative process the value of $\mu(x)$ chosen for the succeeding iteration is a weighted mean:

$$\mu^{(m)} = \eta\mu^{(m-1)} + (1 - \eta)c_2\mu_c^{(m)} 0 \leq x \leq l_b$$

or

$$\mu^{(m)} = \eta\mu^{(m-1)} + (1 - \eta)c_2\mu_b^{(m)} l_b < x \le l$$

with $0 \le \eta \le 1$; $\eta = 0.5$ is a satisfactory value.

The iteration is continued by returning to Step 2. Step 2 in the computational procedure, namely the linear boundary-value problem, is amenable to solution by different methods. Apart from the variational formulation of self-adjoint eigenvalue problems described for instance in Collatz [9], finite difference methods can also be used. The method adopted is as follows:

(a) Two sets of values are assumed for $q_3(0)$ and $q_4(0)$ such that the matrix

$$B_0 = \begin{bmatrix} q_3^{(1)}(0) & q_3^{(2)}(0) \\ q_4^{(1)}(0) & q_4^{(2)}(0) \end{bmatrix} \tag{37}$$

is nonsingular. The B_0 matrix can be chosen to be the unit matrix.

(b) For a chosen value of ω, the system of four simultaneous differential equations is integrated from $x = 0$ to $x = l$. The Runge–Kutta procedure has been used in this investigation. The resulting end conditions must be such that

$$b_1 q_3^{(1)}(l, \omega) + b_2 q_3^{(2)}(l, \omega) = 0,$$

$$b_1 q_4^{(1)}(l, \omega) + b_2 q_4^{(2)}(l, \omega) = 0.$$

A nontrivial solution for b_1 and b_2 can occur only if

$$\det B_l = \begin{vmatrix} q_3^{(1)}(l, \omega) & q_3^{(2)}(l, \omega) \\ q_4^{(1)}(l, \omega) & q_4^{(2)}(l, \omega) \end{vmatrix} = 0. \tag{38}$$

If the determinant is not sufficiently close to 0 this integration is repeated with a different value of ω.

(c) Once the correct value of ω has been obtained, the solution $\mathbf{q}(x)$ can be obtained by combining the results of the two integrations in the ratio

$$\frac{b_2}{b_1} = -\frac{q_3^{(1)}(l, \omega)}{q_3^{(2)}(l, \omega)} = -\frac{q_4^{(1)}(l, \omega)}{q_4^{(2)}(l, \omega)}$$

so that $\mathbf{q}(x) = b_1\mathbf{q}^{(1)}(x, \omega) + b_2\mathbf{q}^{(2)}(x, \omega)$.

5. Discussion of results. The results are shown in Figs. 1 through 3. In Fig. 1 are shown some representative cross-sectional area distributions for different values of the lower bound of μ, the linear mass density. The increase in frequency, keeping the volume constant, in the cases shown is demonstrated in Table 1, where μ_{opt} = optimum linear mass density distribution, μ_u = uniform mass density distribution of a beam with the same total mass, $\xi = \mu_{\text{opt}}/\mu_u$, μ_b = lower bound on linear mass density, $\xi_b = \mu_b/\mu_u$ = lower bound on ξ, μ_0 = density at clamped end, and $\xi_0 = \mu_0/\mu_u$.

The last column in Table 1 is the ratio of the minimum weight cantilever beam to that of the uniform beam, with the same first fundamental natural frequency. The relationship between the two is obtained by noting that for any cantilever of length l

$$\omega^2 = (\text{const.}/l^4)(\alpha/\mu) = c_1\mu/l^4 = c_1 M/l^5$$

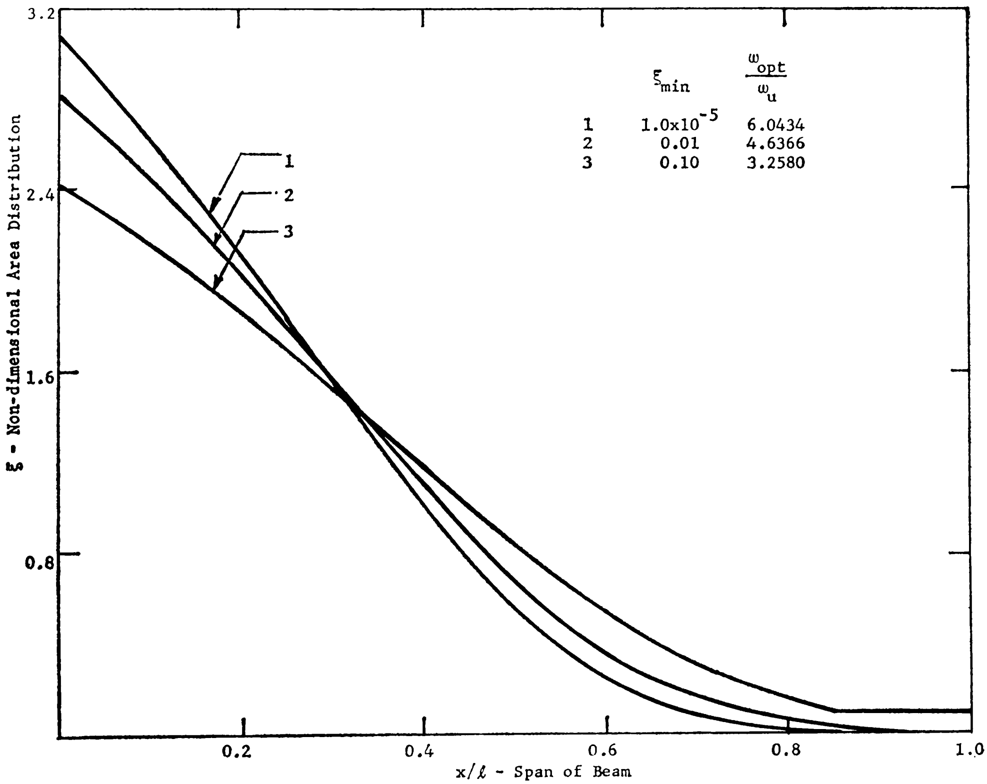

FIG. 1. Optimal cross-sectional area distribution in a vibrating beam.

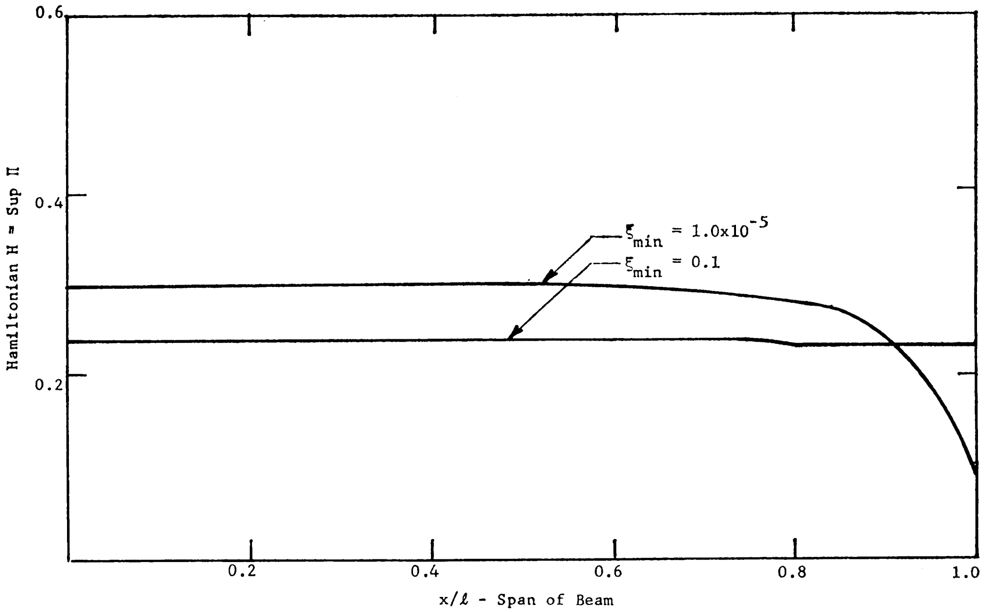

FIG. 2. Distribution of the Hamiltonian function in an optimal vibrating beam.

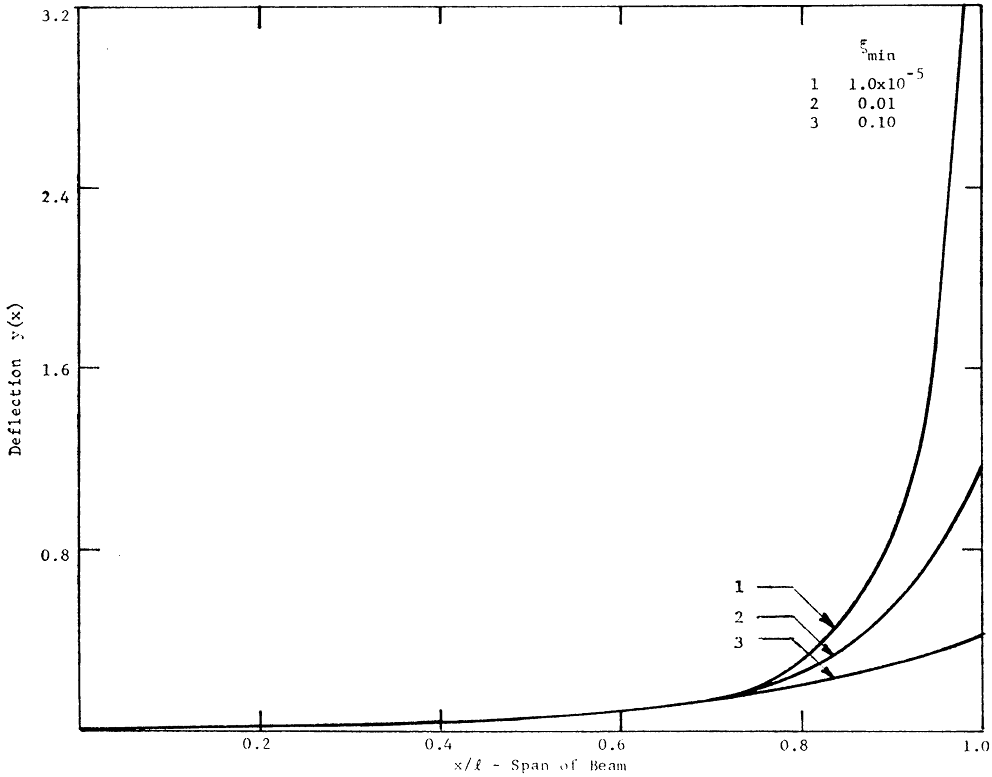

FIG. 3. Lateral deflection mode of optimal vibrating beams.

where M = total mass of the beam, c_1 = constant depending on the cross-sectional shape of the cantilever, $\omega^2 = c_u M_u/l^5 = c_{opt} M_{opt}/l^5$, and $M_{opt}/M_u = c_u/c_{opt} = (\omega_u)^2/\omega_{opt}$.

The resulting savings in volume or mass are seen to be quite spectacular. The interesting feature of this result is the contrast with the case of the simply-supported vibrating beam where the increase in frequency is only 6.6%. It is also obvious that the quantity ω_{opt}/ω_u can be made as high as desired by choosing smaller values of μ_b. However, the actual computational procedure does not permit the choice of arbitrarily small values of μ_b, since the integration of the differential equations involves division by this number.

The iteration procedure in the program is terminated when the difference between successive solutions is an arbitrarily small number. A check is provided by the distribution of the Hamiltonian function Π along the span of the beam. The value of Π should be constant in the region where $\partial\Pi/\partial\mu = 0$. In the case where $\mu_b/\mu_u = 0.1$, Fig. 2 indicates

TABLE 1.

ξ_b	ξ_0	ω_{opt}/ω_u	$M_{opt}/M_u = (\omega_u/\omega_{opt})^2$
0.10	2.44	3.258381	0.09460
0.01	2.83	4.636576	0.04660
1.0×10^{-5}	3.09	6.043409	0.02736

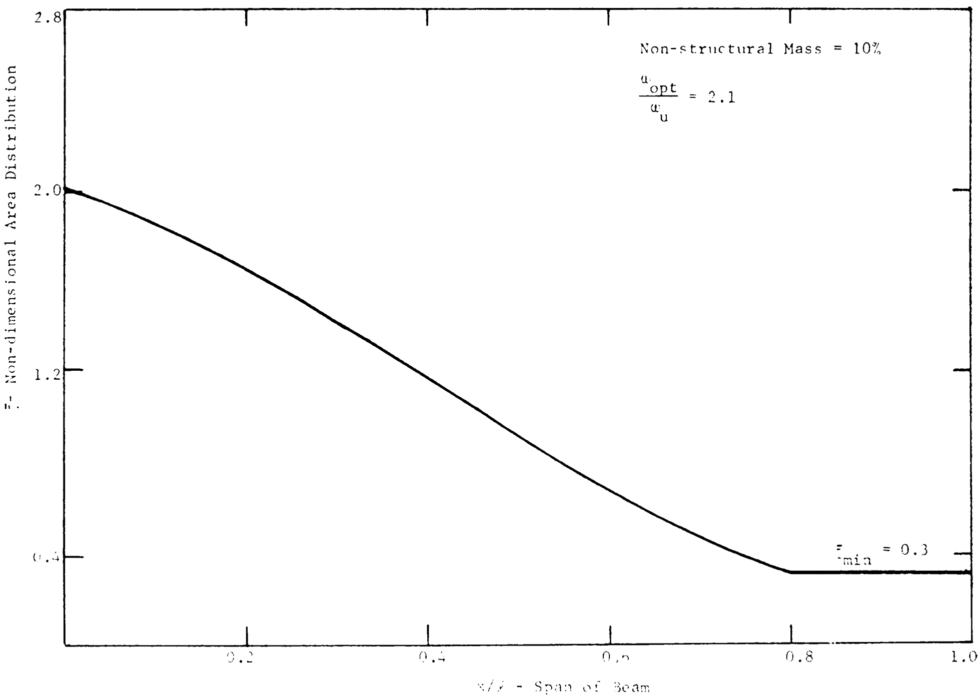

FIG. 4. Optimal area distribution in a vibrating beam.

that Π is indeed constant over the region where $\partial\Pi/\partial\mu = 0$. For the case where $\mu_b/\mu_u = 1.0 \times 10^{-5}$, the value of Π drops off in a region near the free end of the beam where $\partial\Pi/\partial\mu \neq 0$. Fig. 3 shows the wide variation in the lateral deflection configurations of the different optimal beams. Fig. 4 shows a profile of the optimum mass distribution, when a percentage of the material is assumed to be nonstructural but distributed uniformly along the length of the beam.

In conclusion it is necessary to remember that the analysis as carried out is only valid for small deflections, since the simplified expression for curvature is no longer sufficiently exact for large slopes and deflections. Furthermore, the state-space approach and the principle of superposition are used in various stages of the algorithm, the former being essential for the application of Pontryagin's principle. This remark seems necessary in view of cases (2) and (3) in Fig. 3, indicating the occurrence of large slopes in the deflection modes. However, as a consequence of the homogeneous nature of the problem, the analysis yields no information on the absolute magnitudes of y and y' and the figure merely shows the mode shapes.

REFERENCES

[1] L. S. Pontryagin, et al., *The mathematical theory of optimal processes*, English tr., Interscience Publishers, N. Y., 1962

[2] R. M. Brach, *On the extremal fundamental frequencies of vibrating beams*, Int. J. Solids Structures **4**, 667–674 (1968)

[3] B. L. Karihaloo and F. I. Niordson, *Optimum design of vibrating cantilevers*, Rpt. No. 15, Dept. of Solid Mechanics, The Technical University of Denmark.

[4] T. A. Weisshaar, *An application of control theory methods to the optimization of structures having dynamic or aeroelastic constraints*, AFOSR, 70-2862 TR, SUDAAR 412, Dept. of Aeronautics and Astronautics, Stanford University, 1970

[5] F. I. Niordson, *The optimal design of a vibrating beam*, Quart. Appl. Math. **23**, 47–53 (1965)

[6] W. Prager, *Optimality criteria derived from classical extremum principles* in *An introduction to structural optimization*, Study No. 1, Solid Mechanics Division, University of Waterloo (1968)

[7] G. A. Bliss, *The problem of Lagrange in the calculus of variations*, Am. J. Math. **52**, 673–744 (1930)

[8] R. G. Gottleib, *Rapid convergence to optimum solutions using a Min-H strategy*, AIAA J. **5**, 322–329 (1969)

[9] L. Collatz, *The numerical treatment of differential equations*, Springer-Verlag, Berlin, 1960