ON THE SECULAR EQUATION FOR ANISOTROPIC WAVE MOTIONS*

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Summary. In a previous paper [7] it was shown that for waves propagating in certain linear anisotropic mechanical systems there exist universal connections between the phase speeds of harmonic plane waves. Here the results are shown to hold in a wider context.

This note is concerned with the propagation of harmonic plane waves in anisotropic media. We assume that the equations of motion describing the system are of the form

$$\sum_{l=0}^{L} a_{\alpha i p \beta}^{(l)} \frac{\partial^{l+2} u_\beta}{\partial x_i \, \partial x_p \, \partial t^l} + \sum_{m=0}^{M} b_{\alpha i \beta}^{(m)} \frac{\partial^{m+1} u_\beta}{\partial x_i \, \partial t^m} + \sum_{q=0}^{Q} d_{\alpha\beta}^{(q)} \frac{\partial^q u_\beta}{\partial t^q} = 0,$$

$$\alpha, \beta = 1, 2, \cdots r; \quad j, p = 1, 2, 3. \tag{1}$$

Here **a**, **b**, **d** are constants, L, M, Q are arbitrary positive integers, and **u** is assumed sufficiently smooth that all the various derivatives involved in (1) are continuous. The summation convention applies to subscripts as follows: repeated lower-case greek indices are summed from 1 to r and repeated lower-case latin indices summed from 1 to 3. Thus (1) is a system of r differential equations for the r functions u_α of the four independent variables x_i and t.

The motivation for considering systems of the form (1) is that they are of frequent occurrence in mathematical physics. Suitable choices of L, M, Q and of **a**, **b**, **d** lead to systems of equations which occur for example in electromagnetism [1], [2, chap. IV] linearized elasticity [3], thermoelasticity [4], and magnetohydrodynamics [5, p. 749].

Now consider the propagation of plane waves. Let

$$u_\alpha = U_\alpha \exp \iota\omega(S n_i x_i + t), \qquad \alpha = 1, \cdots r, n_i n_i = 1. \tag{2}$$

Here U_α are constants, the angular frequency ω is a constant, assumed positive, and **n** is the direction of the wave. It is assumed that planes of constant phase are also planes of constant amplitude. If S is written $S = S^+ + \iota S^-$, then $1/S^+$ is the speed of propagation of the wave. Inserting (2) into (1) leads to

$$\left\{ \sum_{l=0}^{L} (\iota\omega)^{l+2} a_{\alpha i p \beta}^{(l)} S^2 n_i n_p + \sum_{m=0}^{M} (\iota\omega)^{m+1} b_{\alpha i \beta}^{(m)} S n_i + \sum_{q=0}^{Q} (\iota\omega)^q \, d_{\alpha\beta}^{(q)} \right\} U_\beta = 0. \tag{3}$$

These r equations will have a non-trivial solution for U_β provided the secular equation

$$\det \{ D_{\alpha\beta} V^2 + B_{\alpha i \beta} n_i V + A_{\alpha i p \beta} n_i n_p \} = 0 \tag{4}$$

is satisfied. Here I have written $V = 1/S$ and

$$D_{\alpha\beta} = \sum_{q=0}^{Q} (\iota\omega)^q \, d_{\alpha\beta}^{(q)}, \qquad B_{\alpha i \beta} = \sum_{m=0}^{M} (\iota\omega)^{m+1} b_{\alpha i \beta}^{(m)}, \qquad A_{\alpha i p \beta} = \sum_{l=0}^{L} (\iota\omega)^{l+2} a_{\alpha i p \beta}^{(l)}. \tag{5}$$

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Since we are concerned only with solutions which depend upon both $\mathbf{x}$ and t it will be assumed that $\mathbf{D} \neq \mathbf{0}$. (If $\mathbf{D} = \mathbf{0}$, a solution of (1) is $u_\beta = U_\beta \exp(\iota\omega t)$, where U_β is a constant.) If now for a fixed chosen $\mathbf{n}$ (4) is solved for V, the corresponding $\mathbf{U}$ may be found from (3). In general for a given $\mathbf{n}$ there will be $2r$ roots of (4). Let them be denoted by V_Γ, $\Gamma = 1, 2, \cdots 2r$, or, more explicitly, by $V_\Gamma(\mathbf{n})$. For any of these, if $V_\Gamma^+(\mathbf{n}) < 0$ the wave propagates in the positive $\mathbf{n}$ direction, whilst if $V_\Gamma^+(\mathbf{n}) > 0$ the wave propagates in the negative $\mathbf{n}$ direction. If $V^+(\mathbf{n}) = 0$ the wave does not propagate. In general some of the waves will propagate in one direction, some in the opposite direction and the remainder will be stationary.

Now consider the secular equation (4). Assuming that $\det \mathbf{D} \neq 0$, it is easily seen from (4) that

$$\sum_{\Gamma=1}^{2r} V_\Gamma(\mathbf{n}) = -\mathrm{tr}\,(\mathbf{D}^{-1}\mathfrak{B}),$$
$$\sum_{\substack{\Gamma,\Delta=1\\ \Gamma\neq\Delta}}^{2r} V_\Gamma(\mathbf{n})\,V_\Delta(\mathbf{n}) = \mathrm{tr}\,(\mathbf{D}^{-1}\mathfrak{A}) + [(\mathrm{tr}\,\mathbf{D}^{-1}\mathfrak{B})^2 - \mathrm{tr}\,(\mathbf{D}^{-1}\mathfrak{B})^2]/2, \tag{6}$$

where

$$\mathfrak{B}_{\alpha\beta} = B_{\alpha i\beta}n_i\,, \qquad \mathfrak{A}_{\alpha\beta} = A_{\alpha i\nu\beta}n_in_\nu\,. \tag{7}$$

Hence

$$\sum_{\Gamma=1}^{2r} V_\Gamma^2(\mathbf{n}) = \mathrm{tr}\,(\mathbf{D}^{-1}\mathfrak{B})^2 - 2\,\mathrm{tr}\,(\mathbf{D}^{-1}\mathfrak{A}) = \Psi_{jq}n_jn_q\,, \tag{8}$$

where

$$\Psi_{jq} = D_{\alpha\beta}^{-1}B_{\beta j\phi}\,D_{\phi\sigma}^{-1}B_{\sigma q\alpha} - 2D_{\alpha\beta}^{-1}A_{\beta jq\alpha}\,. \tag{9}$$

If $\mathbf{m}$ and $\mathbf{p}$ are any two other directions such that $\mathbf{n}$, $\mathbf{m}$ and $\mathbf{p}$ form an orthogonal triad of unit vectors then

$$\begin{aligned}\sum_{\Gamma=1}^{2r} \{V_\Gamma^2(\mathbf{n}) + V_\Gamma^2(\mathbf{m}) + V_\Gamma^2(\mathbf{p})\} &= \Psi_{ij}(n_in_j + m_im_j + p_ip_j)\\ &= \Psi_{ii}\,.\end{aligned} \tag{10}$$

Since the right-hand side of (10) is an invariant it follows that if $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ is *any* other triad of mutually orthogonal directions then

$$\sum_{\Gamma=1}^{2r} \{V_\Gamma^2(\mathbf{n}) + V_\Gamma^2(\mathbf{m}) + V_\Gamma^2(\mathbf{p})\} = \sum_{\Gamma=1}^{2r} \{V_\Gamma^2(\mathbf{q}) + V_\Gamma^2(\mathbf{r}) + V_\Gamma^2(\mathbf{s})\}. \tag{11}$$

This is a universal connection, holding irrespective of the form of the $(L + M + Q + 3)$ coefficients $a_{\alpha i\nu\beta}^{(l)}$, $b_{\alpha i\beta}^{(m)}$ and $d_{\alpha\beta}^{(q)}$. This connection is completely general and holds for all systems of the form (1), assuming that $\mathbf{a} \neq \mathbf{0}$, $\mathbf{D} \neq \mathbf{0}$, $\det \mathbf{D} \neq 0$. It should be noted that $\mathbf{n}$, $\mathbf{m}$, $\mathbf{p}$ and $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ are *any* two triads of mutually orthogonal directions.

If $\mathbf{a} = \mathbf{0}$, the secular equation (4) becomes

$$\det\,\{D_{\alpha\beta}V + B_{\alpha i\beta}n_i\} = 0. \tag{12}$$

Assuming that det $\mathbf{D} \neq 0$, it is easily shown that in place of (11) we have the similar universal connection

$$\sum_{\Gamma=1}^{r} \{V_\Gamma^2(\mathbf{n}) + V_\Gamma^2(\mathbf{m}) + V_\Gamma^2(\mathbf{p})\} = \sum_{\Gamma=1}^{r} \{V_\Gamma^2(\mathbf{q}) + V_\Gamma^2(\mathbf{r}) + V_\Gamma^2(\mathbf{s})\} \tag{13}$$

where Γ is now summed from 1 to r.

Universal connections are of value to the experimentalist. For, when the constants determining the system have been obtained experimentally, the universal connection (11) provides a useful cross-check on this data, a cross-check which is moreover easily verified.

A corollary of (11) is that

$$\sum_{\Gamma=1}^{2r} \{V_\Gamma^2(\mathbf{n}) + V_\Gamma^2(\mathbf{m})\} = \sum_{\Gamma=1}^{2r} \{V_\Gamma^2(\mathbf{q}) + V_\Gamma^2(\mathbf{r})\}, \tag{14}$$

where $(\mathbf{n}, \mathbf{m})$ and $(\mathbf{q}, \mathbf{r})$ are *any* two pairs of coplanar mutually orthogonal directions, with a corresponding corollary from (13).

Note that V_Γ will be complex in general. Hence in each of (11) and (14) there are two universal connections, obtained by taking the real amd imaginary parts in the relations.

Finally, note that the universal connections (11) and (14) were derived from the secular equation (4). Clearly the same connections will hold for any other model (e.g. small-amplitude waves in linearized viscoelasticity [6]) which leads to a similar secular equation.

References

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