FUNCTIONAL DIFFERENTIAL EQUATIONS OF PEANO-BAKER TYPE*

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Abstract. The application of the moment method to a class of Fredholm integral equations leads to the study of both an infinite system of ordinary differential equations and a functional differential equation. Sufficient conditions are given that the solution to a truncated system of differential equations approximates the solutions to the original problem. A numerical algorithm for constructing approximate solutions is discussed.

1. Introduction. The principal result given in this paper is an approximatio theorem which provides a justification for an algorithm for constructing approximate solutions of both a functional differential equation of Peano–Baker type and a closely-related denumerable system of ordinary differential equations. Since the algorithm was designed to solve a specific problem in the theory of integral equations, a brief description of the transition to the abstract problem is given as well as the results of some numerical tests.

The functional differential equation of interest here is

$$df/dt = F(f, t)\Theta(t, t_0 ; f), \quad t \in [t_0 , t_1], \quad f(t_0) = f_0 , \tag{1.1}$$

where the solution $f(t)$ is a scalar-valued function, $F(f, t)$ is a smooth $1 \times m$ matrix, and $\Theta(t, t_0 ; f)$ is an integral operator of Peano–Baker type, i.e.

$$\Theta(t, t_0 ; f) = \sum_0^\infty \mathfrak{F}^{(k)}(t, t_0 ; f)\alpha_k . \tag{1.2}$$

The α_k are constant m-vectors and the integral operators $\mathfrak{F}^{(k)}$ are defined recursively:

$$\mathfrak{F}^{(0)} = E, \text{ the identity matrix,}$$

$$\mathfrak{F}^{(k+1)}(t, t_0 ; f) = \int_{t_0}^t B(f(t'))\mathfrak{F}^{(k)}(t', t_0 ; f)\, dt'. \tag{1.3}$$

In (1.3), $B(f)$ is a smooth $m \times m$ matrix.

Integral operators analogous to $\mathfrak{F}^{(k)}$ occur in the Peano–Baker theory of linear systems of ordinary differential equations [2].

The functional differential equation (1.1) was derived from a denumerable system of ordinary differential equations,

$$df/dt = F(f, t)z_0 , \quad f(t_0) = f_0 , \tag{1.4}$$

$$(d/dt)z_j = B(f)z_{j+1} , \quad z_j(t_0) = \alpha_j , \quad j = 0, 1, 2, \cdots \tag{1.5}$$

where z_j and α_j are m-vectors and f is a scalar.

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If we consider f as a known function, then a formal solution to (1.5) is given by

$$z(t) = \sum_{0}^{\infty} \mathfrak{F}^{(k)}(t, t_0 ; f)S^k\alpha, \tag{1.6}$$

where $z = (z_0, z_1, z_2, \cdots)$, $\alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots)$, denote sequences whose elements are m-vectors. The shift operator S is defined by $S\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots)$. As usual, $S^0\alpha = \alpha$, $S^{k+1}\alpha = S(S^k\alpha)$.

Since the first component of the sequence $S^k\alpha$ is the vector α_k, we have

$$z_0(t) = \sum_{0}^{\infty} \mathfrak{F}^{(k)}(t, t_0 ; f)\alpha_k . \tag{1.7}$$

After substituting (1.7) into (1.4), we obtain the functional differential equation (1.1).

Our approximation theorem can be viewed as a solution to the following problem: find sufficient conditions on the vectors $\{\alpha_k\}$ so that the solution to the truncated system,

$$df/dt = F(f, t)z_0 , \quad f(t_0) = f_0 , \tag{1.8}$$

$$(d/dt)z_j = B(f)z_{j+1} , \quad z_j(t_0) = \alpha_j , \quad j = 0, 1, 2, \cdots , M, \quad z_{M+1} \equiv 0, \tag{1.9}$$

approximates at least one solution of the full system (1.4), (1.5). As we shall see, the nature of the approximation and the question of which solution is approximated play a central role in the theory.

The corresponding problem for functional differential equations of Peano–Baker type is to find sufficient conditions on the vectors $\{\alpha_j\}$ so that the solution to

$$df/dt = F(f, t)\Theta_M(t, t_0 ; f), \qquad f(t_0) = f_0 , \qquad \Theta_M = \sum_{0}^{M} \mathfrak{F}^{(k)}(t, t_0 ; f)\alpha_k , \tag{1.10}$$

approximates the (unique) solution to the functional differential equation (1.1).

In the next section, existence and uniqueness theorems will be given for both the functional differential equation (1.1) and the denumerable system (1.4), (1.5).

2. Existence and uniqueness theorems. We shall assume that if $|f - f_0| \leq M$, $|f^* - f_0| \leq M$, $t \in [t_0 , t_1]$, then $B(f)$ and $F(f, t)$ are continuous and are subject to the following:

$$\begin{aligned} ||B(f) - B(f^*)|| &\leq L\,|f - f^*|, \qquad ||B(f)|| \leq K, \\ ||F(f, t) - F(f^*, t)|| &\leq L\,|f - f^*|, \qquad ||F(f, t)|| \leq K. \end{aligned} \tag{2.1}$$

Here, and subsequently, $||\cdot||$ denotes a suitable matrix norm, e.g. a norm of the $m \times 1$ matrix (vector) α_k or of the $m \times m$ matrix $B(f)$. For convenience, we require that $||E|| = 1$.

From (1.3) and (2.1) it follows that

$$||\mathfrak{F}^{(k)}(t, t_0 ; f)|| \leq [K(t - t_0)]^k/k!, \quad t \in [t_0 , t_1]. \tag{2.2}$$

Hence,

$$||\Theta(t, t_0 ; f)|| \leq g(K(t - t_0); \alpha), \tag{2.3}$$

where $g(\tau; \alpha)$ is a monotonic increasing function defined by

$$g(\tau; \alpha) = \sum_0^\infty \tau^k \, ||\alpha_k||/k!. \tag{2.4}$$

For the estimate (2.3) to be useful, the power series (2.4) must have a nonzero radius of convergence. Therefore, we assume that the initial-value vectors $\{\alpha_k\}$ are subject to a growth constraint, namely,

$$\limsup_{k\to\infty} [||\alpha_k||/k!]^{1/k} < \infty. \tag{2.5}$$

This growth constraint is sharp in the sense that it is sufficient for a local approximation theory and that if it is violated, one can construct examples of functional differential equations such as (1.10) whose solutions have no limit as $M \to \infty$.

For later use, we note that if

$$g(\tau; S^i\alpha) = \sum_0^\infty \tau^k \, ||\alpha_{k+i}||/k!, \tag{2.6}$$

and

$$g_i(\tau; \alpha) = \sum_i^\infty \tau^k \, ||\alpha_k||/k!, \tag{2.7}$$

then

$$g(\tau; S^i\alpha) = (d^i/d\tau^i) g_i(\tau; \alpha). \tag{2.8}$$

We shall need the following estimate:

LEMMA: If $f(t)$ and $f^*(t)$ are continuous functions on $[t_0, t_1]$ such that $|f(t) - f_0| \leq M$, $|f^*(t) - f_0| \leq M$, then for all $t \in [t_0, t_1]$,

$$||\Theta(t, t_0; f) - \Theta(t, t_0; f^*)|| \leq Lg(K(t - t_0), S\alpha) \int_{t_0}^t |f(s) - f^*(s)| \, ds. \tag{2.9}$$

Proof: Let $\Phi(t, t_0, \lambda; f)$ and $\Phi(t, t_0, \lambda; f^*)$ denote the fundamental solution matrices of $dy/dt = \lambda B(f(t))y$, $dy^*/dt = \lambda B(f^*(t))y^*$, respectively. Then, by a familiar variation-of-parameters argument, we have

$$\Phi(t, t_0, \lambda, f) - \Phi(t, t_0, \lambda; f^*) = \lambda \int_{t_0}^t \Phi(t, s, \lambda; f)[B(f(s)) - B(f^*(s))]\Phi(s, t_0, \lambda, f^*) \, ds.$$

Since

$$\Phi(t, t_0, \lambda; f) = \sum_0^\infty \lambda^k \mathfrak{F}^{(k)}(t, t_0; f),$$

we can expand in powers of λ, equate coefficients, and obtain

$$\mathfrak{F}^{(k+1)}(t, t_0; f) - \mathfrak{F}^{(k+1)}(t, t_0; f^*)$$
$$= \int_{t_0}^t \sum_0^k \mathfrak{F}^{(k-i)}(t, s; f)[B(f(s)) - B(f^*(s))]\mathfrak{F}^{(i)}(s, t_0; f^*) \, ds.$$

After multiplying by α_{k+1} and summing, we obtain

$$\Theta(t, t_0 ; f) - \Theta(t, t_0 ; f^*)$$
$$= \sum_0^{\infty} \left(\int_{t_0}^{t} \sum_0^{k} \mathfrak{F}^{(k-i)}(t, s; f)[B(f(s)) - B(f^*(s))]\mathfrak{F}^{(i)}(s, t_0 ; f^*)\, ds \right) \alpha_{k+1} .$$

The estimate (2.9) can now be derived with the aid of (2.1), (2.2), and (2.4).

It is now easy to show that the functional differential equation (1.1) has a unique solution.

THEOREM: If the interval $[t_0 , t_1]$ is restricted so that $g(K(t_1 - t_0); \alpha) < \infty$, and

$$K \int_{t_0}^{t_1} g(K(s - t_0); \alpha)\, ds \leq M, \tag{2.9}$$

then there is a unique solution to the functional differential equation

$$df/dt = F(f, t)\Theta(t, t_0 ; f), \quad f(t_0) = f_0 , \quad t \in [t_0 , t_1].$$

Proof: Define a sequence $\{f_n(t)\}$ by $f_0(t) \equiv f_0$,

$$(d/dt)f_{n+1} = F(f_n , t)\Theta(t, t_0 ; f_n), \; t \in [t_0 , t_1], \; f_n(t_0) = f_0 , \; n = 1, 2, \cdots .$$

The bound (2.9) insures that the sequence is well-defined. The convergence and uniqueness proof is standard and is omitted.

Once $f(t)$ is known, we can construct a solution to the denumerable system of ordinary differential equations (1.5) by setting

$$z_j(t) = \sum_0^{\infty} \mathfrak{F}^{(k)}(t, t_0 ; f)\alpha_{k+j} , \qquad j = 0, 1, 2, \cdots . \tag{2.10}$$

From (2.10), we have

$$(d/dt)z_j(t) = B(f(t)) \sum_1^{\infty} \mathfrak{F}^{(k-1)}(t, t_0 ; f)\alpha_{j+k} .$$

Hence (recall $\mathfrak{F}^{(0)} = E$),

$$(d/dt)z_j(t) = B(f(t))z_{j+1}(t), \quad z_j(t_0) = \alpha_j .$$

Since

$$||z_j(t)|| \leq g(K(t - t_0); S^j\alpha) < \infty, \quad t \in [t_0 , t_1], \tag{2.11}$$

the series for z_j and dz_j/dt converge uniformly on $[t_0 , t_1]$. We have proved the following.

THEOREM: If $f(t)$ is a solution to the functional differential equation (1.1) on the interval $[t_0 , t_1]$ and if $g(K(t_1 - t_0); \alpha) < \infty$, then the sequence (2.10) is a solution to the denumerable system of ordinary differential equations (1.5) on $[t_0 , t_1]$.

The following example shows that uniqueness cannot be expected without further restrictions. Consider the sequence of scalar equations,

$$(d/dt)z_j = z_{j+1} , \quad z_j(0) = u^j j!, \quad j = 0, 1, 2, \cdots , \tag{2.12}$$

where u is a positive constant. Then

$$z_0(t) = (1 - ut)^{-1} + w \exp(-t^{-2}), \; w \text{ arbitrary},$$
$$z_{j+1}(t) = (d/dt)z_j(t), \quad j = 0, 1, 2, \cdots$$

is a one-parameter family of solutions on the interval $[0, 1/u)$. It should also be noted that even as simple a system as (2.12) need not have a global solution.

The inequality (2.11) suggests a reasonable restriction on the class of possible solutions to the denumerable system (1.5).

Definition: A sequence of functions $\{z_i(t)\}$ is said to be g-dominated if (2.11) is satisfied.

THEOREM: There is a unique solution of (1.5) within the class of g-dominated sequences.

Proof: If $\{z_i(t)\}$ is a solution to (1.5), then

$$z_i(t) = \alpha_i + \int_{t_0}^{t} B(f(s))z_{i+1}(s)\, ds,$$

$$z_{i+1}(s) = \alpha_{i+1} + \int_{t_0}^{s} B(f(s_1))z_{i+2}(s_1)\, ds_1 ,$$

so

$$z_i(t) = \alpha_i + \int_{t_0}^{t} B(f(s))\alpha_{i+1} + \int_{t_0}^{t} B(f(s)) \int_{t_0}^{s} B(f(s_1))z_{i+2}(s_1)\, ds_1\, ds.$$

By repeated substitutions, we obtain

$$z_i(t) - \sum_0^M \mathfrak{F}^{(k)}(t, t_0\, ; f)\alpha_{k+i} = \int_{t_0}^{t} B(f(s)) \int_{t_0}^{s} \cdots \int_{t_0}^{s_{M-1}} B(f(s_M))z_{i+M+1}(s_M)\, ds_M \cdots ds.$$

It follows from (2.1) and (2.11) that

$$\left\| z_i(t) - \sum_0^M \mathfrak{F}^{(k)}(t, t_0\, ; f)\alpha_{k+i} \right\| \leq K^{M+1} \int_{t_0}^{t} \frac{(t-s)^M}{M!}\, g(K(s - t_0), S^{i+M+1}\alpha)\, ds.$$

By virtue of (2.8), we now have

$$\left\| z_i(t) - \sum_0^M \mathfrak{F}^{(k)}(t, t_0\, ; f)\alpha_{k+i} \right\| \leq g_{M+1}(K(t - t_0); S^i\alpha), \qquad t \in [t_0\, , t_1].$$

We now let M approach infinity and obtain the representation (2.10).

The g-dominated solutions have an important property: they can be constructed by taking componentwise limits of solutions to a finite system of ordinary differential equations. This is discussed in the next section.

3. Approximation theorems. Let us now consider the problem of constructing approximate solutions to the functional differential equation (1.1) and of the related denumerable system of ordinary differential equations (1.4), (1.5). The iterative method which was used to prove the existence of a solution to (1.1) is not an efficient computing algorithm since, among other difficulties, the direct evaluation of the integral operators $\mathfrak{F}^{(k)}(t, t_0\, ; f_n)$ is impractical if k is large. However, since the functional differential equation was derived from the denumerable system, it should not be surprising that truncating the denumerable system leads to an effective computational algorithm. In fact, this study was motivated by use of a truncated system in a specific problem in the theory of integral equations which will be discussed in the next section.

Let $f^M(t)$, $z_j^M(t)$, $j = 0, 1, \cdots, M$, denote the unique solution to the finite system of ordinary differential equations,

$$df/dt = F(f, t)z_0, \quad f(t_0) = f_0, \quad t \in [t_0, t_1],$$
$$(d/dt)z_j = B(f(t))z_{j+1}, \quad z_j(t_0) = \alpha_j, \quad j = 0, 1, \cdots, M, \tag{3.1}$$

where $z_{M+1}(t) \equiv 0$. It follows that $f^M(t)$ (see Sec. 1) is the unique solution of the functional differential equation

$$df/dt = F(f, t)\Theta^M(t, t_0; f), \quad f(t_0) = f_0, \tag{3.2}$$

where

$$\Theta^M(t, t_0; f) = \sum_0^M \mathfrak{F}^{(k)}(t, t_0; f)\alpha_k. \tag{3.3}$$

Furthermore, the $z_j^M(t)$ can be written

$$z_j^M(t) = \sum_0^M \mathfrak{F}^{(k)}(t, t_0; f^M)\alpha_{j+k}, \qquad j = 0, 1, 2, \cdots, M. \tag{3.4}$$

We shall later use the following decomposition and estimates

$$\Theta(t, t_0; f) = \Theta^M(t, t_0; f) + \Theta_{M+1}(t, t_0; f)$$

where (see (2.7))

$$\Theta_{M+1}(t, t_0; f) = \sum_{M+1}^{\infty} \mathfrak{F}^{(k)}(t, t_0; f)\alpha_k, \quad ||\Theta_{M+1}(t, t_0; f)|| \le g_{M+1}(K(t - t_0); \alpha). \tag{3.5}$$

The relationship between the solutions to the truncated problem (3.2) and the full problem (1.1) is the content of the following theorem.

THEOREM: Let $f(t)$ and $f^M(t)$ denote solutions to the functional differential equations

$$df/dt = F(f, t)\Theta(t, t_0; f), \quad df/dt = F(f, t)\Theta^M(t, t_0; f), \quad t \in [t_0, t_1],$$

respectively, with initial conditions $f(t_0) = f^M(t_0) = f_0$. Then if $g(K(t_1 - t_0); \alpha) < \infty$, there exists a positive constant σ such that

$$|f(t) - f^M(t)| \le \sigma g_{M+1}(K(t_1 - t_0); \alpha), \quad t \in [t_0, t_1]. \tag{3.6}$$

Hence

$$\lim_{M\to\infty} |f(t) - f^M(t)| = 0, \quad \text{uniformly on} \quad [t_0, t_1].$$

Proof. We write

$$(d/dt)(f - f^M) = [F(f, t) - F(f^M, t)]\Theta(t, t_0; f) + F(f^M, t)[\Theta^M(t, t_0; f) - \Theta^M(t, t_0; f^M)] + F(f^M, t)\Theta_{M+1}(t, t_0; f).$$

Then

$$f(t) - f^M(t) = \int_{t_0}^t [F(f(s), s) - F(f^M(s), s)]\Theta(s, t; f)\, ds + \int_{t_0}^t F(f^M(s), s)[\Theta^M(s, t_0; f) - \Theta^M(s, t_0; f^M)]\, ds + \int_{t_0}^t F(f^M(s), s)\Theta_{M+1}(s, t_0; f)\, ds.$$

With the aid of the estimates (2.1), (2.9), and (3.5), we obtain

$$|f(t) - f^M(t)| \leq L \int_{t_0}^{t} g(K(s - t_0); \alpha)\, |f(s) - f^M(s)|\, ds$$

$$+ KLg(K(t - t_0); S\alpha) \int_{t_0}^{t} |f(s) - f^M(s)|\, ds + K \int_{t_0}^{t} g_{M+1}(K(s - t_0); \alpha)\, ds.$$

The estimate (3.6) follows immediately.

COROLLARY: For fixed j,

$$\lim_{M\to\infty} ||z_j(t) - z_j^M(t)|| = 0, \quad \text{uniformly on} \quad [t_0 , t_1]. \tag{3.7}$$

Proof: From (2.10) and (3.4), we have

$$z_j(t) - z_j^M(t) = \sum_{0}^{M-j} (\mathfrak{F}^{(k)}(t, t_0 ; f) - \mathfrak{F}^{(k)}(t, t_0 ; f^M))\alpha_{k+j} + \sum_{M-j+1}^{\infty} \mathfrak{F}^{(k)}(t, t_0 ; f)\alpha_{k+j}$$

Hence,

$$||z_j(t) - z_j^M(t)|| \leq L \int_{t_0}^{t} g(K(s - t_0); S^j\alpha)|\, f(s) - f^M(s)|\, ds + g_{M-j+1}(K(t - t_0); S^j\alpha).$$

The desired result (3.7) follows immediately from (3.6).

It should be emphasized that the limit is coordinatewise. Unless severe growth restrictions are placed on the initial-value vectors $\{\alpha_k\}$ it is not possible to define a norm on the space of sequences so that the shift operator is bounded. Because of this, convergence in norm should not be expected.

4. The motivating problem. Let $\varphi(z, t)$ be a solution of the following integral equation:

$$\varphi(z, t) = K_1(t - z) + \int_{-t}^{t} \gamma(z')K(|z - z'|)\varphi(z', t)\, dz', \qquad z \in [-t, t], \quad t \in [0, T],$$

where K and K_1 have integral representations

$$K(u) = \int_0^\infty k(s) \exp[-a(s)u]\, ds,$$

$$K_1(u) = \int_0^\infty k_1(s) \exp[-a(s)u]\, ds, \qquad u \in [0, T].$$

It is assumed that $k(s)$ and $k_1(s)$ are absolutely integrable functions and that $\gamma(s)$ and $a(s)$ are continuous even functions. The task of computing the diagonal functions $\varphi(t, t)$ and $\varphi(-t, t)$ leads to several interesting mathematical problems, e.g., the study of functional differential equations of Peano–Baker type. The purpose of this section is to give a brief description of the transition to the abstract problem.

Wing [3] used the method of invariant imbedding to show that

$$\varphi(t, t) = \int_0^\infty k_1(s)X(t, s)\, ds, \qquad \varphi(-t, t) = \int_0^\infty k_1(s)Y(t, s)\, ds,$$

where $X(t, s)$ and $Y(t, s)$ are solutions of a nonlinear initial-value problem:

$$
\begin{aligned}
(\partial/\partial t)X(t, s) &= 2\gamma(t)Y(t, s) \int_0^\infty k(s')Y(t, s')\, ds', \\
(\partial/\partial t)Y(t, s) &= -2a(s)Y(t, s) + 2\gamma(t)X(t, s) \int_0^\infty k(s')Y(t, s')\, ds', \\
X(0, s) &= Y(0, s) = 1, \qquad t \in [0, T], \quad s \in [0, \infty).
\end{aligned}
\tag{4.1}
$$

Allen [1] developed a computational algorithm for solving such problems by replacing the system (4.1) by a denumerable system of ordinary differential equations. He set

$$
\begin{aligned}
P_j(t) &= \int_0^\infty [a(s)]^j k(s)X(t, s)\, ds, \\
Q_j(t) &= \int_0^\infty [a(s)]^j k(s)Y(t, s)\, ds, \qquad j = 0, 1, 2, \cdots,
\end{aligned}
\tag{4.2}
$$

and showed that X, Y, P_j, Q_j, satisfy the system

$$
\begin{aligned}
(\partial/\partial t)X(t, s) &= 2\gamma(t)Y(t, s)Q_0(t), \\
(\partial/\partial t)Y(t, s) &= -2a(s)Y(t, s) + 2\gamma(t)X(t, s)Q_0(t), \\
X(0, s) &= Y(0, s) = 1,
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
(d/dt)P_j(t) &= 2\gamma(t)Q_j(t)Q_0(t), \\
(d/dt)Q_j(t) &= -2Q_{j+1}(t) + 2\gamma(t)P_j(t)Q_0(t), \quad j = 0, 1, 2, \cdots, \\
& P_j(0) = Q_j(0) = A_j,
\end{aligned}
\tag{4.4}
$$

where

$$
A_j = \int_0^\infty [a(s)]^j k(s)\, ds, \qquad j = 0, 1, 2, \cdots. \tag{4.5}
$$

Allen truncated the system (4.4) and solved the truncated system together with (4.3) (for selected values of s) by a fourth-order Runge–Kutta method. He conjectured that for many problems of physical interest, the exact X, Y solutions of the truncated system approached the X, Y solutions of the full system as the truncation number approached infinity.

We shall now show that the denumerable system (4.4) can be transformed into an equivalent system,

$$
\begin{aligned}
df/dt &= 2\gamma(t)[U_0 \sinh f + V_0 \cosh f], \\
\frac{d}{dt}\begin{bmatrix} U_j \\ V_j \end{bmatrix} &= 2\begin{bmatrix} \sinh^2 f & \sinh f \cosh f \\ -\sinh f \cosh f & -\cosh^2 f \end{bmatrix}\begin{bmatrix} U_{j+1} \\ V_{j+1} \end{bmatrix}, \quad t \in [0, T], \ j = 0, 1, 2, \cdots
\end{aligned}
\tag{4.6}
$$

with initial conditions $f(0) = 0$, $U_j(0) = V_j(0) = A_j$, where the moments A_j are defined by (4.5).

Clearly, the system (4.6) is a special case of (1.4), (1.5); i.e., we set

$$
z_j(t) = \begin{bmatrix} U_j(t) \\ V_j(t) \end{bmatrix}, \qquad \alpha_j = \begin{bmatrix} A_j \\ A_j \end{bmatrix}, \qquad F(f, t) = 2\gamma(t)[\sinh f, \cosh f], \tag{4.7}
$$

and $B(f)$ equal to the 2×2 matrix of (4.6).

The transformation from (4.4) to (4.6) is motivated by the observation that if $Q_0(t)$ were known, then the system (4.4) would be a linear system which could be simplified by a variation of parameters transformation. We set

$$f(t) = 2\int_0^t \gamma(t')Q_0(t')\,dt', \qquad D(f) = \begin{bmatrix} \cosh f & \sinh f \\ \sinh f & \cosh f \end{bmatrix}.$$

It is easy to verify that D is a fundamental solution matrix of the system

$$dp/dt = 2\gamma(t)Q_0(t)q, \quad dq/dt = 2\gamma(t)Q_0(t)p,$$

which is obtained from (4.4) by suppressing the term Q_{i+1}. The variation-of-parameters transformation,

$$\begin{bmatrix} P_i \\ Q_i \end{bmatrix} = D(f)\begin{bmatrix} U_i \\ V_i \end{bmatrix}, \tag{4.8}$$

can now be employed to transform (4.4) into the equivalent system (4.6).

We also set

$$\begin{bmatrix} X(t, s) \\ Y(t, s) \end{bmatrix} = D(f)\begin{bmatrix} W_1(t, s) \\ W_2(t, s) \end{bmatrix} \tag{4.9}$$

and transform the differential equation (4.3) into

$$\begin{aligned} (\partial/\partial t)W_1(t, s) &= 2a(s)\sinh f[\sinh fW_1(t, s) + \cosh fW_2(t, s)], \\ (\partial/\partial t)W_2(t, s) &= -2a(s)\cosh f[\sinh fW_1(t, s) + \cosh fW_2(t, s)]. \end{aligned} \tag{4.10}$$

We shall discuss these equations later.

The moment equations (4.5) are assumed to satisfy the growth constraint

$$\limsup_{j\to\infty} [|A_j|/j!]^{1/j} < \infty. \tag{4.11}$$

Since (4.11) implies (2.5), the approximation theorems of Sec. 3 provide a mathematical justification of Allen's algorithm.

We shall not attempt a detailed study of the class of admissible functions $\{a(s), k(s)\}$ whose moments satisfy (4.11). However, we note that if $a(s)$ is bounded and $k(s)$ is absolutely integrable, or if $a(s)$ is linear and $k(s)$ decays exponentially, then $a(s)$ and $k(s)$ are admissible. The integral equations studied by Wing [3] admitted a less restrictive class of functions, but it is unlikely that Allen's algorithm would be useful in the absence of a growth constraint.

5. Numerical methods. In this section we shall apply a specialized numerical method to the differential equations of the previous section, more precisely, to the equations obtained by truncating the z-subsystem and restricting the variable s to a finite number of values, $\{s_0, s_1, \cdots, s_N\}$. We shall assume that the resulting finite system of ordinary differential equations has a solution on the interval $[0, T]$.

Let

$$H^n(t) = \begin{bmatrix} W_1(t, s_n) \\ W_2(t, s_n) \end{bmatrix}, \qquad H_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

then we have

$$dH^n/dt = a(s_n)G(f)H^n, \quad H^n(0) = H_0, \quad n = 0, 1, 2, \cdots, \tag{5.1}$$

where $G(f)$ is the coefficient matrix of (4.10).

The truncated z-subsystem is written as

$$dz/dt = B(f)\mathcal{J}z, \quad z(0) = \alpha, \tag{5.2}$$

where z and α are M-vectors whose entries are 2-vectors (see (4.7)). The nonzero entries of the matrix $\mathcal{J}$ lie on the superdiagonal and are E, the 2×2 identity matrix. The 2×2 matrix $B(f)$ is the coefficient matrix of (4.6).

Finally, we have

$$df/dt = F(f, t)z_0, \quad f(0) = 0, \tag{5.3}$$

where $F(f, t)$ is defined in (4.7).

For simplicity, we shall let $H^n(t)$, $z(t)$, $f(t)$, denote the numerical solutions of the system (5.1), (5.2), (5.3).

Our computational algorithm parallels our analysis in that $f(t)$ is treated differently from the other dependent variables. It is computed by a power series method so that $B(f(t))$ and $G(f(t))$ could be evaluated at the points required by the fourth-order algorithms used to compute $z(t)$ and $H^n(t)$. In other words, $z(t)$ and $H^n(t)$ are computed as if $f(t)$ were a known function.

Let us assume that $f(t)$ and $z(t)$ have been computed and that a step size Δ has been chosen. Then for $\tau \in [t, t + \Delta]$,

$$f(\tau) = \sum_0^5 f^{(r)}(t)\ \delta^r/r! + O(\delta^6), \qquad \tau = t + \delta, \quad \delta \in [0, \Delta], \tag{5.4}$$

where

$$f^{(r)}(t) = \sum_0^r \binom{r}{i} \frac{d^i}{dt^{r-i}} z_0(t). \tag{5.5}$$

By means of these formulas, $f(\tau)$ can be considered as a known function on $[t, t + \Delta]$.

The computation of $z(t + \Delta)$ can be done in many ways. We decided to experiment with a pseudo-implicit scheme which exploited the strictly upper triangular form of the matrix $\mathcal{J}$ of the z-subsystem (5.2). We set

$$z(t + \Delta) = [\mathcal{E} + \mathcal{A}(t, \Delta)]z(t) + \mathcal{C}(t, \Delta)z(t + \Delta), \tag{5.6}$$

where $\mathcal{A}$ and $\mathcal{C}$ are strictly upper triangular matrices with two constant superdiagonals,

$$\mathcal{A} = a_1\mathcal{J} + a_2\mathcal{J}^2, \qquad \mathcal{C} = c_1\mathcal{J} + c_2\mathcal{J}^2,$$

$$a_1 = \frac{\Delta}{6}[B_0 + 2B_{1/2}], \qquad a_2 = \frac{\Delta^2}{12}[B_{1/2}B_0],$$

$$c_1 = \frac{\Delta}{6}[B_1 + 2B_{1/2}], \qquad c_2 = \frac{\Delta^2}{12}[-B_{1/2}B_1],$$

$$B_0 = B(f(t)), \qquad B_1 = B(f(t + \Delta)), \qquad B_{1/2} = B(f(t + \Delta/2)).$$

The term "pseudo-implicit" is used since if a backward substitution method is used to compute the components of the vector $z(t + \Delta)$, no matrices need be inverted.

The formulas (5.6) were derived by setting

$$\int_t^{t+\Delta} B(f(t'))\mathcal{g}z(t')\,dt' = \frac{\Delta}{6}[B_0\mathcal{g}z(t) + 4B_{1/2}\mathcal{g}z(t+\Delta/2) + B_1\mathcal{g}z(t+\Delta)] + O(\Delta^5)$$

and using a Hermite interpolation formula to compute

$$z(t+\Delta/2) = \tfrac{1}{2}z(t) + \tfrac{1}{2}z(t+\Delta) + \frac{\Delta}{8}\frac{dz}{dt}(t) - \frac{\Delta}{8}\frac{dz}{dt}(t+\Delta) + O(\Delta^4).$$

The step size Δ was selected by comparing the values of df/dt obtained from a power series formula with those obtained by evaluating the right-hand side of (5.3). More explicitly, we compared

$$(df/dt)(t+\Delta) = \sum_0^4 f^{(r+1)}(t)\,\Delta^r/r! \tag{5.7}$$

with

$$(df/dt)(t+\Delta) = F(f(t+\Delta), t+\Delta)z_0(t+\Delta). \tag{5.8}$$

Since the local truncation error in both formulas is $O(\Delta^5)$, they can be used in a consistent step size control algorithm. We felt that error estimates of the z-subsystem were not required since the function $f(t)$ plays the central role in the $X(t, s)$, $Y(t, s)$ computation (see (4.9) and (5.1)).

A program was written to test our computational algorithm. Eqs. (5.2) were integrated using a scheme based on Eqs. (5.6). Eqs. (5.1) were integrated using a fourth-order Runge–Kutta–Fehlberg code with automatic step size adjustment to control local error. The initial conditions α given by Eqs. (4.5) were evaluated by an adaptive Simpson quadrature. The program was executed primarily in single-precision arithmetic on an IBM 360/67; double precision was used only in carrying out the various transformations.

Two examples follow. They were both run for an s value of 0.5 and for $T = 1.0$.

Example 1.

$$k(s) = e^{-5s},$$

$$a(s) = s, \qquad 0 \le s \le 1.$$

Number of moments	$X(1.0, 0.5)$	$Y(1.0, 0.5)$
5	1.334225	0.6904516
10	1.334029	0.6901570
15	1.334029	0.6901570
20	1.334029	0.6901570

Example 2.

$$k(s) = e^{-s^2}$$

$$a(s) = s, \qquad 0 \le s \le 1.$$

Number of moments	$X(1.0, 0.5)$	$Y(1.0, 0.5)$
5	1.079214	0.4395494
10	1.078984	0.4390551
15	1.078984	0.4390552
20	1.078984	0.4390552

Less than one minute of computer time was used for all of the above calculations. Results are accurate to at least five digits as compared with standard methods for solving (4.1) [1]. As a numerical tool, the method appears to be fast, accurate, and numerically stable.

REFERENCES

[1] R. C. Allen, Jr., *On the numerical solution of a certain class of non-linear Cauchy problems arising from integral equation theory*, Computing **7**, 13–16 (1971)

[2] E. L. Ince, *Ordinary differential equations*, Dover, New York, 1944

[3] G. M. Wing, *On certain Fredholm integral equations reducible to initial value problems*, SIAM Review **9**, 655–670 (1967)